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Trisecant lines and Jacobians, II

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Abstract. We prove that an indecomposable principally polarized complex abelian variety X is the Jacobian of a smooth curve if and only if there exist points a, b, c of X whose images under the Kummer map $X \to |2\Theta|^*$ are distinct and collinear, and such that the subgroup of X generated by $a - b$ and $b - c$ is dense in X.

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1. Introduction

Let (X, λ) be a complex principally polarized abelian variety. Symmetric representatives Θ of the polarization λ differ by translations by points of order 2, hence the linear system $|2\Theta|$ is independent of the choice of Θ. It defines a morphism $K: X \to |2\Theta|^*$, whose image is the *Kummer variety* $K(X)$ of X. When (X, λ) is the Jacobian of an algebraic curve, there are infinitely many *trisecants* to $K(X)$, i.e. lines in the projective space $|2\Theta|^*$ that meet $K(X)$ in at least 3 points. Welters conjectured in [W] that the existence of *one* trisecant line to the Kummer variety should characterize Jacobians among all indecomposable principally polarized abelian varieties, thereby giving one answer to the Schottky problem.

The aim of this article is to improve on the results of [D], where a partial answer to this problem was given under additional hypotheses. More precisely, our main theorem implies that *an indecomposable principally polarized abelian variety* (X, λ) *is a Jacobian if and only if there exist points* a, b, c *of* X *such that*

(i) *the subgroup of* X *generated by* $a - b$ *and* $b - c$ *is dense in* X*;*
(ii) *the points* $K(a)$, $K(b)$ *and* $K(c)$ *are distinct and collinear.*

Instead of working on the intersection of a theta divisor with a translate, whose possibly complicated geometry is the source of most difficulties in [AD], [D], [M] and [S], we perform algebraic calculations directly on a theta divisor, which has the advantage of being integral. The point is to prove that the existence of

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one trisecant line implies the existence of a one-dimensional family of such lines. Welters' criterion ([W]) then yields the conclusion.

2. The set up

Let (X, λ) be a complex indecomposable principally polarized abelian variety, let Θ be a symmetric representative of the polarization and let $K\colon X \to |2\Theta|^*$ be the Kummer morphism. Let θ be a non-zero section of $\mathcal{O}_X(\Theta)$. For any $x \in X$, we write Θ_x for the divisor $\Theta + x$ and θ_x for the section $z \mapsto \theta(z - x)$ of $\mathcal{O}_X(\Theta_x)$. If a, b and c are points of X, it is classical that the points $K(a), K(b)$ and $K(c)$ are collinear if and only if there exist complex numbers α, β and γ not all zero such that

$$\alpha\theta_a\theta_{-a} + \beta\theta_b\theta_{-b} + \gamma\theta_c\theta_{-c} = 0.$$

Following Welters, we consider the set

$$V_{a,b,c} = 2\,\{\zeta \in X \mid K(\zeta + a), K(\zeta + b), K(\zeta + c) \text{ are collinear}\},$$

endowed with its natural scheme structure. By [W], Theorem 0.5, (X, λ) *is a Jacobian if and only if there exist distinct points* a, b *and* c *such that* $\dim V_{a,b,c} > 0$. This condition is equivalent to the existence of a sequence $\{D_n\}_{n>0}$ of constant vector fields on X and of a formal curve $\zeta(\varepsilon) = \zeta(0) + \frac{1}{2}D(\varepsilon)$ with $D(\varepsilon) = \Sigma_{n>0}D_n\varepsilon^n$, contained in $V_{a,b,c}$. This in turn is equivalent to a relation of the type

$$\alpha(\varepsilon)\theta_{a+\zeta(\varepsilon)}\theta_{-a-\zeta(\varepsilon)} + \beta(\varepsilon)\theta_{b+\zeta(\varepsilon)}\theta_{-b-\zeta(\varepsilon)} + \gamma(\varepsilon)\theta_{c+\zeta(\varepsilon)}\theta_{-c-\zeta(\varepsilon)} = 0,$$

where $\alpha(\varepsilon), \beta(\varepsilon)$ and $\gamma(\varepsilon)$ are relatively prime elements of $\mathbf{C}[[\varepsilon]]$.

3. The case of a degenerate trisecant

In this section, we prove the following result:

THEOREM 3.1. *Let* (X, λ) *be a complex indecomposable principally polarized abelian variety, let* Θ *be a symmetric representative of the polarization and let* $K\colon X \to |2\Theta|^*$ *be the Kummer morphism. Assume that there exist points* u *and* v *of* X *such that* $2u \neq 0$ *and*

(i) *the points* $K(u)$ *and* $K(v)$ *are distinct and the line that joins them is tangent to* $K(X)$ *at* $K(u)$;

(ii) $\operatorname{codim}_X \bigcap_{r\in\mathbf{Z}} \Theta_{2ru} > 2$.

Then (X, λ) *is isomorphic to the Jacobian of a smooth algebraic curve.*

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Note that condition (ii) in the theorem is equivalent to saying that there are no hypersurfaces in Θ invariant by translation by $2u$; it holds when u generates X.

Proof. As explained in Section 2, it is enough to prove that the scheme $V_{u,-u,v}$ has positive dimension at 0: we look for a sequence $\{D_n\}_{n>0}$ of constant vector fields on X with $D_1 \neq 0$ and relatively prime elements $\alpha(\varepsilon), \beta(\varepsilon)$ and $\gamma(\varepsilon)$ of $\mathbf{C}[[\varepsilon]]$ such that

$$\begin{aligned} P(z,\varepsilon) \;=\; & \alpha(\varepsilon)\theta_{u+D(\varepsilon)/2}\theta_{-u-D(\varepsilon)/2} + \beta(\varepsilon)\theta_{-u+D(\varepsilon)/2}\theta_{u-D(\varepsilon)/2} \\ & +\gamma(\varepsilon)\theta_{v+D(\varepsilon)/2}\theta_{-v-D(\varepsilon)/2} \end{aligned} \tag{3.2}$$

vanishes, with $D(\varepsilon) = \Sigma_{n>0} D_n \varepsilon^n$. This is nothing but equation (1.4) from [D]. It follows from *loc. cit.* that we may assume

$$\alpha(\varepsilon) = 1 + \sum_{n>0} \alpha_n \varepsilon^n, \qquad \beta(\varepsilon) = -1, \qquad \gamma(\varepsilon) = \varepsilon.$$

Write $P(z,\varepsilon) = \Sigma_{n\geqslant 0} P_n \varepsilon^n$, where P_n is a section of $\mathcal{O}_X(2\Theta)$ for each $n \geqslant 0$. One has $P_0 = 0$ and

$$P_1 = \alpha_1 \theta_u \theta_{-u} + \theta_u D_1 \theta_{-u} - \theta_{-u} D_1 \theta_u + \theta_v \theta_{-v}. \tag{3.3}$$

As explained in *loc.cit.*, hypothesis (i) in the theorem is equivalent to the vanishing of P_1 for a suitable D_1 such that $K_*(D_1)$ is tangent at $K(u)$ to the line that joins $K(u)$ and $K(v)$, and a suitable α_1. In general, note that P_n depends only on $\alpha_1, \ldots, \alpha_n$ and $D_1, \ldots, D_n$. Knowing that P_1 vanishes, we need to construct a sequence $\{D_n\}_{n>0}$ of constant vector fields on X and a sequence $\{\alpha_n\}_{n>1}$ of complex numbers such that P_n vanishes for all positive integers n.

It is convenient to set

$$R(z,\varepsilon) = P(z + \tfrac{1}{2}D(\varepsilon), \varepsilon) = \sum_{n>0} R_n(z)\varepsilon^n.$$

We begin by proving a few identities. Note that

$$R(\cdot,\varepsilon) = \alpha(\varepsilon)\theta_u\theta_{-u-D(\varepsilon)} - \theta_{-u}\theta_{u-D(\varepsilon)} + \varepsilon\theta_v\theta_{-v-D(\varepsilon)}. \tag{3.4}$$

Modulo θ_u, we get

$$R(\cdot,\varepsilon) \equiv -\theta_{-u}\theta_{u-D(\varepsilon)} + \varepsilon\theta_v\theta_{-v-D(\varepsilon)}, \tag{3.5}$$

$$R(\cdot,\varepsilon)_{2u} \equiv \alpha(\varepsilon)\theta_{3u}\theta_{u-D(\varepsilon)} + \varepsilon\theta_{2u+v}\theta_{2u-v-D(\varepsilon)}, \tag{3.6}$$

$$R(\cdot,\varepsilon)_{u-v} \equiv \alpha(\varepsilon)\theta_{2u-v}\theta_{-v-D(\varepsilon)} - \theta_{-v}\theta_{2u-v-D(\varepsilon)}. \tag{3.7}$$

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Since P_1 and its translate by $2u$ both vanish, formula (3.3) yields, modulo θ_u,

$$\theta_{-u}D_1\theta_u - \theta_v\theta_{-v} \equiv 0, \tag{3.8}$$

$$\theta_{3u}D_1\theta_u + \theta_{2u-v}\theta_{2u+v} \equiv 0. \tag{3.9}$$

The following result is the main technical step of the proof.

LEMMA 3.10. *Modulo θ_u, one has*

$$\alpha(\varepsilon)R(\cdot,\varepsilon)\theta_{3u}\theta_{-v} + R(\cdot,\varepsilon)_{2u}\theta_{-u}\theta_{-v} + \varepsilon R(\cdot,\varepsilon)_{u-v}\theta_{-u}\theta_{2u+v} \equiv 0.$$

Proof. By (3.5), (3.6) and (3.7), the left-hand side of the expression in the lemma is congruent modulo θ_u to

$$\begin{aligned}&-\alpha(\varepsilon)\theta_{-u}\theta_{u-D(\varepsilon)}\theta_{3u}\theta_{-v} + \varepsilon\alpha(\varepsilon)\theta_v\theta_{-v-D(\varepsilon)}\theta_{3u}\theta_{-v}\\ &\quad+\alpha(\varepsilon)\theta_{3u}\theta_{u-D(\varepsilon)}\theta_{-u}\theta_{-v} + \varepsilon\theta_{2u+v}\theta_{2u-v-D(\varepsilon)}\theta_{-u}\theta_{-v}\\ &\quad+\varepsilon\alpha(\varepsilon)\theta_{2u-v}\theta_{-v-D(\varepsilon)}\theta_{-u}\theta_{2u+v} - \varepsilon\theta_{-v}\theta_{2u-v-D(\varepsilon)}\theta_{-u}\theta_{2u+v}.\end{aligned}$$

All terms cancel out but the second and the fifth. Since (3.8) and (3.9) yield $\theta_v\theta_{3u}\theta_{-v} + \theta_{2u-v}\theta_{-u}\theta_{2u+v} \equiv 0$, the sum vanishes. □

We proceed by induction: let n be an integer $\geqslant 2$ and assume that $\alpha_1, \ldots, \alpha_{n-1}$ and $D_1, \ldots, D_{n-1}$ have been constructed so that $P_1 = \cdots = P_{n-1} = 0$. We want to find a complex number α_n and a tangent vector D_n such that P_n vanishes on X. By [D], Lemma 1.8, it is enough to show that the restriction of P_n to the scheme $\Theta_u \cap \Theta_{-u}$ (which depends only on $\alpha_1, \ldots, \alpha_{n-1}$ and $D_1, \ldots, D_{n-1}$) vanishes.

Our induction hypothesis can be rewritten as $R_1 = \cdots = R_{n-1} = 0$ and $P_n = R_n$. Therefore, we need to prove that R_n vanishes on the scheme $\Theta_u \cap \Theta_{-u}$. Since $\alpha(\varepsilon) \equiv 1$ modulo ε, the identity of the lemma taken modulo ε^{n+1} yields

$$\theta_{-v}[R_n\theta_{3u} + (R_n)_{2u}\theta_{-u}] \equiv 0,$$

modulo θ_u; since Θ_u is integral and $-v \neq u$, we get

$$R_n\theta_{3u} + (R_n)_{2u}\theta_{-u} \equiv 0. \tag{3.11}$$

LEMMA 3.12. *If F is a section of an ample line bundle $\mathcal{L}$ on X such that R_nF vanishes on the scheme $\Theta_u \cap \Theta_{-u}$, then, for any integer r, the section R_nF_{2ru} vanishes on the scheme $\Theta_u \cap \Theta_{-u}$.*

Proof. Recall that $P_n = R_n$ is a section of $\mathcal{O}_X(2\Theta)$, so that R_nF is a section of $\mathcal{L}(2\Theta) \otimes \mathcal{I}_{\Theta_u\cap\Theta_{-u}}$. The Koszul complex yields an exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(\Theta_u) \oplus \mathcal{L}(\Theta_{-u}) \to \mathcal{L}(2\Theta) \otimes \mathcal{I}_{\Theta_u\cap\Theta_{-u}} \to 0.$$

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Because $\mathcal{L}$ is ample, one has $H^1(X, \mathcal{L}) = 0$, and there exist sections B and C such that $R_n F = B\theta_u + C\theta_{-u}$. It follows that $(R_n)_{2u} F_{2u} \equiv B_{2u}\theta_{3u} (\text{mod } \theta_u)$. Multiplying (3.11) by F_{2u}, we get

$$R_n F_{2u}\theta_{3u} + B_{2u}\, \theta_{3u}\theta_{-u} \equiv 0 \quad (\text{mod } \theta_u).$$

Since $2u \neq 0$ and Θ_u is integral, θ_{3u} is not a zero divisor modulo θ_u and we get $R_n F_{2u} \equiv 0\ (\text{mod}(\theta_u, \theta_{-u}))$. By a similar reasoning, $R_n F_{-2u} \equiv 0\ (\text{mod}(\theta_u, \theta_{-u}))$. □

(3.13) The lemma implies that $R_n\theta_{u+2ru}$ vanishes on $\Theta_u \cap \Theta_{-u}$ for all r. Because of hypothesis (ii), it follows that R_n vanishes on each primary component of codimension 2 of $\Theta_u \cap \Theta_{-u}$; since this scheme has no embedded components, R_n vanishes on it. This concludes the proof of the theorem. □

It follows from (3.5) that $R_n\theta_{-v}$, hence also its image $R_n\theta_v$ by the involution $x \mapsto -x$, vanish on the scheme $\Theta_u \cap \Theta_{-u}$. Hypothesis (ii) in Theorem 3.1 can therefore be relaxed to

$$\operatorname{codim}_X \bigcap_{r\in\mathbf{Z}} (\Theta_u \cap \Theta_v \cap \Theta_{-v})_{2ru} > 2.$$

4. The case of a non-degenerate trisecant

In this section, we prove, under an extra hypothesis, that the existence of a non-degenerate trisecant line implies the existence of a degenerate trisecant of the type studied in Section 3.

THEOREM 4.1. *Let (X, λ) be an indecomposable principally polarized abelian variety, let Θ be a symmetric representative of the polarization and let $K\colon X \to |2\Theta|^*$ be the Kummer morphism. Assume that there exist points a, b and c of X such that*

(i) *the points $K(a), K(b)$ and $K(c)$ are distinct and collinear;*

(ii) $\operatorname{codim}_X \bigcap_{\substack{p,q,r\in\mathbf{Z}\\ p+q+r=0}} \Theta_{pa+qb+rc} > 2.$

Then (X, λ) is isomorphic to the Jacobian of a smooth algebraic curve.

Note that condition (ii) in the theorem is equivalent to saying that there are no hypersurfaces in Θ invariant by translation by $a - b$ and $b - c$; it holds when $a - b$ and $b - c$ together generate X.

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Proof. Instead of proving that $V_{a,b,c}$ has positive dimension at 0, we will proceed as follows. As explained in Section 2, condition (i) translates into the existence of nonzero complex numbers α, β and γ such that

$$\alpha\theta_a\theta_{-a} + \beta\theta_b\theta_{-b} + \gamma\theta_c\theta_{-c} = 0. \tag{4.2}$$

For any x in X, we will write P^x for $\theta_{a+b+c}\theta_{-x}$. Our first aim is to show that P^c vanishes on the scheme $\Theta_a \cap \Theta_b$.

LEMMA 4.3. *One has*

$$P^c_{a-b}\theta_b + P^c\theta_{2a-b} \equiv 0 \pmod{\theta_a}.$$

Proof. Equation (4.2) and its translates by $(a+c)$ and $(a-b)$ yield, modulo θ_a,

$$\beta\theta_b\theta_{-b} + \gamma\theta_c\theta_{-c} \equiv 0,$$

$$\alpha\theta_{2a+c}\theta_c + \beta\theta_{a+b+c}\theta_{a-b+c} \equiv 0,$$

$$\alpha\theta_{2a-b}\theta_{-b} + \gamma\theta_{a-b+c}\theta_{a-b-c} \equiv 0.$$

It follows that, still modulo θ_a

$$\begin{aligned}&\alpha\beta\theta_{-b}\left(P^c_{a-b}\theta_b + P^c\theta_{2a-b}\right)\\ &\quad\equiv \theta_{2a+c}\theta_{a-b-c}\left(-\alpha\gamma\theta_c\theta_{-c}\right) + \theta_{a+b+c}\theta_{-c}\left(-\beta\gamma\theta_{a-b+c}\theta_{a-b-c}\right)\\ &\quad\equiv -\gamma\theta_{a-b-c}\theta_{-c}\left(\alpha\theta_{2a+c}\theta_c + \beta\theta_{a+b+c}\theta_{a-b+c}\right) \equiv\ 0.\end{aligned}$$

Since Θ is integral and $-b \neq a$, the section θ_{-b} is not a zero divisor modulo θ_a and the lemma follows. □

LEMMA 4.4. *If F is a section of an ample line bundle on X such that P^cF vanishes on the scheme $\Theta_a \cap \Theta_b$, then, for any integer s, the section $P^cF_{s(a-b)}$ also vanishes on the scheme $\Theta_a \cap \Theta_b$.*

Proof. Since a and b play the same role, it is enough to prove that P^cF_{a-b} vanishes on $\Theta_a \cap \Theta_b$. As in the proof of Lemma 3.12, there exist sections B and C such that $P^cF = B\theta_a + C\theta_b$. Then $P^c_{a-b}F_{a-b} \equiv B_{a-b}\theta_{2a-b} \pmod{\theta_a}$. Using Lemma 4.3, we get

$$P^cF_{a-b}\theta_{2a-b} + B_{a-b}\theta_{2a-b}\,\theta_b \equiv 0 \pmod{\theta_a}.$$

Since $2a-b \neq a$ and Θ_a is irreducible, we can divide out by θ_{2a-b}, and the lemma is proved. □

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LEMMA 4.5. *If F is a section of an ample line bundle on X, then P^cF vanishes on the scheme $\Theta_a \cap \Theta_b$ if and only if P^bF vanishes on the scheme $\Theta_a \cap \Theta_c$.*

Proof. As in the proof of Lemma 3.12, write $\theta_{a+b+c}\theta_{-c}F \equiv B\theta_b \pmod{\theta_a}$. We get

$$\begin{aligned}\gamma B\theta_b\theta_c &\equiv \theta_{a+b+c}\gamma\theta_c\theta_{-c}F \\ &\equiv -\theta_{a+b+c}\beta\theta_b\theta_{-b}F = -\beta P^bF\theta_b \pmod{\theta_a},\end{aligned}$$

where we used (4.2). Since Θ_a is irreducible and $a \neq b$, the lemma is proved. □

We combine the last two lemmas to get, for all integers r and s,

$$\begin{aligned} & P^c\, F \equiv 0 \pmod{(\theta_a,\theta_b)} \\ \Longrightarrow\quad & P^b\, F \equiv 0 \pmod{(\theta_a,\theta_c)} \\ \Longrightarrow\quad & P^b\, F_{r(a-c)} \equiv 0 \pmod{(\theta_a,\theta_c)} \\ \Longrightarrow\quad & P^c\, F_{r(a-c)} \equiv 0 \pmod{(\theta_a,\theta_b)} \\ \Longrightarrow\quad & P^c\, F_{r(a-c)+s(a-b)} \equiv 0 \pmod{(\theta_a,\theta_b)}. \end{aligned}$$

It follows in particular that $P^c\theta_{a+r(a-c)+s(a-b)}$ vanishes on $\Theta_a \cap \Theta_b$ for all integers r and s. As in (3.13), hypothesis (ii) in the theorem shows that

$$P^c \text{ vanishes on the scheme } \Theta_a \cap \Theta_b \tag{4.6}$$

(hence also P^a on $\Theta_b \cap \Theta_c$, and P^b on $\Theta_c \cap \Theta_a$).

Let u be any point of X such that $2u = a - b$, and set $v = u - a - c$. Translating (4.6) by $(-u - b)$, we get that $\theta_v\theta_{-v}$ vanishes on $\Theta_u \cap \Theta_{-u}$. As explained in [D], this is equivalent to the existence of a complex number α_1 and a tangent vector D_1 to X such that

$$\alpha_1\theta_u\theta_{-u} + \theta_u D_1\theta_{-u} - \theta_{-u}D_1\theta_u + \theta_v\theta_{-v} = 0. \tag{4.7}$$

In other words, the line that joins $K(u)$ and $K(v)$ is tangent to $K(X)$ at $K(u)$. Note that we cannot apply Theorem 3.1 directly, since hypothesis (ii) may not be satisfied. However, we will still follow the same method, i.e. we will show that the scheme $V_{a,b,-c}$ (which is a translate of $V_{u,-u,v}$) has positive dimension at $(-a-b)$, but we will need to prove at the same time that $V_{a,-b,c}$ has positive dimension at the point $(-a-c)$.

Let n be an integer $\geqslant 1$. As in Section 3, the scheme $V_{a,b,-c}$ contains a scheme isomorphic to $\mathbf{C}[\varepsilon]/\varepsilon^{n+1}$ and concentrated at $(-a-b)$ if and only if one can find complex numbers $\alpha_1, \ldots, \alpha_n$ and tangent vectors $D_1, \ldots, D_n$ such that $R_1, \ldots, R_n$, defined in Section 3, vanish (α_1 and D_1 are the same as in (4.7),

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and R_1 is the left-hand side of that equation). Similarly, the scheme $V_{a,-b,c}$ contains a scheme isomorphic to $\mathbf{C}[\varepsilon]/\varepsilon^{n+1}$ and concentrated at $(-a-c)$ if and only if there exist complex numbers $\alpha'_1, \ldots, \alpha'_n$ and tangent vectors $D'_1, \ldots, D'_n$ such that $R'_1, \ldots, R'_n$ vanish.

We proceed as in Section 3: let n be an integer $\geqslant 2$ and assume that $\alpha_1, \ldots, \alpha_{n-1}$, $\alpha'_1, \ldots, \alpha'_{n-1}$ and $D_1, \ldots, D_{n-1}, D'_1, \ldots, D'_{n-1}$ have been constructed so that $R_1, \ldots, R_{n-1}$, $R'_1, \ldots, R'_{n-1}$ vanish on X. As in the proof of theorem 3.1, it is enough to show that the restriction of R_n to the scheme $\Theta_a \cap \Theta_b$ (which depends only on $\alpha_1, \ldots, \alpha_{n-1}$ and $D_1, \ldots, D_{n-1}$), and the restriction of R'_n to $\Theta_a \cap \Theta_c$ (which depends only on $\alpha'_1, \ldots, \alpha'_{n-1}$ and $D'_1, \ldots, D'_{n-1}$) both vanish.

LEMMA 4.8. *One has*

$$(-\gamma)^n R_n \theta_c - \beta^n R'_n \theta_b \equiv -\theta_b \theta_c \big((-\gamma)^n D_n - \beta^n D'_n\big)\theta_a \pmod{\theta_a}.$$

Proof. Formula (3.4) translates into

$$R(\cdot, \varepsilon) = \alpha(\varepsilon)\theta_a \theta_{b-D(\varepsilon)} - \theta_b \theta_{a-D(\varepsilon)} + \varepsilon\theta_{-c}\theta_{a+b+c-D(\varepsilon)}, \tag{4.9}$$

$$R'(\cdot, \varepsilon) = \alpha'(\varepsilon)\theta_a \theta_{c-D'(\varepsilon)} - \theta_c \theta_{a-D'(\varepsilon)} + \varepsilon\theta_{-b}\theta_{a+b+c-D'(\varepsilon)}. \tag{4.10}$$

For $0 < s \leqslant n$, let $\mathcal{P}(s)$ be the property "$\beta^t D'_t = (-\gamma)^t D_t$ whenever $0 < t < s$," or equivalently $D'(\beta\varepsilon) \equiv D(-\gamma\varepsilon) \pmod{\varepsilon^s}$. Assume $\mathcal{P}(s)$ holds; using (4.9), (4.10) and (4.2), we get

$$\begin{aligned} &R(\cdot, -\gamma\varepsilon)\theta_c - R'(\cdot, \beta\varepsilon)\theta_b \\ &\quad \equiv -\theta_b\theta_c\big((-\gamma)^s D_s - \beta^s D'_s\big)\theta_a \pmod{(\theta_a, \varepsilon^{s+1})}. \end{aligned} \tag{4.11}$$

Assume $s < n$; then R and R' vanish modulo ε^{s+1}, we get $(-\gamma)^s D_s - \beta^s D'_s = 0$ and $\mathcal{P}(s+1)$ holds. Since $\mathcal{P}(1)$ is empty, this proves that $\mathcal{P}(n)$ holds, hence also formula (4.11) for $s = n$. □

LEMMA 4.12. *Let F be a section of an ample line bundle on X. Then $R_n F$ vanishes on the scheme $\Theta_a \cap \Theta_b$ if and only if $R'_n F$ vanishes on the scheme $\Theta_a \cap \Theta_c$.*

Proof. As in the proof of Lemma 3.12, we can write $R_n F \equiv B\theta_b \pmod{\theta_a}$. Multiplying the congruence of Lemma 4.8 by F, we get

$$\begin{aligned} &(-\gamma)^n B\theta_b\theta_c - \beta^n R'_n F \theta_b \\ &\quad \equiv -\theta_b\theta_c F\big((-\gamma)^n D_n - \beta^n D'_n\big)\theta_a \pmod{\theta_a}. \end{aligned}$$

Since Θ_a is irreducible and $a \neq b$, one can divide out by θ_b. This proves the Lemma. □

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By Lemma 3.12, $R'_n\theta_{a+r(a-c)}$ vanishes on $\Theta_a \cap \Theta_c$ for all integers r; by Lemma 4.12, this implies that $R_n\theta_{a+r(a-c)}$ vanishes on $\Theta_a \cap \Theta_b$ for all r, and by Lemma 3.12 again, so does $R_n\theta_{a+r(a-c)+s(a-b)}$ for all r and s. Hypothesis (ii) in the theorem implies, as in (3.13), that R_n vanishes on $\Theta_a \cap \Theta_b$, which concludes the proof of the theorem. □

5. Complements

In this short Section, we will indicate how to combine the techniques used here with those of [D] to get better results in the degenerate case when the theta divisor is not too singular. We will use the following lemma, inspired by Proposition 2.6 in [D].

LEMMA 5.1. *Let (X, λ) be a principally polarized abelian variety and let Θ be a representative of the polarization. Let x be a non-torsion element of X and assume that Z is a component of $\Theta \cap \Theta_x$ such that Z_{red} is contained in $\bigcap_{r\in\mathbf{Z}} \Theta_{rx}$. Assume that $\mathrm{codim}_X(Z \cap \mathrm{Sing}\,\Theta) > 3$. Then Z is reduced.*

Proof. Since $Z_{\mathrm{red}} + rx$ is contained in $\Theta \cap \Theta_x$ for all integers r, so is $Z_{\mathrm{red}} + A$, where A is the neutral component of the closed subgroup generated by x. It follows that $Z_{\mathrm{red}} + A = Z_{\mathrm{red}}$, hence Z_{red} contains a translate A' of A that satisfies $\mathrm{codim}_{A'}(A' \cap \mathrm{Sing}\,\Theta) \geqslant 2$. If Z is not reduced, it is contained in the singular locus of $\Theta \cap \Theta_x$, hence so is A'. By the Jacobian criterion, the $D\theta_x/D\theta$, for $D \in T_0A$, define a section of $\mathcal{O}_{A'}(\Theta_x - \Theta)$ which is regular outside of the closed subset $A' \cap \mathrm{Sing}\,\Theta$. Since this subset has codimension $\geqslant 2$ in A', the line bundle $\mathcal{O}_{A'}(\Theta_x - \Theta)$ is trivial. This means that x is in the kernel of the restriction homomorphism $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(A')$, hence so is A. The composed homomorphism $A \to \mathrm{Pic}^0(A)$ is therefore zero. Since it is the morphism associated with the restriction of the polarization λ to A, this implies $A = 0$, which contradicts the fact that x is not torsion. Hence Z is reduced. □

In the case of a degenerate trisecant, this lemma allows us to prove the following improvement on Theorem 2.2 of [D].

THEOREM 5.2. *Let (X, λ) be a complex indecomposable principally polarized abelian variety, let Θ be a symmetric representative of the polarization and let $K\colon X \to |2\Theta|^*$ be the Kummer morphism. Assume that there exist points u and v of X such that*

(i) *the points $K(u)$ and $K(v)$ are distinct and the line that joins them is tangent to $K(X)$ at $K(u)$;*
(ii) *the point u is not torsion;*
(iii) $\mathrm{codim}_X\left(\mathrm{Sing}\,\Theta \cap \bigcap_{r\in\mathbf{Z}} \Theta_{2ru}\right) > 3$.

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Then (X, λ) is isomorphic to the Jacobian of a smooth non-hyperelliptic algebraic curve.

Note that by [BD], condition (i) implies $\operatorname{codim}_X \operatorname{Sing}\Theta \leqslant 4$. On the other hand, the indecomposability of (X, λ) implies $\operatorname{codim}_X \operatorname{Sing}\Theta \geqslant 3$ ([EL]): if hypothesis (iii) fails, there is a component of $\operatorname{Sing}\Theta$ of codimension 3 in X, invariant by the abelian subvariety generated by u.

Proof. We keep the notation of the proof of Theorem 3.1. The point is to show that R_n vanishes on the scheme $\Theta_u \cap \Theta_{-u}$. Let Z be a primary component of $\Theta_u \cap \Theta_{-u}$; it has codimension 2. If Z_{red} is *not* contained in $\bigcap_{r\in\mathbf{Z}} \Theta_{u+2ru}$, Lemma 3.12 implies that R_n vanishes on Z. Otherwise, Lemma 5.1 implies that Z is reduced. On page 9 of [D], it is proved that R_n^2 vanishes on $\Theta_u \cap \Theta_{-u}$. Since Z is reduced, it follows that R_n vanishes on Z. Hence R_n vanishes on all primary components of $\Theta_u \cap \Theta_{-u}$, which proves the theorem. □

This approach does not seem to work in the non-degenerate case.

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