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ON α-LIKE RADICALS OF RINGS

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Abstract

Let α be any radical of associative rings. A radical γ is called α-like if, for every α-semisimple ring A, the polynomial ring $A[x]$ is γ-semisimple. In this paper we describe properties of α-like radicals and show how they can be used to solve some open problems in radical theory.

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1. Introduction

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. The fundamental definitions and properties of radicals can be found in [1] and [10]. A class μ of rings is called hereditary if μ is closed under ideals. If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. As usual, for a radical γ, the γ radical of a ring A is denoted by $\gamma(A)$ and the class of all γ-semisimple rings is denoted by $\mathcal{S}(\gamma)$. The class of all prime rings is denoted by π and $\beta = \mathcal{U}(\pi)$ denotes the prime radical. The notation $I \triangleleft A$ means that I is a two-sided ideal of a ring A. An ideal I of a ring A is called essential in A if $I \cap J \neq 0$ for every nonzero two-sided ideal J of A. A ring A is called an essential extension of a ring I if I is an essential ideal of A. A class μ of rings is called essentially closed if $\mu = \mu_k$, where $\mu_k = \{A : A \text{ is an essential extension of some } I \in \mu\}$ is the essential cover of μ. A hereditary and essentially closed class of prime rings is called a special class and the upper radical generated by a special class is called a special radical. Given a ring A, the polynomial ring over A in a commuting indeterminate x is denoted by $A[x]$. We say that a radical γ has the Amitsur property if $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$ for every ring A. A radical γ is called polynomially

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extensible if $A[x] \in \gamma$ for every ring $A \in \gamma$. It is well known [10, Proposition 4.9.21] that γ is polynomially extensible if and only if $\gamma = \gamma_x$, where $\gamma_x = \{A : A[x] \in \gamma\}$.

A radical α is said to satisfy the polynomial equation if $\alpha(A[x]) = (\alpha(A))[x]$ for every ring A. It was proved in [11] that α satisfies the polynomial equation if and only if it is polynomially extensible and has the Amitsur property.

A radical γ is called prime-like [18] if $A[x] \in \mathcal{S}\gamma$ for any prime ring A. The importance of prime-like radicals stems from the fact that, as was shown in [18], they allow us to easily construct pairs of distinct special radicals that coincide on simple rings and on polynomial rings. This answers a question posed by Ferrero [19]. Also, Gardner's [8, Problem 1] long-standing open question whether $\beta = \mathcal{U}(*_k)$, is equivalent to the question whether the radical $\mathcal{U}(*_k)$ is prime-like, where $*$ denotes the class of all $*$-rings (see [3–5, 12]), that is, semiprime rings R such that $R/I \in \beta$ for every nonzero ideal I of R.

It was shown in [18] that a radical γ is prime-like if and only if $A[x] \in \mathcal{S}\gamma$ for every semiprime ring A. Inspired by this fact, we introduce the following definition.

Definition 1.1. Let α be any radical. We say that a radical γ is α-like if $A[x] \in \mathcal{S}\gamma$ for any $A \in \mathcal{S}\alpha$.

Alpha-like radicals with α satisfying the polynomial equation were introduced and studied in [7] where they were used to easily construct pairs of distinct special radicals that meet Ferrero's conditions [19].

In this paper we study properties of α-like radicals for any radical α. We generalise some results of [7]. In particular, we characterise α-like radicals and give sufficient conditions for a radical γ to be α-like. For every proper radical α—that is, a radical $\alpha \neq$ {all rings}—we construct a strictly ascending chain of radicals $\gamma_i \supsetneqq \alpha$ that are not α-like. This answers a question posed in [7]. Strong radicals are those containing all one-sided radical ideals. Since the class of strong radicals is not a sublattice of the lattice of all radicals [16], there are radicals that do not contain largest strong radicals. We use α-like radicals to construct those that do. This allows us to reformulate the famous Koethe problem which asks whether the nil radical is strong. All these give a reason for studying α-like radicals. We look at α-like radicals from the lattice theory point of view. We prove that the collection $\mathbb{L}_\alpha$ of all α-like radicals is a complete sublattice of the lattice of all radicals. This allows us to show that for any proper radical α, there exists a unique proper largest α-like radical. We prove that for a radical $\alpha \supsetneqq \beta$, the lattice $\mathbb{L}_\alpha$ is not atomic. We show, however, that for every proper radical α, the complete lattice $\mathbb{L}_{h\alpha}$ of all hereditary α-like radicals is atomic and we describe its atoms.

2. Main results

In this paper α denotes any radical. We will start with some examples and properties of α-like radicals.

It was shown in [7] that any radical α with the Amitsur condition is α-like. We will now show that it is not so in general.

Example 2.1. Consider the special radical $\gamma = \mathcal{U}(\{Z_p\})$, where p is a prime integer and Z_p denotes the field of integers modulo p. We have $Z_p \in \mathcal{S}(\gamma)$ but $Z_p[x] \notin \mathcal{S}(\gamma)$ because $0 \neq x^p - x \in \gamma(Z_p[x])$ since every $a \in Z_p$ satisfies the polynomial equation $x^p - x = 0$ and, γ being a special radical, $\gamma(Z_p[x]) = \cap\{I \lhd Z_p[x] : Z_p[x]/I \simeq Z_p\}$. Thus γ is not γ-like.

Proposition 2.2. *If a radical γ is α-like, then $\gamma_x \subseteqq \alpha$ but the converse does not hold in general.*

Proof. Let $A \in \gamma_x$ and suppose $A \notin \alpha$. Then $A[x] \in \gamma$ and $0 \neq B = A/(\alpha(A)) \in \mathcal{S}(\alpha)$. But, since γ is α-like, this implies that $B[x] \in \mathcal{S}(\gamma)$. On the other hand, $B[x] = (A/(\alpha(A)))[x] \simeq A[x]/(\alpha(A))[x] \in \gamma$ in view of $A[x] \in \gamma$. This implies that $B[x] = 0$, a contradiction.

To see that the converse does not hold, consider again $\gamma = \mathcal{U}(\{Z_p\})$. Since for any radical ρ, $\rho_x \subseteq \rho$ [10, Proposition 4.9.17 (ii)], we have, in particular, that $\gamma_x \subseteqq \gamma$ but γ is not γ-like as Example 2.1 shows. □

However, we have the following generalisation of [7, Theorem 2.9].

Corollary 2.3. *A radical γ with the Amitsur property is α-like if and only if $\gamma_x \subseteqq \alpha$.*

Proof. If $\gamma_x \subseteqq \alpha$, then $\mathcal{S}(\alpha) \subseteq \mathcal{S}(\gamma_x)$. Thus if $A \in \mathcal{S}(\alpha)$, then $A \in \mathcal{S}(\gamma_x)$ which, in view of [20, Theorem 3.5], implies that $A[x] \in \mathcal{S}\gamma$ since γ has the Amitsur property. Thus γ is α-like.

The converse follows from Proposition 2.2. □

A hereditary radical γ is said to be subidempotent if the radical class γ consists of idempotent rings.

It follows from [14, Proposition 4.1] that every subidempotent radical γ has the Amitsur property and $\gamma_x = \{0\}$. Thus Corollary 2.3 implies the following corollary.

Corollary 2.4. *Every subidempotent radical γ is α-like for every radical α.*

We also have another corollary.

Corollary 2.5. *Let γ be a radical with the Amitsur property. Then any radical $\tau \subseteq \gamma$ is γ-like.*

Proof. Let $A \in \mathcal{S}(\gamma)$. Then $A \in \mathcal{S}(\gamma_x)$ since $\gamma_x \subseteqq \gamma$ implies that $\mathcal{S}(\gamma) \subseteq \mathcal{S}(\gamma_x)$. But, since γ is a radical with the Amitsur property, it follows from [20, Theorem 3.5] that $A[x] \in \mathcal{S}\gamma$. This implies that $A[x] \in \mathcal{S}\tau$ because $\mathcal{S}\gamma \subseteq \mathcal{S}\tau$ as $\tau \subseteq \gamma$. Thus τ is γ-like. □

Let $\mathcal{N}$ be the nil radical, $\mathcal{J}$ the Jacobson radical, $\mathcal{G}$ the Brown–McCoy radical and $\psi = \mathcal{U}(\mathcal{P})$, where $\mathcal{P}$ is the class of all prime rings A such that every nonzero ideal of A contains a nonzero element from the centre of A. It was shown in [10, Proposition 4.9.27] that $\mathcal{J}_x \subseteq \mathcal{N}$ and it was proved in [15] that $\psi = \mathcal{G}_x$. Since $\mathcal{N}$, $\mathcal{J}$ and $\mathcal{G}$ are radicals with the Amitsur property [10], Corollary 2.3 implies the following example.

Example 2.6. The radical $\mathcal{J}$ is $\mathcal{N}$-like and $\mathcal{G}$ is ψ-like.

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Lemma 2.7. *If σ and γ are radicals such that $\sigma \subseteq \gamma$ and γ is α-like, then σ is also α-like.*

Proof. Let $A \in \mathcal{S}(\alpha)$. Then $A[x] \in \mathcal{S}\gamma$ since γ is α-like. But, since $\sigma \subseteq \gamma$ implies that $\mathcal{S}(\gamma) \subseteq \mathcal{S}(\sigma)$, it follows that $A[x] \in \mathcal{S}(\sigma)$. This shows that σ is α-like. □

Our next result is a generalisation of [7, Theorem 2.9].

Theorem 2.8. *A radical γ is α-like if and only if $\gamma(A[x]) \subseteq (\alpha(A))[x]$ for any ring A.*

Proof. Let γ be α-like and suppose that $\gamma(A[x]) \nsubseteq (\alpha(A))[x]$ for some ring A. Then $\gamma(B[x]) = 0$ for any $B \in S\alpha$. In particular, for $B = A/(\alpha(A))$, we have $0 = \gamma(A/(\alpha(A))[x]) \cong \gamma(A[x]/(\alpha(A))[x])$. On the other hand, since $\gamma(A[x]) \nsubseteq (\alpha(A))[x]$,

$$0 \neq \frac{(\alpha(A))[x] + \gamma(A[x])}{(\alpha(A))[x]} \cong \frac{\gamma(A[x])}{\gamma(A[x]) \cap (\alpha(A))[x]} \in \gamma.$$

But, since $((\alpha(A))[x] + \gamma(A[x]))/\alpha(A)[x] \lhd A[x]/(\alpha(A))[x]$,

$$0 \neq \frac{(\alpha(A))[x] + \gamma(A[x])}{(\alpha(A))[x]} \subseteq \gamma\Big(\frac{A[x]}{(\alpha(A))[x]}\Big) = 0,$$

and we have a contradiction.

Conversely, suppose that $\gamma(A[x]) \subseteq (\alpha(A))[x]$ for any ring A. Let $A \in S\alpha$. Then $\alpha(A) = 0$. Hence $\gamma(A[x]) \subseteq (\alpha(A))[x] = 0[x] = 0$, which shows that γ is α-like. □

Alpha-like radicals can be used to easily identify radicals that are not polynomially extensible, as our next result shows.

Proposition 2.9. *If a radical γ is α-like, then any radical $\rho \subseteq \gamma$ with $\mathcal{S}(\alpha) \cap \rho \neq 0$ is not polynomially extensible.*

Proof. Let $0 \neq A \in \mathcal{S}(\alpha) \cap \rho$. Then, since γ is α-like, $A[x] \in \mathcal{S}\gamma \subseteq \mathcal{S}\rho$. Thus $A \in \rho$ but $A[x] \notin \rho$, which shows that ρ is not polynomially extensible. □

Example 2.10. Let $\widehat{l_W}$ be the smallest special radical containing the nonnil Jacobson radical $*$-ring $W = \{2x/(2y+1) : x, y \in \mathbb{Z} \text{ and } (2x, 2y+1) = 1\}$ (see [3, 4, 12]). Then, since $\mathcal{J}$ is $\mathcal{N}$-like and $\widehat{l_W} \subseteq \mathcal{J}$ and $W \in \mathcal{S}(\mathcal{N}) \cap \widehat{l_W}$, it follows from Proposition 2.9 that $\widehat{l_W}$ is not polynomially extensible.

We say that radicals γ and τ are like each other if γ is τ-like and τ is γ-like. Our next result shows how to construct them.

Corollary 2.11. *Let γ be a radical with the Amitsur property. If τ is a radical such that $\gamma_x \subseteq \tau \subseteq \gamma$, then τ and γ are like each other. In particular, γ and γ_x are like each other.*

Proof. Let $A \in \mathcal{S}(\gamma)$. Then, since $\gamma_x \subseteq \tau \subseteq \gamma$ implies that $\mathcal{S}(\gamma) \subseteq \mathcal{S}(\tau) \subseteq \mathcal{S}(\gamma_x)$, it follows that $A \in \mathcal{S}(\gamma_x)$. But, as γ is a radical with the Amitsur property, it then follows from [20, Theorem 3.5] that $A[x] \in \mathcal{S}\gamma \subseteq \mathcal{S}(\tau)$. Thus τ is γ-like.

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To show that γ is τ-like, we will first show that $\gamma(A[x]) \subseteq (\gamma_x(A))[x]$ for any ring A. Indeed, since γ is a radical with the Amitsur property, $(\gamma(A[x]) \cap A)[x] = \gamma(A[x]) \in \gamma$ for every ring A. This means that $\gamma(A[x]) \cap A \in \gamma_x$ and, since $\gamma(A[x]) \cap A \triangleleft A$, it follows that $\gamma(A[x]) \cap A \subseteq \gamma_x(A)$. Then, $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x] \subseteq (\gamma_x(A))[x]$ for any ring A.

Now, if $A \in \mathcal{S}(\tau) \subseteq \mathcal{S}(\gamma_x)$ then $\gamma_x(A) = 0$ and so $\gamma(A[x]) \subseteq (\gamma_x(A))[x] = 0[x] = 0$, which means that $A[x] \in \mathcal{S}\gamma$ and shows that γ is τ-like. □

Since $\mathcal{J}_x \subseteq \mathcal{N} \subseteq \mathcal{J}$ and $\mathcal{G}_x = \psi \subseteq \mathcal{G}$, Corollary 2.11 gives us the following example.

Example 2.12. The radicals $\mathcal{N}$ and $\mathcal{J}$ are like each other and so are ψ and $\mathcal{G}$.

Corollary 2.13. *Let α and γ be radicals satisfying the polynomial equation. Then α and γ are like each other if only if $\alpha(A[x]) = \gamma(A[x])$ for every ring A.*

Proof. If α and γ are like each other, then it follows from Theorem 2.8 that $\gamma(A[x]) \subseteq (\alpha(A))[x]$ and $\alpha(A[x]) \subseteq (\gamma(A))[x]$ for every ring A. But, since both α and γ satisfy the polynomial equation, $(\alpha(A))[x] = \alpha(A[x])$ and $(\gamma(A))[x] = \gamma(A[x])$ for every ring A. Thus $\alpha(A[x]) = \gamma(A[x])$ for every ring A.

Conversely, let $\alpha(A[x]) = \gamma(A[x])$ for every ring A. Then $\gamma(A[x]) \subseteq \alpha(A[x]) = (\alpha(A))[x]$ since α satisfies the polynomial equation. This, in view of Theorem 2.8, means that γ is α-like. Similarly, $\alpha(A[x]) \subseteq \gamma(A[x]) = (\gamma(A))[x]$ since γ satisfies the polynomial equation. This, in view of Theorem 2.8, means that α is γ-like. Thus α and γ are like each other. □

Theorem 2.14. *Let α and γ be radicals with the Amitsur property. Then α and γ are like each other if and only if $\alpha_x \cup \gamma_x \subseteq \gamma \cap \alpha$.*

Proof. Since both α and γ are radicals with the Amitsur property and they are like each other, Corollary 2.3 implies that $\gamma_x \subseteq \alpha$ and $\alpha_x \subseteq \gamma$. But, as $\gamma_x \subseteq \gamma$ and $\alpha_x \subseteq \alpha$, it follows that $\gamma_x \subseteq \gamma \cap \alpha$ and $\alpha_x \subseteq \gamma \cap \alpha$. Consequently, $\alpha_x \cup \gamma_x \subseteq \gamma \cap \alpha$.

Conversely, if $\alpha_x \cup \gamma_x \subseteq \gamma \cap \alpha$, then $\gamma_x \subseteq \alpha_x \cup \gamma_x \subseteq \gamma \cap \alpha \subseteq \alpha$ and $\alpha_x \subseteq \alpha_x \cup \gamma_x \subseteq \gamma \cap \alpha \subseteq \gamma$. But, since both α and γ are radicals with the Amitsur property, it follows from Corollary 2.3 that γ is α-like and α is γ-like. Thus α and γ are like each other. □

In what follows, for any class σ of rings, the lower radical generated by σ is denoted by $\mathcal{L}(\sigma)$. A radical γ is said to be small if $\mathcal{L}(\gamma \cup \tau)$ is proper for any proper radical τ. It was proved in [9] that the lower radical generated by a set of rings is small.

In [7] it was noted that for some radicals α, there exist radicals $\gamma \supsetneq \alpha$ that are not α-like and the question was asked whether this is so for any radical α. Our next two results answer this question.

Theorem 2.15. *For every proper radical α, there exists a strictly ascending chain of proper radicals $\gamma_i \supsetneq \alpha$, $i = 1, \ldots, n$, which are not α-like.*

Proof. Since α is a proper radical, there exists $0 \neq A_1 \in S\alpha$. Then $A_1[x] \notin \alpha$ since otherwise A_1, being a homomorphic image of $A_1[x]$, would be in α, giving

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a contradiction. But $A_1[x] \in \mathcal{L}(A_1[x]) \subseteq \mathcal{L}(\alpha \cup \mathcal{L}(A_1[x]))$. Thus $\alpha \subsetneq \alpha_1 = \mathcal{L}(\alpha \cup \mathcal{L}(A_1[x]))$ and it follows from [9] that $\alpha_1 \neq \{\text{all rings}\}$. So, arguing as before, there exists $0 \neq A_2 \in S\alpha_1$. Then again $A_2[x] \notin \alpha_1$ which shows that $\alpha_1 \subsetneq \alpha_2 = \mathcal{L}(\alpha \cup \mathcal{L}(A_1[x]) \cup \mathcal{L}(A_2[x]))$ and, by [9], we again obtain $\alpha_2 \neq \{\text{all rings}\}$. Continuing this process, we get a strictly increasing chain of proper radicals $\alpha \subsetneq \alpha_1 \subsetneq \alpha_2 \subsetneq \cdots \subsetneq \alpha_i \subsetneq \alpha_{i-1} \subsetneq \cdots$, where $\alpha_i = \mathcal{L}(\alpha \cup \mathcal{L}(A_1[x]) \cup \cdots \cup \mathcal{L}(A_i[x])) \neq \{\text{all rings}\}$ and $0 \neq A_i \in \mathcal{S}\alpha_{_{i-1}}$ for $i = 1, 2, 3, \ldots$. Moreover, every radical α_i in this chain is not α-like because the ring $0 \neq A_i \in \mathcal{S}\alpha_{i-1} \subseteq \mathcal{S}\alpha$ but, since $0 \neq A_i[x] \in \mathcal{L}(A_i[x]) \subseteq \alpha_i$, it follows that $A_i[x] \notin \mathcal{S}\alpha_i$. □

Note that if $\alpha = \{\text{all rings}\}$, then 0 is the only ring in $\mathcal{S}\alpha$. Therefore $A[x] = 0[x] = 0 \in \mathcal{S}\gamma$ for every $A \in \mathcal{S}\alpha$. Thus we have the following corollary.

Corollary 2.16. *Every radical γ is α-like for $\alpha = \{$all rings$\}$.*

We will now use α-like radicals to construct radicals that contain largest strong radicals.

We say that a radical γ satisfies condition (z) if, for every ring A, $A \in \gamma$ implies that $A^o \in \gamma$, where A^o is the zero-ring built on the additive group of A.

Theorem 2.17. *Let α be a radical such that, for some α-like left and right strong radical γ which satisfies condition (z), $M_2(A) \in \alpha$ implies that $A[x] \in \gamma$, where $M_2(A)$ denotes the ring of all 2×2 matrices with entries from A. Then γ_x is the largest left and right strong radical contained in α.*

Proof. Since γ is α-like, Proposition 2.2 implies that $\gamma_x \subseteqq \alpha$. Since γ is a left and right strong radical satisfying condition (z), it follows from [21] that so is γ_x. Now, let ρ be a left and right strong radical contained in α and let $A \in \rho$. Clearly $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a right ideal of $\left(\begin{smallmatrix} A^1 & 0 \\ A^1 & 0 \end{smallmatrix}\right)$, where A^1 is the Dorroh extension of A to a ring with unity. Since $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \simeq A \in \rho$ and ρ is right strong, the ideal $\left(\begin{smallmatrix} A & 0 \\ A & 0 \end{smallmatrix}\right)$ of $\left(\begin{smallmatrix} A^1 & 0 \\ A^1 & 0 \end{smallmatrix}\right)$ generated by $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is also in ρ. But $\left(\begin{smallmatrix} A & 0 \\ A & 0 \end{smallmatrix}\right)$ is a left ideal of $\left(\begin{smallmatrix} A^1 & A^1 \\ A^1 & A^1 \end{smallmatrix}\right)$ so, since ρ is left strong, it follows that the ideal $\left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$ of $\left(\begin{smallmatrix} A^1 & A^1 \\ A^1 & A^1 \end{smallmatrix}\right)$ generated by $\left(\begin{smallmatrix} A & 0 \\ A & 0 \end{smallmatrix}\right)$ is also in ρ. Now, since $M_2(A) \in \rho \subseteq \alpha$, our assumption implies that $A[x] \in \gamma$, which means that $A \in \gamma_x$. Thus $\rho \subseteq \gamma_x$. So γ_x is indeed the largest left and right strong radical contained in α. □

Our next result gives a necessary and sufficient condition for the Koethe problem to have a positive solution.

Corollary 2.18. *A radical $\mathcal{N}$ is left and right strong if and only if $\mathcal{N} = \mathcal{J}_x$.*

Proof. First we will show that $\mathcal{J}$ and $\mathcal{N}$ satisfy the assumptions of Theorem 2.17. It is well known [10] that $\mathcal{J}$ is left and right strong and Example 2.6 shows that $\mathcal{J}$ is $\mathcal{N}$-like. Moreover, $\mathcal{J}$ satisfies condition (z) since $\beta \subseteq \mathcal{J}$. Let $M_2(A) \in \mathcal{N}$. Then it follows from [10, Proof of Theorem 4.9.13] that $M_n(A) \in \mathcal{N}$ for every $n = 1, 2, \ldots$. But this, in view of [13], implies that $A[x] \in \mathcal{J}$. Hence $\mathcal{J}$ and $\mathcal{N}$ satisfy the assumptions of Theorem 2.17.

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So, taking $\alpha = \mathcal{N}$ and $\gamma = \mathcal{J}$ in Theorem 2.17, we can conclude that $\mathcal{J}_x$ is the largest left and right strong radical contained in $\mathcal{N}$. Thus, since $\mathcal{J}_x \subseteq \mathcal{N}$, it follows that, if $\mathcal{N}$ is left and right strong, then $\mathcal{N} = \mathcal{J}_x$. Conversely, if $\mathcal{N} = \mathcal{J}_x$, then $\mathcal{N}$ is left and right strong since, by [21], so is $\mathcal{J}_x$ as $\mathcal{J}$ is a left and right strong radical which satisfies condition (z). □

It is well known [1, 17] that the collection $\mathbb{L}$ of all radicals forms a complete lattice with respect to inclusion of radical classes, where the meet and the join of a family of radicals γ_i, $i \in I$, are defined by $\wedge_{i\in I}\gamma_i = \cap_{i\in I}\gamma_i$ and $\vee_{i\in I}\gamma_i = \mathcal{L}(\cup_{i\in I}\gamma_i)$, respectively. We will now consider some sublattices of $\mathbb{L}$ that consist of α-like radicals.

Proposition 2.19. *For any radical α, the collection $\mathbb{L}_\alpha$ of all α-like radicals is a complete sublattice of the lattice $\mathbb{L}$ and $\phi = \mathcal{U}(\{\text{all rings}\})$ is its smallest element.*

Proof. Every ring is in $\mathcal{S}(\phi)$—in particular, $A[x] \in \mathcal{S}(\phi)$ for every $A \in \mathcal{S}(\alpha)$—which shows that $\phi \in \mathbb{L}_\alpha$. Clearly ϕ is the smallest element of $\mathbb{L}_\alpha$.

Let γ_i, $i \in I$, be a family of α-like radicals.

Let $\Gamma = \vee_{i\in I}\gamma_i$. We will show that Γ is α-like. Let $A \in \mathcal{S}\alpha$ and suppose that $A[x] \notin \mathcal{S}\Gamma$. Then $\Gamma(A[x]) \neq 0$. Hence there exists a nonzero subring I_1 such that $I_1 \trianglelefteq \cdots \trianglelefteq I_n = A[x]$ and I_1 is a homomorphic image of a ring $B \in \gamma_i$ for some i. Therefore $I_1 \in \gamma_i$ and, since semisimple classes are hereditary, it follows that $\gamma_i(A[x]) \neq 0$. On the other hand, since γ_i is α-like for each $i \in I$, $\gamma_i(A[x]) = 0$, a contradiction. It follows that $\Gamma(A[x]) = 0$ for every $A \in \mathcal{S}\alpha$, which means that Γ is α-like.

Let $\Delta = \wedge_{i\in I}\gamma_i$. To see that Δ is also α-like observe that, since $\Delta \subseteq \gamma_i$ for all $i \in I$ and since each γ_i is α-like, it follows that, for every $A \in \mathcal{S}\alpha$, $\Delta(A[x]) \subseteq \gamma_i(A[x]) = 0$. □

Lemma 2.20. *The trivial radical $\gamma = \{\text{all rings}\}$ is not α-like for any proper radical α.*

Proof. Suppose that $\gamma = \{\text{all rings}\}$ is α-like for some proper radical α. Then, since α is proper, there exists $0 \neq A \in S\alpha$. But then, since γ is α-like, it follows that $0 \neq A[x] \notin \gamma$. On the other hand, $A[x] \in \gamma$ since $\gamma = \{\text{all rings}\}$ and we have a contradiction which ends the proof. □

Proposition 2.19 and Lemma 2.20 imply the following corollary.

Corollary 2.21. *For any proper radical α, $\gamma = \vee\{\gamma_i : \gamma_i \in \mathbb{L}_\alpha\}$ is a unique largest proper α-like radical.*

Proposition 2.22. *If $\alpha \supsetneqq \beta$ is a radical with the Amitsur property, then the lattice $\mathbb{L}_\alpha$ is not atomic.*

Proof. Let $\gamma \subseteq \beta$ be a radical that does not contain atoms of the lattice $\mathbb{L}$. Such a radical was constructed in [2] by Beidar. Then, as $\beta \subseteq \alpha$, we have $\gamma \subseteq \alpha$ and, since α is a radical with the Amitsur property, it follows from Corollary 2.5 that γ is α-like. Suppose that γ contains some atom σ of the lattice $\mathbb{L}_\alpha$. Then σ is not an atom of the lattice $\mathbb{L}$ and therefore there exists a radical $\rho \neq \{0\}$ such that $\rho \subsetneqq \sigma$. But, since $\sigma \subseteq \gamma \subseteq \alpha$, it follows that $\rho \subseteq \alpha$. So, since α is a radical with the Amitsur

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property, Corollary 2.5 implies that ρ is α-like, which is impossible as σ is an atom of $\mathbb{L}_\alpha$. Thus γ does not contain atoms of $\mathbb{L}_\alpha$, which shows that the lattice $\mathbb{L}_\alpha$ is not atomic. □

PROPOSITION 2.23. *The collection $\mathbb{L}_{h\alpha}$ of all hereditary α-like radicals is a complete atomic sublattice of the lattice $\mathbb{L}$. Any atom of $\mathbb{L}_{h\alpha}$ is of the form $\mathcal{L}(\{\mathbb{Z}_p^0\})$ or $\mathcal{L}(\{A\})$, where $\mathbb{Z}_p^0$ is a zero-ring on a cyclic additive group $\mathbb{Z}_p$ of prime order p and A is a simple idempotent ring.*

PROOF. Let γ_i, $i \in I$, be a family of hereditary α-like radicals. It follows from [17] and Proposition 2.19 that $\wedge_{i\in I}\gamma_i$ and $\vee_{i\in I}\gamma_i$ are hereditary and α-like radicals. Hence $\mathbb{L}_{h\alpha}$ is a complete sublattice of the lattice $\mathbb{L}$.

Let $\{0\} \neq \gamma \in \mathbb{L}_{h\alpha}$. Then, since γ is hereditary, it follows from [17] that γ contains $\mathbb{Z}_p^0$ for some prime p or a nonzero simple idempotent ring A. Then $\gamma \supseteq \rho$ where ρ denotes one of the following radicals $\mathcal{L}(\mathbb{Z}_p^0)$ or $\mathcal{L}(A)$ and, since γ is α-like, it follows, by Lemma 2.7, that ρ is α-like. Moreover, since ρ is also a hereditary radical, it follows that $\rho \in \mathbb{L}_{h\alpha}$. Since ρ is an atom of the lattice of all hereditary radicals [17], ρ is also an atom of the lattice $\mathbb{L}_{h\alpha}$, which ends the proof. □

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