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RESEARCH ARTICLE

# Minimal subdynamics and minimal flows without characteristic measures

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**Abstract**
Given a countable group $G$ and a $G$-flow $X$, a probability measure $\mu$ on $X$ is called characteristic if it is $\mathrm{Aut}(X, G)$-invariant. Frisch and Tamuz asked about the existence of a minimal $G$-flow, for any group $G$, which does not admit a characteristic measure. We construct for every countable group $G$ such a minimal flow. Along the way, we are motivated to consider a family of questions we refer to as minimal subdynamics: Given a countable group $G$ and a collection of infinite subgroups $\{\Delta_i : i \in I\}$, when is there a faithful $G$-flow for which every $\Delta_i$ acts minimally?

## Contents



Given a countable group $G$ and a faithful $G$-flow $X$, we write $\mathrm{Aut}(X, G)$ for the group of homeomorphisms of $X$ which commute with the $G$-action. When $G$ is abelian, $\mathrm{Aut}(X, G)$ contains a natural copy of $G$ resulting from the $G$-action, but in general this need not be the case. Much is unknown about how the properties of $X$ restrict the complexity of $\mathrm{Aut}(X, G)$; for instance, Cyr and Kra [1] conjecture that when $G = \mathbb{Z}$ and $X \subseteq 2^{\mathbb{Z}}$ is a minimal, 0-entropy subshift, then $\mathrm{Aut}(X, \mathbb{Z})$ must be amenable. In fact, no counterexample is known even when restricting to any two of the three properties 'minimal', '0-entropy' or 'subshift'. In an effort to shed light on this question, Frisch and Tamuz [3] define a probability measure $\mu$ on $X$ to be *characteristic* if it is $\mathrm{Aut}(X, G)$-invariant. They show that 0-entropy subshifts always admit characteristic measures. More recently, Cyr and Kra [2] provide several examples of flows which admit characteristic measures for nontrivial reasons, even in cases where $\mathrm{Aut}(X, G)$ is nonamenable. Frisch and Tamuz asked (Question 1.5, [3]) whether there exists, for any countable group $G$, some minimal $G$-flow without a characteristic measure. We give a strong affirmative answer.

**Theorem 0.1.** *For any countably infinite group $G$, there is a free minimal $G$-flow $X$ so that $X$ does not admit a characteristic measure. More precisely, there is a free $(G \times F_2)$-flow $X$ which is minimal as a $G$-flow and with no $F_2$-invariant measure.*

We remark that the $X$ we construct will not in general be a subshift.

Over the course of proving Theorem 0.1, there are two main difficulties to overcome. The first difficulty is a collection of dynamical problems we refer to as *minimal subdynamics*. The general template of these questions is as follows.

**Question 0.2.** Given a countably infinite group $\Gamma$ and a collection $\{\Delta_i : i \in I\}$ of infinite subgroups of $\Gamma$, when is there a faithful (or essentially free, or free) minimal $\Gamma$-flow for which the action of each $\Delta_i$ is also minimal? Is there a natural space of actions in which such flows are generic?

In [8], the author showed that this was possible in the case $\Gamma = G \times H$ and $\Delta = G$ for any countably infinite groups $G$ and $H$. We manage to strengthen this result considerably.

**Theorem 0.3.** *For any countably infinite group $\Gamma$ and any collection $\{\Delta_n : n \in \mathbb{N}\}$ of infinite normal subgroups of $\Gamma$, there is a free $\Gamma$-flow which is minimal as a $\Delta_n$-flow for every $n \in \mathbb{N}$.*

In fact, what we show when proving Theorem 0.3 is considerably stronger. Recall that given a countably infinite group $\Gamma$, a subshift $X \subseteq 2^\Gamma$ is *strongly irreducible* if there is some finite symmetric $D \subseteq \Gamma$ so that whenever $S_0, S_1 \subseteq \Gamma$ satisfy $DS_0 \cap S_1 = \emptyset$ (i.e., $S_0$ and $S_1$ are *D-apart*), then for any $x_0, x_1 \in X$, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. Write $\mathcal{S}$ for the set of strongly irreducible subshifts, and write $\overline{\mathcal{S}}$ for its Vietoris closure. Frisch, Tamuz and Vahidi-Ferdowsi [5] show that in $\overline{\mathcal{S}}$, the minimal subshifts form a dense $G_\delta$ subset. In our proof of Theorem 0.3, we show that the shifts in $\overline{\mathcal{S}}$ which are $\Delta_n$-minimal for each $n \in \mathbb{N}$ also form a dense $G_\delta$ subset.

This brings us to the second main difficulty in the proof of Theorem 0.1. Using this stronger form of Theorem 0.3, one could easily prove Theorem 0.1 by finding a strongly irreducible $F_2$-subshift which does not admit an invariant measure. This would imply the existence of a strongly irreducible $(G \times F_2)$-subshift without an $F_2$-invariant measure. As not admitting an $F_2$-invariant measure is a Vietoris-open condition, the genericity of $G$-minimal subshifts would then be enough to obtain the desired result. Unfortunately, whether such a strongly irreducible subshift can exist (for any nonamenable group) is an open question. To overcome this, we introduce a flexible weakening of the notion of a strongly irreducible shift.

The paper is organized as follows. Section 1 is a very brief background section on subsets of groups, subshifts and strong irreducibility. Section 2 introduces the notion of a UFO, a useful combinatorial gadget for constructing shifts where subgroups act minimally; Theorem 0.3 answers Question 3.6 from [8]. Section 3 introduces the notion of $\mathcal{B}$-irreducibility for any group $H$, where $\mathcal{B} \subseteq \mathcal{P}_f(H)$ is a right-invariant collection of finite subsets of $H$. When $H = F_2$, we will be interested in the case when $\mathcal{B}$ is the collection of finite subsets of $F_2$ which are connected in the standard left Cayley graph. Section 4 gives the proof of Theorem 0.1.

## 1. Background

Let $\Gamma$ be a countably infinite group. Given $U, S \subseteq \Gamma$ with $U$ finite, then we call $S$ a (one-sided) *$U$-spaced* set if for every $g \neq h \in S$ we have $h \notin Ug$, and we call $S$ a *$U$-syndetic* set if $US = \Gamma$. A *maximal $U$-spaced set* is simply a $U$-spaced set which is maximal under inclusion. We remark that if $S$ is a maximal $U$-spaced set, then $S$ is $(U \cup U^{-1})$-syndetic. We say that sets $S, T \subseteq \Gamma$ are (one-sided) *$U$-apart* if $US \cap T = \emptyset$ and $S \cap UT = \emptyset$. Notice that much of this discussion simplifies when $U$ is symmetric, so we will often assume this. Also, notice that the properties of being $U$-spaced, maximal $U$-spaced, $U$-syndetic and $U$-apart are all right invariant.

If $A$ is a finite set or *alphabet*, then $\Gamma$ acts on $A^\Gamma$ by *right shift*, where given $x \in A^\Gamma$ and $g, h \in \Gamma$, we have $(g{\cdot}x)(h) = x(hg)$. A *subshift* of $A^\Gamma$ is a nonempty, closed, $\Gamma$-invariant subset. Let $\mathrm{Sub}(A^\Gamma)$ denote the space of subshifts of $A^\Gamma$ endowed with the Vietoris topology. This topology can be described as follows. Given $X \subseteq A^\Gamma$ and a finite $U \subseteq \Gamma$, the set of *$U$-patterns* of $X$ is the set $P_U(X) = \{x|_U : x \in X\} \subseteq A^U$. Then the typical basic open neighborhood of $X \in \mathrm{Sub}(A^\Gamma)$ is the set $N_U(X) := \{Y \in \mathrm{Sub}(A^\Gamma) : P_U(Y) = P_U(X)\}$, where $U$ ranges over finite subsets of $\Gamma$.

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A subshift $X \subseteq A^\Gamma$ is *U-irreducible* if for any $x_0, x_1 \in X$ and any $S_0, S_1 \subseteq \Gamma$ which are $U$-apart, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. If $X$ is $U$-irreducible and $V \supseteq U$ is finite, then $X$ is also $V$-irreducible. We call $X$ *strongly irreducible* if there is some finite $U \subseteq \Gamma$ with $X$ $U$-irreducible. By enlarging $U$ if needed, we can always assume $U$ is symmetric. Let $\mathcal{S}(A^\Gamma) \subseteq \mathrm{Sub}(A^\Gamma)$ denote the set of strongly irreducible subshifts of $A^\Gamma$, and let $\overline{\mathcal{S}}(A^\Gamma)$ denote the closure of this set in the Vietoris topology.

More generally, if $2^\mathbb{N}$ denotes Cantor space, then $\Gamma$ acts on $(2^\mathbb{N})^\Gamma$ by right shift exactly as above. If $k < \omega$, we let $\pi_k\colon 2^\mathbb{N} \to 2^k$ denote the restriction to the first $k$ entries. This induces a factor map $\tilde{\pi}_k\colon (2^\mathbb{N})^\Gamma \to (2^k)^\Gamma$ given by $\tilde{\pi}_k(x)(g) = \pi_k(x(g))$; we also obtain a map $\overline{\pi}_k\colon \mathrm{Sub}((2^\mathbb{N})^\Gamma) \to \mathrm{Sub}((2^k)^\Gamma)$ (where $2^k$ is viewed as a finite alphabet) given by $\overline{\pi}_k(X) = \tilde{\pi}_k[X]$. The Vietoris topology on $\mathrm{Sub}((2^\mathbb{N})^\Gamma)$ is the coarsest topology making every such $\overline{\pi}_k$ continuous. We call a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ *strongly irreducible* if for every $k < \omega$, the subshift $\overline{\pi}_k(X) \subseteq (2^k)^\Gamma$ is strongly irreducible in the ordinary sense. We let $\mathcal{S}((2^\mathbb{N})^\Gamma) \subseteq \mathrm{Sub}((2^\mathbb{N})^\Gamma)$ denote the set of strongly irreducible subflows of $(2^\mathbb{N})^\Gamma$, and we let $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ denote its Vietoris closure.

The idea of considering the closure of the strongly irreducible shifts has it roots in [4]. This is made more explicit in [5], where it is shown that in $\overline{\mathcal{S}}(A^\Gamma)$, the minimal subflows form a dense $G_\delta$ subset. More or less the same argument shows that the same holds in $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ (see [6]). Recall that a $\Gamma$-flow $X$ is *free* if for every $g \in \Gamma \setminus \{1_\Gamma\}$ and every $x \in X$, we have $gx \neq x$. The main reason for considering a Cantor space alphabet is the following result, which need not be true for finite alphabets.

**Proposition 1.1.** *In $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$, the free flows form a dense $G_\delta$ subset.*

*Proof.* Fixing $g \in \Gamma$, the set $\{X \in \mathrm{Sub}((2^\mathbb{N})^\Gamma) : \forall x \in X\, (gx \neq x)\}$ is open; indeed, if $X_n \to X$ is a convergent sequence in $\mathrm{Sub}((2^\mathbb{N})^\Gamma)$ and $x_n \in X_n$ is a point fixed by $g$, then passing to a subsequence, we may suppose $x_n \to x \in X$, and we have $gx = x$. Intersecting over all $g \in \Gamma \setminus \{1_\Gamma\}$, we see that freeness is a $G_\delta$ condition.

Thus, it remains to show that freeness is dense in $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$. To that end, we fix $g \in \Gamma \setminus \{1_\Gamma\}$ and show that the set of shifts in $\mathcal{S}((2^\mathbb{N})^\Gamma)$ where $g$ acts freely is dense. Fix $X \in \mathcal{S}((2^\mathbb{N})^\Gamma)$, $k < \omega$ and a finite $U \subseteq \Gamma$; so a typical open set in $\mathcal{S}((2^\mathbb{N})^\Gamma)$ has the form $\{X' \in \mathcal{S}((2^\mathbb{N})^\Gamma) : P_U(\overline{\pi}_k(X')) = P_U(\overline{\pi}_k(X))\}$. We want to produce $Y \in \mathrm{Sub}((2^\mathbb{N})^\Gamma)$ which is strongly irreducible, $g$-free and with $P_U(\overline{\pi}_k(Y)) = P_U(\overline{\pi}_k(X))$. In fact, we will produce such a $Y$ with $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$.

Let $D \subseteq \Gamma$ be a finite symmetric set containing $g$ and $1_\Gamma$. Setting $m = |D|$, consider the subshift $\mathrm{Color}(D, m) \subseteq m^\Gamma$ defined by

$$\mathrm{Color}(D, m) := \{x \in m^\Gamma : \forall i < m\, [x^{-1}(\{i\}) \text{ is } D\text{-spaced}]\}.$$

A greedy coloring argument shows that $\mathrm{Color}(D, m)$ is nonempty and $D$-irreducible. Moreover, $g$ acts freely on $\mathrm{Color}(D, m)$. Inject $m$ into $2^{\{k,\ldots,\ell-1\}}$ for some $\ell > k$ and identify $\mathrm{Color}(D, m)$ as a subflow of $(2^{\{k,\ldots,\ell-1\}})^\Gamma$. Then $Y := \overline{\pi}_k(X) \times \mathrm{Color}(D, m) \subseteq (2^\ell)^\Gamma \subseteq (2^\mathbb{N})^\Gamma$, where the last inclusion can be formed by adding strings of zeros to the end. Then $Y$ is strongly irreducible, $g$-free and $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$. □

## 2. UFOs and minimal subdynamics

Much of the construction will require us to reason about the product group $G \times F_2$. So for the time being, fix countably infinite groups $\Delta \subseteq \Gamma$. For our purposes, $\Gamma$ will be $G \times F_2$, and $\Delta$ will be $G$, where we identify $G$ with a subgroup of $G \times F_2$ in the obvious way. However, for this subsection, we will reason more generally.

**Definition 2.1.** Let $\Delta \subseteq \Gamma$ be countably infinite groups. A finite subset $U \subseteq \Gamma$ is called a $(\Gamma, \Delta)$-*UFO* if for any maximal $U$-spaced set $S \subseteq \Gamma$, we have that $S$ meets every right coset of $\Delta$ in $\Gamma$.

We say that the inclusion of groups $\Delta \subseteq \Gamma$ *admits UFOs* if for every finite $U \subseteq \Gamma$, there is a finite $V \subseteq \Gamma$ with $V \supseteq U$ which is a $(\Gamma, \Delta)$-UFO.

As a word of caution, we note that the property of being a $(\Gamma, \Delta)$-UFO is not upwards closed.

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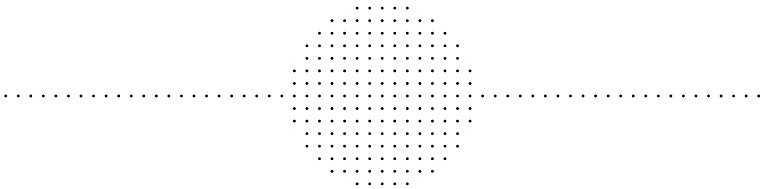

***Figure 1.*** *Sighting in Roswell; a* $(\mathbb{Z}\times\mathbb{Z}, \mathbb{Z}\times\{0\})$*-UFO subset of* $\mathbb{Z}\times\mathbb{Z}$.

The terminology comes from considering the case of a product group, that is, $\Gamma = \mathbb{Z}\times\mathbb{Z}$ and $\Delta = \mathbb{Z}\times\{0\}$. Figure 1 depicts a typical UFO subset of $\mathbb{Z}\times\mathbb{Z}$.

**Proposition 2.2.** *Let* $\Delta$ *be a subgroup of* $\Gamma$. *If* $|\bigcap_{u\in U} u\Delta u^{-1}|$ *is infinite for every finite set* $U \subseteq \Gamma$, *then* $\Delta \subseteq \Gamma$ *admits UFOs. In particular, if* $\Delta$ *contains an infinite subgroup that is normal in* $\Gamma$, *then* $\Delta \subseteq \Gamma$ *admits UFOs.*

*Proof.* We prove the contrapositive. So assume that $\Delta \subseteq \Gamma$ does not admit UFOs. Let $U \subseteq \Gamma$ be a finite symmetric set such that no finite $V \subseteq \Gamma$ containing $U$ is a $(\Gamma, \Delta)$-UFO. Let $D \subseteq \Delta$ be finite, symmetric and contain the identity. It will suffice to show that $C = \bigcap_{u\in U} uDu^{-1}$ satisfies $|C| \le |U|$.

Set $V = U \cup D^2$. Since $V$ is not a $(\Gamma, \Delta)$-UFO, there is a maximal $V$-spaced set $S \subseteq \Gamma$ and $g \in \Gamma$ with $S \cap \Delta g = \varnothing$. Since $S$ is $V$-spaced and $u^{-1}C^2u \subseteq D^2 \subseteq V$, the set $C_u = (uS) \cap (Cg)$ is $C^2$-spaced for every $u \in U$. Of course, any $C^2$-spaced subset of $Cg$ is empty or a singleton, so $|C_u| \le 1$ for each $u \in U$. On the other hand, since $S$ is maximal we have $VS = \Gamma$, and since $S \cap \Delta g = \varnothing$ we must have $Cg \subseteq US$. Therefore, $|C| = |Cg| = \sum_{u\in U} |C_u| \le |U|$. □

In the spaces $\overline{\mathcal{S}}(k^\Gamma)$ and $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$, the minimal flows form a dense $G_\delta$. However, when $\Delta \subseteq \Gamma$ is a subgroup, we can ask about the properties of members of $\overline{\mathcal{S}}(k^\Gamma)$ and $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ viewed as $\Delta$-flows.

**Definition 2.3.** Given a subshift $X \subseteq k^\Gamma$ and a finite $E \subseteq \Gamma$, we say that $X$ is $(\Delta, E)$-*minimal* if for every $x \in X$ and every $p \in P_E(X)$, there is $g \in \Delta$ with $(gx)|_E = p$. Given a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ and $n \in \mathbb{N}$, we say that $X$ is $(\Delta, E, n)$-*minimal* if $\overline{\pi}_n(X) \subseteq (2^n)^\Gamma$ is $(\Delta, E)$-minimal. When $\Delta = \Gamma$, we simply say that $X$ is $E$-*minimal* or $(E, n)$-*minimal*.

The set of $(\Delta, E)$-minimal flows is open in $\mathrm{Sub}(k^\Gamma)$, and $X \subseteq k^\Gamma$ is minimal as a $\Delta$-flow iff it is $(\Delta, E)$-minimal for every finite $E \subseteq \Gamma$. Similarly, the set of $(\Delta, E, n)$-minimal flows is open in $\mathrm{Sub}((2^\mathbb{N})^\Gamma)$, and $X \subseteq (2^\mathbb{N})^\Gamma$ is minimal as a $\Delta$-flow iff it is $(\Delta, E, n)$ minimal for every finite $E \subseteq \Gamma$ and every $n \in \mathbb{N}$.

In the proof of Proposition 2.4, it will be helpful to extend conventions about the shift action to subsets of $\Gamma$. If $U \subseteq \Gamma$, $g \in G$ and $p \in k^U$, we write $g\cdot p \in k^{Ug^{-1}}$ for the function where given $h \in Ug^{-1}$, we have $(g\cdot p)(h) = p(hg)$.

**Proposition 2.4.** *Suppose* $\Delta \subseteq \Gamma$ *are countably infinite groups and that* $\Delta \subseteq \Gamma$ *admits UFOs. Then* $\{X \in \overline{\mathcal{S}}(k^\Gamma) : X$ *is minimal as a* $\Delta -$*flow*$\}$ *is a dense* $G_\delta$ *subset. Similarly,* $\{X \in \overline{\mathcal{S}}(2^\mathbb{N})^\Gamma :$ $X$ *is minimal as a* $\Delta -$*flow*$\}$ *is a dense* $G_\delta$ *subset.*

*Proof.* We give the arguments for $k^\Gamma$, as those for $(2^\mathbb{N})^\Gamma$ are very similar.

It suffices to show for a given finite $E \subseteq \Gamma$ that the collection of $(\Delta, E)$-minimal flows is dense in $\overline{\mathcal{S}}(k^\Gamma)$. By enlarging $E$ if needed, we can assume that $E$ is symmetric.

Consider a nonempty open $O \subseteq \overline{\mathcal{S}}(k^\Gamma)$. By shrinking $O$ and/or enlarging $E$ if needed, we can assume that for some $X \in \mathcal{S}(k^\Gamma)$, we have $O = N_E(X) \cap \overline{\mathcal{S}}(k^\Gamma)$. We will build a $(\Delta, E)$-minimal shift $Y$ with $Y \in N_E(X) \cap \mathcal{S}(k^\Gamma)$. Fix a finite symmetric $D \subseteq \Gamma$ so that $X$ is $D$-irreducible. Then fix a finite $U \subseteq \Gamma$ which is large enough to contain an $EDE$-spaced set $Q \subseteq U \cap \Delta$ of cardinality $|P_E(X)|$, and enlarging $U$ if needed, choose such a $Q$ with $EQ \subseteq U$. Fix a bijection $Q \to P_E(X)$ by writing $P_E(X) = \{p_g : g \in Q\}$. Because $X$ is $D$-irreducible, we can find $\alpha \in P_U(X)$ so that $(gq)|_E = p_g$ for every $g \in Q$. By Proposition 2.2, fix a finite $V \subseteq \Gamma$ with $V \supseteq UDU$ which is a

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$(\Gamma, \Delta)$-UFO. We now form the shift

$$Y = \{y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ so that } \forall g \in T\ (g\cdot y)|_U = \alpha\}.$$

Because $V = UDU$ and $X$ is $D$-irreducible, we have that $Y \neq \emptyset$. In particular, for any maximal $V$-spaced set $T \subseteq \Gamma$, we can find $y \in Y$ so that $(gy)|_U = \alpha$ for every $g \in T$. We also note that $Y \in N_E(X)$ by our construction of $\alpha$.

To see that $Y$ is $(\Delta, E)$-minimal, fix $y \in Y$ and $p \in P_E(Y)$. Suppose this is witnessed by the maximal $V$-spaced set $T \subseteq \Gamma$. Because $V$ is a $(\Gamma, \Delta)$-UFO, find $h \in \Delta \cap T$. So $(hy)|_U = \alpha$. Now, suppose $g \in Q$ is such that $p = p_g$. We have $(ghy)|_E = (g \cdot ((hy)|_U)|_E = p_g$.

To see that $Y \in \mathcal{S}(k^\Gamma)$, we will show that $Y$ is $DUVUD$-irreducible. Suppose $y_0, y_1 \in Y$ and $S_0, S_1 \subseteq \Gamma$ are $DUVUD$-apart. For each $i < 2$, fix $T_i \subseteq \Gamma$ a maximal $V$-spaced set which witnesses that $y_i$ is in $Y$. Set $B_i = \{g \in T_i : DUg \cap S_i \neq \emptyset\}$. Notice that $B_i \subseteq UDS_i$. It follows that $B_0 \cup B_1$ is $V$-spaced, so extend to a maximal $V$-spaced set $B$. It also follows that $S_i \cup UB_i \subseteq U^2DS_i$. Since $V \supseteq UDU$ and by the definition of $B_i$, the collection of sets $\{S_i \cup UB_i : i < 2\} \cup \{Ug : g \in B \setminus (B_0 \cup B_1)\}$ is pairwise $D$-apart. By the $D$-irreducibility of $X$, we can find $y \in X$ with $y|_{S_i \cup UB_i} = y_i|_{S_i \cup UB_i}$ for each $i < 2$ and with $(gy)|_U = \alpha$ for each $g \in B \setminus (B_0 \cup B_1)$. Since $B_i \subseteq T_i$, we actually have $(gy)|_U = \alpha$ for each $g \in B$. So $y \in Y$ and $y|_{S_i} = y_i|_{S_i}$ as desired. □

*Proof of Theorem 0.3.* By Proposition 2.4, the generic member of $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma)$ is minimal as a $\Delta_n$-flow for each $n \in \mathbb{N}$, and by Proposition 1.1, the generic member of $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma$ is free. □

In contrast to Theorem 0.1, the next example shows that Question 0.2 is nontrivial to answer in full generality.

**Theorem 2.5.** *Let $G = \sum_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$, and let $X$ be a $G$ flow with infinite underlying space. Then there exists an infinite subgroup $H$ such that $X$ is not minimal as an $H$ flow.*

*Proof.* We may assume that $X$ is a minimal $G$-flow, as otherwise we may take $H = G$. We construct a sequence $X \supsetneq K_0 \supsetneq K_1 \supsetneq \cdots$ of proper, nonempty, closed subsets of $X$ and a sequence of group elements $\{g_n : n \in \mathbb{N}\}$ so that by setting $K = \bigcap_{\mathbb{N}} K_n$ and $H = \langle g_n : n \in \mathbb{N}\rangle$, then $K$ will be a minimal $H$-flow. Start by fixing a closed, proper subset $K_0 \subsetneq X$ with nonempty interior. Suppose $K_n$ has been created and is $\langle g_0, \ldots, g_{n-1}\rangle$-invariant. As $X$ is a minimal $G$-flow, the set $S_n := \{g \in G : \mathrm{Int}(gK_n \cap K_n) \neq \emptyset\}$ is infinite. Pick any $g_n \in S_n \setminus \{1_G\}$, and set $K_{n+1} = g_nK_n \cap K_n$. As $g_n^2 = 1_G$, we see that $K_{n+1}$ is $g_n$-invariant, and as $G$ is abelian, we see that $K_{n+1}$ is also $g_i$-invariant for each $i < n$. It follows that $K$ will be $H$-invariant as desired. □

Before moving on, we give a conditional proof of Theorem 0.1, which works as long as some nonamenable group admits a strongly irreducible shift without an invariant measure. It is the inspiration for our overall construction.

**Proposition 2.6.** *Let $G$ and $H$ be countably infinite groups, and suppose that for some $k < \omega$ and some strongly irreducible flow $Y \subseteq k^H$ that $Y$ does not admit an $H$-invariant measure. Then there is a minimal $G$-flow which does not admit a characteristic measure.*

*Proof.* Viewing $Z = k^G \times Y$ as a subshift of $k^{G\times H}$, then $Z$ is strongly irreducible and does not admit an $H$-invariant probability measure. The property of not possessing an $H$-invariant measure is an open condition in $\mathrm{Sub}(k^{G\times H})$; indeed, if $X_n \to X$ is a convergent sequence in $\mathrm{Sub}(k^{G\times H})$ and $\mu_n$ is an $H$-invariant probability measure supported on $X_n$, then by passing to a subsequence, we may suppose that the $\mu_n$ weak*-converge to some $H$-invariant probability measure $\mu$ supported on $X$. By Proposition 2.4, we can therefore find $X \subseteq k^{G\times H}$ which is minimal as a $G$-flow and which does not admit an $H$-invariant measure. As $H$ acts by $G$-flow automorphisms on $X$, we see that $X$ does not admit a characteristic measure. □

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Unfortunately, the question of if there exists any countable group $H$ and a strongly irreducible $H$-subshift $Y$ with no $H$-invariant measure is an open problem. Therefore, our construction proceeds by considering the free group $F_2$ and defining a suitable weakening of strongly irreducible subshift which is strong enough for $G$-minimality to be generic in $(G \times F_2)$-subshifts but weak enough for $(G \times F_2)$-subshifts without $F_2$-invariant measures to exist.

## 3. Variants of strong irreducibility

In this section, we investigate a weakening of strong irreducibility that one can define given any right-invariant collection $\mathcal{B}$ of finite subsets of a given countable group. For our overall construction, we will consider $F_2$ and $G \times F_2$, but we give the definitions for any countably infinite group $\Gamma$. Write $\mathcal{P}_f(\Gamma)$ for the collection of finite subsets of $\Gamma$.

**Definition 3.1.** Fix a right-invariant subset $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$. Given $k \in \mathbb{N}$, we say that a subshift $X \subseteq k^\Gamma$ is *$\mathcal{B}$-irreducible* if there is a finite $D \subseteq \Gamma$ so that for any $m < \omega$, any $B_0, \ldots, B_{m-1} \in \mathcal{B}$, and any $x_0, \ldots, x_{m-1} \in X$, if the sets $\{B_0, \ldots, B_{m-1}\}$ are pairwise $D$-apart, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < m$. We call $D$ the *witness* to $\mathcal{B}$-irreducibility. If we have $D$ in mind, we can say that $X$ is *$\mathcal{B}$-$D$-irreducible*.

We say that a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ is *$\mathcal{B}$-irreducible* if for each $k \in \mathbb{N}$, the subshift $\overline{\pi}_k(X) \subseteq (2^k)^\Gamma$ is $\mathcal{B}$-irreducible.

We write $\mathcal{S}_\mathcal{B}(k^\Gamma)$ or $\mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the set of $\mathcal{B}$-irreducible subflows of $k^\Gamma$ or $(2^\mathbb{N})^\Gamma$, respectively, and we write $\overline{\mathcal{S}}_\mathcal{B}(k^\Gamma)$ or $\overline{\mathcal{S}}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the Vietoris closures.

**Remark.**

1. If $\mathcal{B}$ is closed under unions, it is enough to consider $m = 2$. However, this will often not be the case.
2. By compactness, if $X \subseteq k^\Gamma$ is $\mathcal{B}$-$D$-irreducible, $\{B_n : n < \omega\} \subseteq \mathcal{B}$ is pairwise $D$-apart, and $\{x_n : n < \omega\} \subseteq X$, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$.
3. If $\mathcal{B} \subseteq \mathcal{B}'$, then $\mathcal{S}_{\mathcal{B}'}(k^\Gamma) \subseteq \mathcal{S}_\mathcal{B}(k^\Gamma)$ and $\mathcal{S}_{\mathcal{B}'}((2^\mathbb{N})^\Gamma) \subseteq \mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$

When $\mathcal{B}$ is the collection of all finite subsets of $H$, then we recover the notion of a strongly irreducible shift. Again, we consider Cantor space alphabets to obtain freeness.

**Proposition 3.2.** *For any right-invariant collection $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$, the generic member of $\overline{\mathcal{S}}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ is free.*

*Proof.* Analyzing the proof of Proposition 1.1, we see that the only properties that we need of the collections $\mathcal{S}_\mathcal{B}(k^\Gamma)$ and $\mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the proof to generalize are that they are closed under products and contain the flows $\mathrm{Color}(D, m)$. If $k, \ell \in \mathbb{N}$ an $X \subseteq k^\Gamma$ and $Y \subseteq \ell^\Gamma$ are $\mathcal{B}$-$D$-irreducible and $\mathcal{B}$-$E$-irreducible for some finite $D, E \subseteq \Gamma$, then $X \times Y \subseteq (k \times \ell)^\Gamma$ will be $\mathcal{B}$-$(D \cup E)$-irreducible. And as $\mathrm{Color}(D, m)$ is strongly irreducible, it is $\mathcal{B}$-irreducible. □

Now, we consider the group $F_2$. We consider the left Cayley graph of $F_2$ with respect to the standard generating set $A := \{a, b, a^{-1}, b^{-1}\}$. We let $d\colon F_2 \times F_2 \to \omega$ denote the graph metric. Write $D_n = \{s \in F_2 : d(s, 1_{F_2}) \leq n\}$.

**Definition 3.3.** Given $n$ with $1 \leq n < \omega$, we set

$$\mathcal{B}_n = \{D \in \mathcal{P}_f(F_2) : \text{ connected components of } D \text{ are pairwise } D_n\text{-apart}\}.$$

Write $\mathcal{B}_\omega$ for the collection of finite, connected subsets of $F_2$.

**Proposition 3.4.** *Suppose $X \subseteq k^{F_2}$ is $\mathcal{B}_\omega$-irreducible. Then there is some $n < \omega$ for which $X$ is $\mathcal{B}_n$-irreducible.*

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*Proof.* Suppose $X$ is $\mathcal{B}_\omega$-$D_n$-irreducible. We claim $X$ is $\mathcal{B}_n$-$D_n$-irreducible. Suppose $m < \omega$, $B_0, \ldots, B_{m-1} \in \mathcal{B}_n$ are pairwise $D_n$-apart, and $x_0, \ldots, x_{m-1} \in X$. For each $i < m$, we suppose $B_i$ has $n_i$-many connected componenets, and we write $\{C_{i,j} : j < n_i\}$ for these components. Then the collection of connected sets $\bigcup_{i<m}\{C_{i,j} : j < n_i\}$ is pairwise $D_n$-apart. As $X$ is $\mathcal{B}_\omega$-$D_n$-irreducible, we can find $y \in X$ so that for each $i < m$ and $j < n_i$, we have $y|_{C_{i,j}} = x_i|_{C_{i,j}}$. Hence, $y|_{B_i} = x_i|_{B_i}$, showing that $X$ is $\mathcal{B}_n$-$D_n$-irreducible. □

We now construct a $\mathcal{B}_\omega$-irreducible subshift with no $F_2$-invariant measure. We consider the alphabet $A^2$ and write $\pi_0, \pi_1 \colon A^2 \to A$ for the projections. We set

$$X_{pdox} = \{x \in (A^2)^{F_2} : \forall g, h \in F_2 \, \forall i, j < 2 \\ (i, g) \neq (j, h) \Rightarrow \pi_i(x(g)) \cdot g \neq \pi_j(x(h)) \cdot h\}.$$

More informally, the flow $X_{pdox}$ is the space of '2-to-1 paradoxical decompositions' of $F_2$ using $A$. We remark that here, our decomposition need not be a partition of $F_2$; we just ask for disjoint $S_0, S_1 \subseteq F_2$ such that for every $g \in G$ and $i < 2$, we have $Ag \cap S_i \neq \emptyset$. This is in some sense the prototypical example of an $F_2$-shift with no $F_2$-invariant measure.

**Lemma 3.5.** *$X_{pdox}$ has no $F_2$-invariant measure.*

*Proof.* For $u \in A^2$ set $Y_u = \{x \in X_{pdox} : x(1_G) = u\}$. Notice that if $y \in Y_u$, $i < 2$ and $x = \pi_i(u)y$, then $x(\pi_i(u)^{-1}) = y(1_G) = u$. Consequently, if $u, v \in A^2$, $x \in \pi_i(u)Y_u \cap \pi_j(v)Y_v$ then, since $x \in X_{pdox}$ and

$$\pi_i(x(\pi_i(u)^{-1}))\pi_i(u)^{-1} = 1_G = \pi_j(x(\pi_j(v)^{-1}))\pi_j(v)^{-1},$$

we must have that $(i, \pi_i(u)) = (j, \pi_j(v))$, and hence also

$$\pi_{1-i}(u) = \pi_{1-i}(x(\pi_i(u)^{-1})) = \pi_{1-j}(x(\pi_j(v)^{-1})) = \pi_{1-j}(v).$$

Therefore, $\pi_i(u)Y_u \cap \pi_j(v)Y_v = \varnothing$ whenever $(i, u) \neq (j, v)$.

If $\mu$ were an invariant Borel probability measure on $X_{pdox}$, then we would have

$$2\mu(X_{pdox}) = 2\sum_{u \in A^2} \mu(Y_u) = \sum_{i<2}\sum_{u \in A^2} \mu(\pi_i(u)Y_u) \leq \mu(X)$$

which is a contradiction. □

When proving that $X_{pdox}$ is $\mathcal{B}_\omega$-irreducible, note that $D_1 = A \cup \{1_{F_2}\}$.

**Proposition 3.6.** *$X_{pdox}$ is $\mathcal{B}_\omega$-$D_4$-irreducible.*

*Proof.* The proof will use a 2-to-1 instance of Hall's matching criterion [7] which we briefly describe. Fix a bipartite graph $\mathbb{G} = (V, E)$ with partition $V = V_0 \sqcup V_1$. Given $S \subseteq V_0$, write $N_{\mathbb{G}}(S) = \{v \in V_1 : \exists u \in S (u, v) \in E\}$. Then the matching condition we need states that if for every finite $S \subseteq V_0$, we have $|N_{\mathbb{G}}(S)| \geq 2S$, then there is $E' \subseteq E$ so that in the graph $\mathbb{G}' := (V, E')$, $d_{\mathbb{G}'}(u) = 2$ for every $u \in V_0$.

Let $B_0, \ldots, B_{k-1} \in \mathcal{B}_\omega$ be pairwise $D_4$-apart. Let $x_0, \ldots, x_{k-1} \in X_{pdox}$. To construct $y \in X_{pdox}$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < k$, we need to verify a 2-to-1 Hall's matching criterion on every finite subset of $F_2 \setminus \bigcup_{i<k} B_i$. Call $s \in F_2$ *matched* if for some $i < k$, some $g \in B_i$ and some $j < 2$, we have $s = \pi_j(x_i(g)) \cdot g$. So we need for every finite $E \in \mathcal{P}_f(F_2 \setminus \bigcup_{i<k} B_i)$ that $AE$ contains at least $2|E|$-many unmatched elements. Towards a contradiction, let $E \in \mathcal{P}_f(F_2 \setminus \bigcup_{i<k} B_i)$ be a minimal failure of the Hall condition.

In the left Cayley graph of $F_2$, given a reduced word $w$ in alphabet $A = \{a, b, a^{-1}, b^{-1}\}$, write $N_w$ for the set of reduced words which *end* with $w$. Now, find $t \in E$ (let us assume the leftmost character of $t$ is $a$) so that all of $E \cap N_{at}$, $E \cap N_{bt}$ and $E \cap N_{b^{-1}t}$ are empty. If any two of $at$, $bt$ and $b^{-1}t$ is an unmatched point in $AE$, then $E \setminus \{t\}$ is a smaller failure of Hall's criterion. So there must be

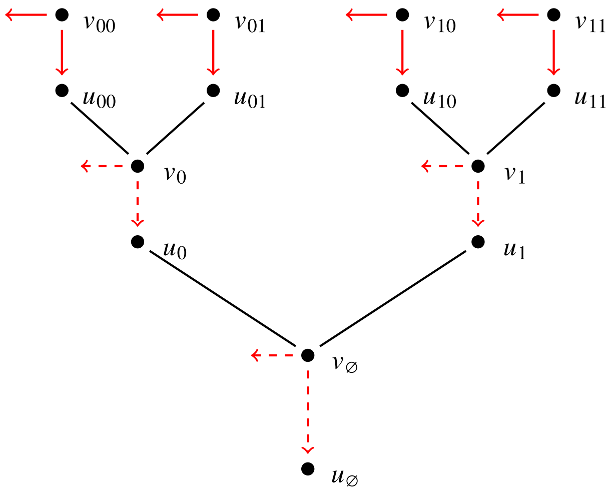


***Figure 2.*** *A pair of outgoing edges, drawn in solid red, is chosen at each of $v_{00}$, $v_{01}$, $v_{10}$ and $v_{11}$. Edges which must consequently be oriented in a particular direction are indicated with dashed red arrows. Most importantly, $v_\varnothing$ is forced to direct an edge to $u_\varnothing$. By considering the generalization of this picture for any length of binary string, we see that $X_{pdox}$ cannot be $D_n$-irreducible for any $n \in \mathbb{N}$.*

some $i < k$, some $g \in B_i$ and some $j < 2$, we have $\pi_j(x_i(g)) \cdot g \in \{at, bt, b^{-1}t\}$. Let us suppose $\pi_j(x_i(g)) \cdot g = at$. Note that since $g \notin E$, we must have $g \in \{bat, a^2t, b^{-1}at\}$. But then since $B_i$ is connected, we have $D_1 B_i \cap \{bt, b^{-1}t\} = \emptyset$, and since the other $B_q$ are at least distance 5 from $B_i$, we have $D_1 B_q \cap \{bt, b^{-1}t\} = \emptyset$ for every $q \in k \setminus \{i\}$. In particular, $bt$ and $b^{-1}t$ are unmatched points in $AE$, a contradiction. □

We remark that $X_{pdox}$ is not $D_n$-irreducible for any $n \in \mathbb{N}$. See Figure 2.

## 4. The construction

Our goal for the rest of the paper is to use $X_{pdox}$ to build a subflow of $(2^{\mathbb{N}})^{G\times F_2}$ which is free, $G$-minimal and with no $F_2$-invariant measure. In what follows, given an $F_2$-coset $\{g\} \times F_2$, we endow this coset with the left Cayley graph for $F_2$ using the generating set $A$ exactly as above. We extend the definition of $\mathcal{B}_n$ to refer to finite subsets of any given $F_2$-coset.

**Definition 4.1.** Given $n$ with $1 \le n \le \omega$, we set

$$\mathcal{B}_n^* = \{D \in \mathcal{P}_f(G \times F_2) : \text{ for each } F_2 - \text{coset } C, D \cap C \in \mathcal{B}_n\}.$$

Given $y \in k^{G\times F_2}$ and $g \in G$, we define $y_g \in k^{F_2}$ where given $s \in F_2$, we set $y_g(s) = y(g,s)$. If $X \subseteq k^{F_2}$ is $\mathcal{B}_n$-irreducible, then the subshift $X^G \subseteq k^{G\times F_2}$ is in $\mathcal{S}_{\mathcal{B}_n^*}$, where we view $X^G$ as the set $\{y \in k^{G\times F_2} : \forall g \in G\ (y_g \in X)\}$. In particular, $(X_{pdox})^G$ is $\mathcal{B}_4^*$-irreducible. By encoding $(X_{pdox})^G$ as a subshift of $(2^m)^{G\times F_2}$ for some $m \in \mathbb{N}$ and considering $\tilde{\pi}_m^{-1}((X_{pdox})^G) \subseteq (2^{\mathbb{N}})^{G\times F_2}$, we see that there is a $\mathcal{B}_4^*$-irreducible subflow of $(2^{\mathbb{N}})^{G\times F_2}$ for which the $F_2$-action doesn't fix a measure. It follows that such subflows constitute a nonempty open subset of $\Phi := \overline{\bigcup_n \mathcal{S}_{\mathcal{B}_n^*}((2^{\mathbb{N}})^{G\times F_2})}$. Combining the next result with Proposition 3.2, we will complete the proof of Theorem 0.1.

**Proposition 4.2.** *With $\Phi$ as above, the $G$-minimal flows are dense $G_\delta$ in $\Phi$.*

*Proof.* We show the result for $\Phi_k := \overline{\bigcup_n \mathcal{S}_{\mathcal{B}_n^*}(k^{G\times F_2})}$ to simplify notation; the proof in full generality is almost identical.

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We only need to show density. To that end, fix a finite symmetric $E \subseteq G \times F_2$ which is connected in each $F_2$-coset. It is enough to show that the $(G, E)$-minimal subshifts are dense in $\Phi_k$. Fix some nonempty open $O \subseteq \Phi_k$. By enlarging $E$ and/or shrinking $O$, we may assume that for some $n < \omega$ and $X \in \mathcal{S}_{\mathcal{B}_n^*}(k^{G\times F_2})$ that $O = \{X' \in \Phi_k : P_E(X') = P_E(X)\}$. We will build a $(G, E)$-minimal subshift $Y \subseteq k^{G\times F_2}$ so that $P_E(Y) = P_E(X)$ and so that for some $N < \omega$, we have $Y \in \mathcal{S}_{\mathcal{B}_N^*}(k^{G\times F_2})$.

Recall that $D_n \subseteq F_2$ denotes the ball of radius $n$. Fix a finite, symmetric $D \subseteq G \times F_2$ so that $\{1_G\} \times D_{2n} \subseteq D$ and $X$ is $\mathcal{B}_n^*$-$D$-irreducible. Find a finite symmetric $U_0 \subseteq G$ with $1_G \subseteq U_0$ and $r < \omega$ so that upon setting $U = U_0 \times D_r \subseteq G \times F_2$, then $U$ is large enough to contain an $EDE$-spaced set $Q \subseteq G$ with $EQ \subseteq U$. As $X$ is $\mathcal{B}_n^*$-$D$-irreducible, there is a pattern $\alpha \in P_U(X)$ so that $\{(g\alpha)|_E : g \in Q\} = P_E(X)$.

Let $V \supseteq UD^2U$ be a $(G \times F_2, G)$-UFO. We remark that for most of the remainder of the proof, it would be enough to have $V \supseteq UDU$; we only use the stronger assumption $V \supseteq UD^2U$ in the proof of the final claim. Consider the following subshift:

$$Y = \{y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ so that } \forall g \in T\ (gy)|_U = \alpha\}.$$

The proof that $Y$ is nonempty and $(G, E)$-minimal is exactly the same as the analogous proof from Proposition 2.4. Note that by construction, we have $P_E(Y) = P_E(X)$.

We now show that $Y \in \mathcal{S}_{\mathcal{B}_N^*}(k^{G\times F_2})$ for $N = 4r + 3n$. Set $W = DUVUD$. We show that $Y$ is $\mathcal{B}_N^*$-$W$-irreducible. Suppose $m < \omega$, $y_0, \ldots, y_{m-1} \in Y$ and $S_0, \ldots, S_{m-1} \in \mathcal{B}_N^*$ are pairwise $W$-apart. Suppose for each $i < m$ that $T_i \subseteq G \times F_2$ is a maximal $V$-spaced set which witness that $y_i \in Y$. Set $B_i = \{g \in T_i : DUg \cap S_i \neq \emptyset\}$. Then $\bigcup_{i<m} B_i$ is $V$-spaced, so enlarge to a maximal $V$-spaced set $B \subseteq G \times F_2$.

For each $i < m$, we enlarge $S_i \cup UB_i$ to $J_i \in \mathcal{B}_n^*$ as follows. Suppose $C \subseteq G \times F_2$ is an $F_2$-coset. Each set of the form $C \cap Ug$ is connected. Since $S_i \in \mathcal{B}_N^*$, it follows that given $g \in B_i$, there is at most one connected component $\Theta_{C,g}$ of $S_i \cap C$ with $Ug \cap \Theta_{C,g} = \emptyset$, but $Ug \cap D_n\Theta_{C,g} \neq \emptyset$. We add the line segment in $C$ connecting $\Theta_{C,g}$ and $Ug$. Upon doing this for each $g \in B_i$ and each $F_2$-coset $C$, this completes the construction of $J_i$. Observe that $J_i \subseteq D_{n-1}S_i \cap UB_i$.

**Claim.** *Let $C$ be an $F_2$-coset, and suppose $Y_0$ is a connected component of $S_i \cap C$. Let $Y$ be the connected component of $J_i \cap C$ with $Y_0 \subseteq Y$. Then $Y \subseteq D_{2r+n}Y_0$. In particular, if $Y_0 \neq Z_0$ are two connected components of $S_i \cap C$, then $Y_0$ and $Z_0$ do not belong to the same component of $J_i \cap C$.*

*Proof.* Let $L = \{x_j : j < \omega\} \subseteq C$ be a ray with $x_0 \in Y_0$ and $x_j \notin Y_0$ for any $j \geq 1$. Then $\{j < \omega : x_j \in J_i \cap C\}$ is some finite initial segment of $\omega$. We want to argue that for some $j \leq 2r + n + 1$, we have $x_j \notin J_i \cap C$. First, we argue that if $x_n \in J_i \cap C$, then $x_n \in UB_i$. Otherwise, we must have $x_n \in D_{n-1}S_i$. But since $x_n \notin D_{n-1}Y_0$, there must be another component $Y_1$ of $S_i \cap C$ with $x_n \in D_nY_1$. But this implies that $Y_0$ and $Y_1$ are not $D_{2n-1}$-apart, a contradiction since $2n - 1 \leq 4r - 3n = N$.

Fix $g \in B_i$ with $x_n \in Ug$. Let $q < \omega$ be least with $q > n$ and $x_q \notin U_g$. We must have $q \leq 2r + n + 1$. We claim that $x_q \notin J_i \cap C$. Towards a contradiction, suppose $x_q \in J_i \cap C$. We cannot have $x_q \in UB_i$, so we must have $x_q \in D_{n-1}S_i$. But now there must be some component $Y_1$ of $S_i \cap C$ with $x_q \in D_{n-1}Y_1$. But then $D_{2r+2n}Y_0 \cap Y_1 \neq \emptyset$, a contradiction as $Y_0$ and $Y_1$ are $D_N$-apart. This concludes the proof that $Y \subseteq D_{2r+n}Y_0$.

Now, suppose $Y_0 \neq Z_0$ are two connected components of $S_i \cap C$. Then $Y_0$ and $Z_0$ are $N$-apart. In particular, $Z_0 \not\subseteq D_{2r+n}Y_0$, so cannot belong to the same connected component of $J_i \cap C$ as $Y_0$. □

**Claim.** $J_i \in \mathcal{B}_n^*$.

*Proof.* Fix an $F_2$-coset $C$ and two connected components $Y \neq Z$ of $J_i \cap C$. By the previous claim, each of $Y$ and $Z$ can only contain at most one nonempty component of $S_i \cap C$. The claim will be proven after considering three cases.

1. First, suppose each of $Y$ and $Z$ contain a nonempty component of $S_i \cap C$, say $Y_0 \subseteq Y$ and $Z_0 \subseteq Z$. Then since $Y_0$ and $Z_0$ are $D_{4r+3n}$-apart, the previous claim implies that $Y$ and $Z$ are $D_n$-apart.

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2. Now, suppose $Y$ contains a nonempty component $Y_0$ of $S_i \cap C$ and that $Z$ does not. Then for some $g \in B_i$, we have $Z = Ug \cap C$. Towards a contradiction, suppose $D_n Y \cap Ug \neq \emptyset$. Let $L = \{x_j : j \leq M\}$ be the line segment connecting $Y$ and $Ug$ with $L \cap Y = \{x_0\}$ and $L \cap Ug = \{x_M\}$. We must have $M \leq n$. We cannot have $x_0 \in UB_i$, so we must have $x_0 \in D_{n-1}S_i$. This implies that $x_0 \in D_{n-1}Y_0$. We cannot have $x_0 \in Y_0$, as otherwise, we would have connected $Y_0$ and $Ug \cap C$ when constructing $J_i$. It follows that for some $h \in B_i$, we have that $x_0$ is on the line segment $L' = \{x'_j : j \leq M'\}$ connecting $Y_0$ and $Uh \cap C$, and we have $M' \leq n$. But this implies that $Ug \cap D_{2n}Uh \neq \emptyset$, a contradiction since $V \supseteq UDU$ and $D \supseteq D_{2n}$.
3. If neither $Y$ nor $Z$ contain a component of $S_i \cap C$, then there are $g \neq h \in B_i$ with $Y = Uh \cap C$ and $Z = Ug \cap C$. It follows that $Y$ and $Z$ are $D_n$-apart. □

**Claim.** *Suppose $i \neq j < m$. Then $J_i$ and $J_j$ are $D$-apart.*

*Proof.* We have that $J_i \subseteq D_{n-1}S_i \cup UB_i$, and likewise for $j$. As $UB_i \subseteq U^2DS_i$ and as $D \supseteq D_{2n}$, we have $J_i \subseteq U^2DS_i$, and likewise for $j$. As $S_i$ and $S_j$ are $W$-apart and as $V \supseteq UDU$, we see that $J_i$ and $J_j$ are $D$-apart. □

**Claim.** *Suppose $g \in B \setminus \bigcup_{i<m} B_i$. Then $Ug$ and $J_i$ are $D$-apart for any $i < m$.*

*Proof.* As $g \notin B_i$, we have $U_g$ and $S_i$ are $D$-apart. Also, for any $h \in B$ with $g \neq h$, we have that $Ug$ and $Uh$ are $D$-apart. Now, suppose $DUg \cap J_i \neq \emptyset$. If $x \in DUg \cap J_i$, then on the coset $C = F_2 x$, $x$ must belong on the line between a component of $S_i \cap C$ and $Uh$ for some $h \in B_i$. Furthermore, we have $x \in D_{n-1}Uh$. But since $D_{2n} \subseteq D$, this contradicts that $Ug$ and $Uh$ are $D^2$-apart (using the full assumption $V \supseteq UD^2U$). □

We can now finish the proof of Proposition 4.2. The collection $\{J_i : i < m\} \cup \{Ug : g \in B \setminus (\bigcup_{i<m} B_i)\}$ is a pairwise $D$-apart collection of members of $\mathcal{B}_n^*$. As $X$ is $\mathcal{B}_n^*$-$D$-irreducible, we can find $y \in X$ with $y|_{J_i} = y_i|_{J_i}$ for each $i < m$ and with $(gy)|_U = \alpha$ for each $g \in B \setminus (\bigcup_{i<m} B_i)$. As $J_i \supseteq UB_i$ and since $B_i \subseteq T_i$, we actually have $(gy)|_U = \alpha$ for each $g \in B$. As $B$ is a maximal $V$-spaced set, it follows that $y \in Y$ and $y|_{S_i} = y_i|_{S_i}$ as desired. □

**Competing interest.** The authors have no competing interest to declare.

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