J. Austral. Math. Soc. (Series A) **58** (1995), 219–221

ON ELEMENTARY AMENABLE GROUPS OF FINITE HIRSCH NUMBER

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(Received 15 June 1992; revised 15 August 1992)

Communicated by H. Lausch

Abstract

We give an alternative short proof of a recent theorem of J.A. Hillman and P.A. Linnell that an elementary amenable group with finite Hirsch number has, modulo its locally finite radical, a soluble normal subgroup with index and derived length bounded only in terms of the Hirsch number of the group.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20F19.

For any group G, let $\tau(G)$ denote its unique maximal locally finite normal subgroup. In [2] Hillman and Linnell prove that there are integer-valued functions $d(h)$ and $M(h)$ of h only such that for any elementary amenable group G with finite Hirsch number $h(G) = h$ there is a soluble normal subgroup of $G/\tau(G)$ with derived length at most $d(h)$ and index at most $M(h)$, the case where $h(G) \leq 3$ being dealt with in [1]. Here we offer an alternative short proof of this theorem, a proof that also makes explicit a little more of the structure of these groups. The phrase 'bounded by an integer-valued function of n only' we shorten to 'n-bounded'.

(a) *Let G be a group with $\tau(G) = \langle 1 \rangle$ and with a torsion-free abelian normal subgroup A of finite rank r such that G/A is locally finite. Then G has a torsion-free abelian normal subgroup with rank r and r-bounded index.*

Set $C = C_G(A)$. Then C is centre by locally-finite, so C' is locally finite (Schur) and $C' = \langle 1 \rangle$. Then C is torsion-free abelian, necessarily of rank r. Also G/C is isomorphic to a locally finite subgroup of $GL(r, \mathbb{Q})$ and therefore has r-bounded order ([6, 9.33 ii & iii]).

https://doi.org/10.1017/S1446788700038258 Published online by Cambridge University Press

(b) *Let T be a normal subgroup of finite order m of a group G such that G/T is torsion-free abelian of rank $r < \infty$. Then G has a torsion-free abelian normal subgroup of finite rank r and index dividing $m^{2r+1}.m!$.*

Set $C = C_G(T)$. Then $(G : C)$ divides $m!$. Also C is nilpotent of class at most 2 with $|C'|$ dividing m. Hence $A = C^m$ is abelian, so A^m is torsion-free abelian, necessarily of rank r. Also $(C : A^m)$ divides m^{2r+1}.

Let $\mathfrak{X}_h$ denote the class of all groups G with a series of finite length whose factors are locally finite or torsion-free abelian, such that the sum $h(G)$ of the ranks of the torsion-free abelian factors is at most h. Here $h(G)$ is an invariant of G. By (a minor extension of) a theorem of Mal´cev ([4, Theorem 3]) the factor $G/\tau(G)$ of such a group G has a poly torsion-free abelian, characteristic subgroup of finite index. Stronger still is the following.

(c) *There is an integer-valued function $i(h)$ of h only such that a group G in $\mathfrak{X}_h$ has a characteristic subgroup S with $(G : S) \leq i(h)$ and $S/\tau(S)$ poly torsion-free abelian of derived length at most h. The maximal soluble normal subgroup of $G/\tau(G)$ has derived length at most $h + i(h)$ and index at most $i(h)$.*

We induct on h. Clearly we may assume that $\langle 1 \rangle = \tau(G) < G$. Then G has normal subgroups $G_1 \leq H_1$ with G/H_1 locally finite and H_1/G_1 torsion-free abelian of rank $r \geq 1$. By (a) we may assume that $(G : H_1)$ is r-bounded. Apply induction to $G_1 \in \mathfrak{X}_{h-r}$. There exists a characteristic subgroup S_1 of G_1 with S_1 poly torsion-free abelian of $(h - r)$-bounded index in G_1. Certainly S_1 is normal in G, so apply (b) to H_1/S_1. We obtain T_1 normal in H_1 with T_1 poly torsion-free abelian and $(H_1 : T_1)$ h-bounded. Set $S = G^{(G:T_1)}$. Then S is a characteristic poly torsion-free abelian subgroup of G and $(G : S) \leq (G : T_1)^{h+1}$ is h-bounded. Trivially S has derived length at most $h(S) = h(G) \leq h$.

For any group G define the subgroups $G^{[i]}$ of G for integers $i \geq 0$ by $G^{[0]} = G$, $G^{[1]}/G' = \tau(G/G')$ and inductively $G^{[i+1]} = (G^{[i]})^{[1]}$. Then $S^{[h]}$ in (c) is locally finite. If H is a subgroup of G then $H' \leq G'$, so $H^{[1]} \leq G^{[1]}$ and a simple induction yields that $H^{[i]} \leq G^{[i]}$ for each i. Also if $g \in G$ then $g \in G^{[1]}$ if and only if there exist elements x_i and y_i of G and a positive integer n such that $g^n = \prod [x_i, y_i]$. Thus $G^{[i]} = \bigcup_X X^{[i]}$, where X ranges over the finitely generated subgroups of G.

(d) *$\mathfrak{X}_h$ is locally closed.*

Let $G \in L\mathfrak{X}_h$. Then *(c)* and the usual inverse limit argument (cf. [3, Section 1.K]) shows that G has a normal subgroup S of finite index at most $i(h)$ such that $S^{[h]} = \bigcup_{\text{f.g.} X \leq S} X^{[h]}$ is locally finite. If $h(G) > h$ then $h(X) > h$ for some finitely generated subgroup X of G. Consequently $h(G) \leq h$ and $G \in \mathfrak{X}_h$.

(e) $\mathfrak{X}_h \mathfrak{X}_k \leq \mathfrak{X}_{h+k}$.

This is immediate from the definition of $\mathfrak{X}_h$. Clearly every group in $\mathfrak{X}_h$ is elementary amenable with Hirsch number (that is, Hirsch length in the sense of [2]) at most h and so $\bigcup_{h \geq 0} \mathfrak{X}_h$ lies in the class of elementary amenable groups of finite Hirsch number.

The converse follows easily from a trivial induction, using (d) and (e). Thus:-

(f) $\bigcup_{h \geq 0} \mathfrak{X}_h$ *is the class of elementary amenable groups of finite Hirsch number and (c) gives the theorem of* [2].

Let G be an elementary amenable group with finite Hirsch number. Then in the notation of [5] the group $G/\tau(G)$ is a finite extension of a torsion-free $\mathfrak{S}_1$-group and from this much follows. For example (c) can be strengthened as below (cf. [5, 10.33, p.169]).

(g) *There is an integer-valued function $j(h)$ of h only such that a group G in $\mathfrak{X}_h$ has characteristic subgroups $\tau(G) \leq N \leq M$ with $N/\tau(G)$ torsion-free nilpotent, M/N free abelian of finite rank and $(G : M)$ at most $j(h)$.*

For example, with S as in (c), set $M = \tau(G) \cdot \bigcap_i C_S(S^{[i]}/S^{[i+1]})$ and use Mal´cev's Theorem ([6, 3.6]).

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https://doi.org/10.1017/S1446788700038258 Published online by Cambridge University Press