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THE ETA INVARIANT AND EQUIVARIANT SPIN^C BORDISM FOR SPHERICAL SPACE FORM GROUPS

PETER B. GILKEY

0. Introduction. A finite group G is a spherical space form group if it admits a fixed point free representation $\tau:G \rightarrow U(k)$ for some k; for the remainder of this paper, we assume G is such a group. The eta invariant of Atiyah et al [**2**] defines $\mathbf{Q}/\mathbf{Z}$ valued invariants of equivariant bordism. In [**6**], we showed the eta invariant completely detects the reduced equivariant unitary bordism groups $\tilde{\Omega}_*^U(BG)$ and completely detects all but the 2-primary part of the reduced equivariant Spin^C bordism groups $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$. The coefficient ring Ω_*^U is without torsion; all the torsion in $\Omega_*^{\mathrm{Spin}^C}$ is of order 2. The prime 2 plays a distinguished role in the discussion of equivariant Spin^C bordism and $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ is quite different from $\tilde{\Omega}_*^U(BG)$ at the prime 2. Let $\ker_*(\eta, G)$ denote the kernel of all eta invariants and let $\ker_*(SW, G)$ denote the kernel of the $\mathbf{Z}_2$-equivariant Stiefel-Whitney numbers (see Section 1 for details). Then:

THEOREM 0.1. *Let* $M \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$. *If* $M \in \ker_*(\eta, G) \cap \ker_*(SW, G)$, $M = 0$.

It is worth explaining the use of the word "equivariant" in this context. A G-structure on M is equivalent to a principal G-bundle $G \mapsto P \mapsto M$. This gives a free G-action on P preserving the Spin^C structure. Conversely, given such an action of G on P, we can form the quotient $M = P/G$. The eta invariant on M can be computed equivariantly in terms of the G-eta invariant on P.

If A is an Abelian group, let $A_{(p)}$ be the p-primary part. If p is odd, the eta invariant alone completely detects $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(p)}$; at the prime 2, we also need the equivariant Stiefel-Whitney numbers to detect manifolds arising from $\mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})$. We will prove Theorem 0.1 first for p-groups and then use the transfer and induction maps to derive the general case. The p-Sylow subgroups of G are cyclic for p odd and if $p = 2$ are either cyclic or generalized quaternionic Q_v (see [**12**]). Theorem 0.1 at odd primes is proved in [**6**] so in this paper we will concentrate on $p = 2$.

We can determine the additive structure of the 2-primary part of these bordism groups analytically. Let $bu_*(BG)$ be the connective K-theory groups. In the fourth section, we will define an embedding of $bu_*(BG)$

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into $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ using suitably chosen spherical space forms. Let CP^j be complex projective space given the cannonical Spin^C structure inherited from the natural unitary structure. Then

$$\{\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)/\text{torsion}\} \otimes \mathbf{Z}_2 = \mathbf{Z}_2[CP^1, CP^2, CP^4, \dots, CP^{2k}, \dots]$$

(see [**11**]) Cartesian product makes $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ into an $\Omega_*^{\mathrm{Spin}^C}$ module. Let

$$X^{4k} = CP^{2k} - (CP^1)^k.$$

THEOREM 0.2. *Let G be a spherical space form group. Then*:

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} \cong bu_*(BG)_{(2)} \otimes \mathbf{Z}[X^4, \dots, X^{4k}, \dots] \oplus \ker_*(\eta, G)$$
$$\ker_*(\eta, G) \cong \tilde{H}_*(BG, \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

Remark. This analytic splitting is functorial and is preserved by transfer and induction.

A byproduct of our discussion will be an expression for the connective K-theory $bu_*(BG)$ in terms of the representation theory. Let $R(G)$ be the group representation ring of G and let $R_0(G)$ be the augmentation ideal. It is well known that

$$bu_{2k-1}(B\mathbf{Z}_n) = R_0(\mathbf{Z}_n)/R_0(\mathbf{Z}_n)^{k+1} = \tilde{K}(S^{2k+1}/\mathbf{Z}_n).$$

However, if $G = Q_\nu$, it does not seem as well known:

THEOREM 0.3. *Let $I = (\tau - 2)R(Q_\nu)$ for $\tau:Q_\nu \mapsto SU(2)$ fixed point free.*

$$bu_{4k-5}(BQ_\nu) \cong I/I^k \quad \textit{and}$$
$$bu_{4k-3}(BQ_\nu) \cong R_0(Q_\nu)/I^k \cong \tilde{K}(S^{4k-1}\mathrm{S}^{4k-1}/\tau(Q_\nu)).$$

In the first section, we discuss the eta invariant and in the second section we discuss the Smith homomorphism to establish notation and to review various results we shall need. The $E_{p,q}^2$ term of the bordism spectral sequence is $\tilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})$. The terms arising from $\tilde{H}_{\mathrm{odd}}(BG; \Omega_q^{\mathrm{Spin}^C})$ can be described by products of spherical space forms with appropriate Spin^C manifolds. The terms arising from $\tilde{H}_{\mathrm{even}}(BG; \mathrm{Tor}(\Omega_q^{\mathrm{Spin}^C}))$ are more difficult to describe and are the obstacle to using standard methods to show the bordism spectral sequence collapses. In the third section, we will construct manifolds representing these classes and show the bordism spectral sequence collapses. We will then prove Theorem 0.1 by induction on the dimension using the Smith homomorphism. In the fourth section, we will define $A_*(G) \subseteq \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ as the span of suitably chosen spherical space forms; the $A_*(G)$ will be invariant under transfer and induction and under the Smith homomorphism. We will use the pairing provided by the eta invariant to relate the $A_*(G)$ to the representation theory. We will show

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} \cong A_*(G) \otimes \mathbf{Z}[X^4, X^8, \dots] \oplus \ker_*(\eta, G) \quad \text{and}$$

$$\ker_*(\eta, G) \cong \tilde{H}_*(BG; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

In [**4**], we constructed a topological splitting of $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)}$ using the Anderson, Brown, and Peterson [**1**] splitting of the spectrum $\underline{\mathrm{MSpin}^C}$. We compare these two formulas to show $A_*(G) \cong bu_*(BG)$ and complete the proof of Theorems 0.2 and 0.3.

This paper completes the work begun in [**3**, **6**] on the relationship between the eta invariant and equivariant bordism. It is a pleasure to acknowledge the contributions of A. Bahri at many points to this work.

1. The eta invariant. Let Spin(m) be the universal cover group of the special orthogonal group for $m > 2$; define Spin(m) in terms of Clifford algebras for $m = 1, 2$. Let

$$\mathrm{Spin}^C(m) = \mathrm{Spin}(m) \times U(1)/\mathbf{Z}_2$$

and let $\gamma(g, \lambda) = \lambda^2$ define a representation in $U(1)$. The forgetful homomorphism $U(m) \mapsto SO(2m)$ lifts to $\mathrm{Spin}^C(2m)$ and the determinant representation lifts to γ. Let

$$W^* = H^*(BO; \mathbf{Z}_2) = \mathbf{Z}_2[w_j]$$

be the algebra of Stiefel-Whitney classes. A real vector bundle V has a Spin^C structure if $w_1(V) = 0$ and if $w_2(V)$ lifts to an integral class. M is a Spin^C manifold if the tangent space $T(M)$ admits a Spin^C structure. Let $\Omega_k^{\mathrm{Spin}^C}$ be the bordism group of compact smooth k-dimensional manifolds modulo the subgroup which bound; Cartesian product makes $\Omega_k^{\mathrm{Spin}^C}$ into a graded ring. Evaluation on $T(M)$ defines a natural pairing

$$W^k \otimes \Omega_k^{\mathrm{Spin}^C} \mapsto \mathbf{Z}_2.$$

Let $\ker_*(SW) \subseteq \Omega_*^{\mathrm{Spin}^C}$ be the kernel of this pairing; $\ker_*(SW)$ is also the kernel of the forgetful functor from $\Omega_*^{\mathrm{Spin}^C}$ to Ω_*^0.

Let $c_1(M)$ be the Chern class of the bundle defined by γ over M and let $p_k(M)$ be the Pontrjagin classes. The characteristic numbers formed from the $\{c_1, p_k\}$ are the Chern/Pontrjagin numbers of M and are bordism invariants. Let Λ^k be the complexified exterior representations of $SO(m)$; extend the Λ^k to $\mathrm{Spin}^C(m)$ using the natural projection. Let

$$R(\mathrm{Spin}^C) = \mathbf{Z}[\gamma, \Lambda^k]$$

be the free polynomial algebra on these variables; $R(\mathrm{Spin}^C)$ is not the full representation ring of Spin^C. If $\theta \in R(\mathrm{Spin}^C)$ and if $M \in \Omega_{\mathrm{even}}^{\mathrm{Spin}^C}$, let index$(\theta, M) \in \mathbf{Z}$ be the index of the Spin^C complex over M with coefficients in the virtual bundle defined by θ; set index$(\theta, M) = 0$ if M is odd dimensional. The index is a bordism invariant vanishing on torsion classes. Let $\ker_*(\text{index}, \text{Spin}^C)$ be the kernel of these invariants. We refer to [**1**] and [**11**] for the proof of

THEOREM 1.1. *Let* $P_* = \mathbf{Z}[CP^1, CP^2, \dots, CP^{2k}, \dots]$.

(a) $\Omega_*^{\text{Spin}^C}$ *is a commutative ring. All the torsion has order* 2. *The Stiefel-Whitney numbers and rational Chern/Pontrjagin numbers completely detect* $\Omega_*^{\text{Spin}^C} \cdot P_*$ *embeds in* $\Omega_*^{\text{Spin}^C}$ *and*

$$\Omega_*^{\text{Spin}^C} \otimes \mathbf{Z}_2 = P_* \otimes \mathbf{Z}_2 \oplus \text{Tor}(\Omega_*^{\text{Spin}^C}) \times \ker(\text{index}, \text{Spin}^C) \cap \ker(SW) = 0.$$

(b) *If* $\text{index}(\theta, M) \equiv 0(n)\ \forall\ \theta \in R(\text{Spin}^C)$, *then*

$$M \in n\Omega_*^{\text{Spin}^C} + \text{Tor}(\Omega_*^{\text{Spin}^C}).$$

Remark. (b) is the Hattori-Stong theorem; it shows that modulo torsion, all relations among Spin^C characteristic classes are given by the index theorem. In particular, if $M \in P_*$, then

$$\{M \in 2^w P_*\} \Leftrightarrow \{\text{index}(\theta, M) \equiv 0(2^w)\ \forall\ \theta \in R(\text{Spin}^C)\}.$$

Theorem 0.1 of this paper is the generalization of the Hattori-Stong theorem to equivariant bordism.

Let $\Omega_*^{\text{Spin}^C}(BG)$ be the equivariant bordism groups. Decompose

$$\Omega_*^{\text{Spin}^C}(BG) = \tilde{\Omega}_*^{\text{Spin}^C}(BG) \oplus \Omega_*^{\text{Spin}^C}.$$

Cartesian product makes $\Omega_*^{\text{Spin}^C}(BG)$ and $\tilde{\Omega}_*^{\text{Spin}^C}(BG)$ into $\Omega_*^{\text{Spin}^C}$ modules. Let

$$W^*(BG) = H^*(BG; \mathbf{Z}_2) \otimes H^*(BO; \mathbf{Z}_2)$$

be the algebra of G equivariant Stiefel-Whitney classes and let

$$\ker_*(SW, G) \subseteq \tilde{\Omega}_*^{\text{Spin}^C}(BG)$$

be the kernel of the natural pairing

$$W^m(BG) \otimes \tilde{\Omega}_m^{\text{Spin}^C}(BG) \mapsto \mathbf{Z}_2.$$

Since $\text{Ker}_*(SW, G) \neq 0$ in general, the equivariant Stiefel-Whitney numbers do not suffice to detect $\tilde{\Omega}_*^{\text{Spin}^C}(BG)$ even if G is a 2-group.

The eta invariant is an analytic invariant of equivariant bordism. Let M be a smooth compact Riemannian manifold of dimension m without boundary and let D be a self-adjoint elliptic differential operator on M. If $\lambda \in \mathbf{R}$, let $E(D, \lambda)$ be the eigenspace of D corresponding to λ and define

$$\eta(s, D) = \{\dim E(D, 0) + \Sigma_\lambda \dim E(D, \lambda)\ \text{sign}(\lambda)\ |\lambda|^{-s}\}/2$$

as a measure of the spectral asymmetry of D. The series converges to define a holomorphic function of s for $\text{Re}(s) \gg 0$. It has a meromorphic extension to $\mathbf{C}$ with isolated simple poles. The value at 0 is regular. Let

$$\eta(D) = \eta(0, D) \in \mathbf{R}/\mathbf{Z}$$

be a measure of the spectral asymmetry of D. If M is an odd dimensional Spin^C manifold, let $N = M \times [0, \infty)$. Let Q be the operator of the Spin^C complex on N and decompose $Q = \partial/\partial t + D$ for $t \in [0, \infty)$. D is the tangential operator of the Spin^C complex and is a self-adjoint elliptic first order differential operator over M. If

$$\theta \in R_0(G) \otimes R(\mathrm{Spin}^C),$$

let $\eta(\theta, M) \in \mathbf{R}/\mathbf{Z}$ be the eta invariant of D with coefficients in the bundle defined by θ. Extend η to be zero if M is even dimensional when the operator D is not defined. Since we consider representations of G of virtual dimension 0, the local terms in the Atiyah-Patodi-Singer index theorem vanish and η extends to an invariant in bordism; see [**2**, **8**] for details:

LEMMA 1.2. $\eta:R_0(G) \otimes R(\mathrm{Spin}^C) \otimes \Omega_*^{\mathrm{Spin}^C}(BG) \mapsto \mathbf{Q}/\mathbf{Z}$.

Remark. $\eta(\theta, M) = 0\ \forall\ \theta$ if $M \in \Omega_*^{\mathrm{Spin}^C}$ since the G structure is trivial, but as we will be concentrating on the reduced bordism groups for the most part, we shall usually restrict η to $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ and define $\ker_*(\eta, G) \subseteq \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ as the kernel of this pairing.

The eta invariant has several functorial properties we describe in the following three lemmas. First, it behaves nicely with respect to products and is closely related to the ordinary index. Let

$$s(\gamma) = \gamma \otimes \gamma \quad \text{and} \quad s(\Lambda^k) = \Sigma_{i+j=k}\Lambda^i \otimes \Lambda^j$$

define a coproduct on $R(\mathrm{Spin}^C)$. If $\theta \in R_0(G) \otimes R(\mathrm{Spin}^C)$, decompose

$$(1 \otimes s)(\theta) = \Sigma_i a_i \otimes b_i$$

for $a_i \in R_0(G) \otimes R(\mathrm{Spin}^C)$ and $b_i \in R(\mathrm{Spin}^C)$. Then

$$\theta(M \times N) = \Sigma_i a_i(M) \otimes b_i(N)$$

for $M \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ and $N \in \Omega_*^{\mathrm{Spin}^C}$. We refer to [**8**] for

LEMMA 1.3. *With the notation established above,*

$$\eta(\theta, M \times N) = \Sigma_i \eta(a_i, M) \cdot \mathrm{index}(b_i, N).$$

Let $C(G)$ be the space of complex class functions on G; the map $\rho \mapsto \mathrm{Tr}(\rho)$ embeds $R(G)$ in $C(G)$ with $R(G) \otimes \mathbf{C} = C(G)$. If f, $g \in C(G)$, let

$$(f, g)_G = \{\Sigma_{x \in G} f(x)g(x)\}/|G|$$

define a non-degenerate symmetric associative pairing which is integer valued on $R(G)$. If $H \subseteq G$, then induction and transfer define maps in homology, cohomology, and equivariant bordism compatible with the bordism spectral sequence (see [**5**] and [**10**]):

$$i_*:H_*(BH;\ -) \mapsto H_*(BG;\ -) \quad t_*:H_*(BG;\ -) \mapsto H_*(BH;\ -)$$
$$i^*:H^*(BG;\ -) \mapsto H^*(BH;\ -) \quad t^*:H^*(BH;\ -) \mapsto H^*(BG;\ -)$$
$$i:\tilde{\Omega}_*^{\mathrm{Spin}^C}(BH) \mapsto \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG) \qquad t:\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG) \mapsto \tilde{\Omega}_*^{\mathrm{Spin}^C}(BH).$$

Let restriction $r:R(G) \mapsto R(H)$ make $R(H)$ into an $R(G)$ module and let $\mathrm{ind}:R(H) \mapsto R(G)$ and $\mathrm{ind}:C(H) \mapsto C(G)$ be the $R(G)$ module morphism

$$(r(f), g)_H = (f, \mathrm{ind}(g))_G \quad \forall f \in C(G) \quad \forall g \in C(H)$$

given by Frobenius reciprocity. With respect to the pairing defined by the eta invariant, ind is the dual of t and r is the dual of i; with respect to the pairing defining the equivariant Stiefel-Whitney numbers, i^* is the dual of i_* and t^* is the dual of t_*.

LEMMA 1.4. *Let* $H \subset G$. *Then*:

(a) *If* $M \in \Omega_*^{\mathrm{Spin}^C}(BH)$, $\theta \in R_0(G) \otimes R(\mathrm{Spin}^C)$, *and* $x \in W^*(BG)$, *then*

$$\eta(\theta, i(M)) = \eta((r \otimes 1)(\theta), M) \quad \textit{and}$$
$$(x, i(M)) = ((i^* \otimes 1)(x), M).$$

(b) *If* $M \in \Omega_*^{\mathrm{Spin}^C}(BG)$, $\theta \in R_0(H) \otimes R(\mathrm{Spin}^C)$, *and* $x \in W^*(BH)$, *then*

$$\eta(\theta, t(M)) = \eta((\mathrm{ind} \otimes 1)(\theta), M) \quad \textit{and}$$
$$(x, t(M)) = ((t^* \otimes 1)(x), M).$$

(c) $i:\ker_*(SW, H) \cap \ker_*(\eta, H) \mapsto \ker_*(SW, G) \cap \ker_*(\eta, G)$ *and*

$t:\ker_*(SW, G) \cap \ker_*(\eta, G) \mapsto \ker_*(SW, H) \cap \ker_*(\eta, H)$.

Proof. The assertions concerning equivariant Stiefel-Whitney classes follow by duality; we refer to [**6**] for the assertions regarding η; (c) follows from (a, b).

If M is a compact Riemannian manifold of constant curvature 1, then M is a spherical space form. If $\tau:G \mapsto U(k)$ is a fixed point free representation of G, let

$$N(G, \tau) = S^{2k-1}/\tau(G)$$

be the resulting spherical space form; if $k > 1$ then

$$\pi_1(N(G, \tau)) = G.$$

All odd dimensional spherical space forms arise in this way; the only even dimensional spherical space forms are the sphere S^{2k} and real projective space RP^{2k}. $N(G, \tau)$ inherits a natural stable unitary and Spin^C

structure. Since $N(G, \tau)$ is odd dimensional, it bounds in Ω_*^U and hence in $\Omega_*^{\mathrm{Spin}^C}$ so

$$N(G, \tau) \in \widetilde{\Omega}_*^{\mathrm{Spin}^C}(BG).$$

The eta invariant for spherical space forms is given by Dedekind sums. If $\tau:G \mapsto U(k)$ is a fixed point free, define $\alpha, \beta \in C(G)$ by

$$\alpha(\tau) = \det(\tau - I)/\det(\tau) \in R_0(G)$$

$$\beta(\tau)(1) = 0 \quad \text{and} \quad \beta(\tau)(x) = \alpha(\tau)(x)^{-1} \quad \text{for } x \neq 1;$$

α and β are multiplicative with respect to direct sums. If $\theta \in R_0(G)$, $\alpha\beta\theta = \theta$. It follows from the arguments of Atiyah et al [**2**, see II-2.9] that

LEMMA 1.5. *If $\tau:G \mapsto U(k)$ is fixed point free and $\theta \in R_0(G)$, then*

$$\eta(\theta, N(G, \tau)) = (\theta, \beta(\tau))_G \in \mathbf{R}/\mathbf{Z}.$$

Remark. If $M = N(G, \tau)$ and $\theta \in R_0(G) \otimes R(\mathrm{Spin}^C)$, we can find $\psi \in R_0(G)$ with $\theta(M) = \psi(M)$. Therefore $\eta(\theta, M) = \eta(\psi, M)$ is given by Dedekind sums. Let $S_*(G)$ be the $\Omega_*^{\mathrm{Spin}^C}$ submodule of $\Omega_*^{\mathrm{Spin}^C}(BG)$ generated by $\{iN(H, \tau)\}$ as H ranges over the subgroups of G and τ ranges over the fixed point free representations of H. $S_*(G)$ contains the image of $\widetilde{\Omega}_*^U(BG)$ under the forgetful homomorphism. The Atiyah-Singer index theorem and Lemmas 1.3, 1.4, and 1.5 enable us to compute η on $S_*(G)$ combinatorially.

There is a close relationship between the eta and K-theory which we will need in the proof of Theorem 0.2. We refer to [7] for

LEMMA 1.6. *Let $\tau:G \mapsto U(k)$ be fixed point free and let $I = \alpha(\tau)R(G)$.*
(a) *If $\theta \in R_0(G)$, then $\theta \in I$ if and only if*

$$\eta(\rho\theta, N(G, \tau)) = 0 \ \forall \ \rho \in R_0(G).$$

(b) *$\widetilde{K}(N(G, \tau)) = R_0(G)/I$. $\widetilde{K}(N(G, \tau_1)) \cong \widetilde{K}(N(G, \tau_2))$ if* $\dim(\tau_1) = \dim(\tau_2)$.

Remark. If $\theta \in R_0(Q_\nu)$, then the map

$$\rho \mapsto \eta(\theta\rho, N(G, \tau))$$

extends to a map

$$\widetilde{K}(N(G, \tau)) = R_0(Q_\nu)/I \mapsto \mathbf{Q}/\mathbf{Z}$$

and conversely all maps

$$\widetilde{K}(N(G, \tau)) \mapsto \mathbf{Q}/\mathbf{Z}$$

arise in this way. This gives a perfect pairing

$$\eta:\tilde{K}(N(G, \tau)) \otimes \tilde{K}(N(G, \tau)) \mapsto \mathbf{Q}/\mathbf{Z}.$$

2. The Smith homomorphism. If G is a spherical space form group, then G has periodic homology and the $\tilde{H}_*(BG; \mathbf{Z})$ are all finite groups. The $E^2_{p,q}$ term in the bordism spectral sequence is $\tilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})$. Consequently

LEMMA 2.1. *If G is a spherical space form group, then* $|\tilde{\Omega}_m^{\mathrm{Spin}^C}(BG)|$ *divides* $|\oplus_{p+q=m}\tilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})|$.

Remark. In fact the bordism spectral sequence degenerates so this estimate is sharp as we shall see in Section 3.

Let $\tau:G \mapsto U(k)$ be fixed point free. Embed $N(G; j\tau)$ in $N(G; (j+1)\tau)$ using the first jk complex coordinates. The classifying space BG is the limit of the $N(G; j\tau)$ under these inclusions. Let $M \in \tilde{\Omega}_m^{\mathrm{Spin}^C}(BG)$ and let $f:M \mapsto N(G; j\tau)$ be the classifying map for j large. Make f transverse to $N(g; (j - 1)\tau)$ and let

$$\Delta(M) = f^{-1}(N(G; (j - 1)\tau));$$

Δ depends of course on the particular τ chosen. The normal bundle of $\Delta(M)$ in M corresponds to the complex representation τ and inherits a natural Spin^C structure. Δ extends as a map in bordism from $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ to the unreduced bordism group $\Omega_{*-2k}^{\mathrm{Spin}^C}(BG)$. This is the crucial difference between unitary and Spin^C bordism. Although M bounds, $\Delta(M)$ need not bound. Since $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$ is a finite group, Δ takes values in

$$\tilde{\Omega}_{*-2k}^{\mathrm{Spin}^C}(BG) \oplus \mathrm{Tor}(\Omega_{*-2k}^{\mathrm{Spin}^C})$$

and is an $\Omega_*^{\mathrm{Spin}^C}$ module morphism. We extend Δ as zero on the direct summand

$$\Omega_*^{\mathrm{Spin}^C} \subseteq \Omega_*^{\mathrm{Spin}^C}(BG)$$

since the G structure is trivial. Let $[M] \in H_m(M; \mathbf{Z})$ be the fundamental class and let

$$\mu(M) = f_*[M] \in H_m(BG; \mathbf{Z}).$$

μ commutes with transfer and induction; $\mu(\Delta M)$ is given by cap product by the Euler class $c_k(\tau)$ with $\mu(M)$. We refer to [**5**, **6**] for:

LEMMA 2.2. *Let* $\tau_i:G \mapsto U(k)$ *be fixed point free define* Δ_i.

(a) *If* $H \subseteq G$, *then* Δ_i *commutes with induction and transfer.*
(b) $\Delta_1(N(G, \tau_1 \oplus \tau_2)) = N(G, \tau_2)$.
(c) *If* Δ_3 *corresponds to* $\tau_1 \oplus \tau_2$, *then* $\Delta_3 = \Delta_1\Delta_2 = \Delta_2\Delta_1$.

The equivariant Stiefel-Whitney classes are well behaved with respect to the Smith homomorphism.

LEMMA 2.3. *Let* $\tau:G \mapsto U(k)$ *be fixed point free and let* $M \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)$. *Let* $N = \Delta(M)$ *and let* $j:N \mapsto M$ *be the inclusion.*

(a) *If* $\theta \in H^{*-2k}(M; \mathbf{Z}_2)$, *then* $(\theta \cdot c_k(\tau))(M) = (j^*\theta)(N)$.

(b) $\Delta(\ker_*(SW, G)) \subseteq \ker_{*-2k}(SW, G) \subseteq \tilde{\Omega}_{*-2k}^{\mathrm{Spin}^C}(BG)$.

Proof. (a) is true since N is the Poincaré dual of the Euler class $c_k(\tau)$. If $x \in W^{*-2k}(BG)$, let

$$\theta = x(M) \in H^{*-2k}(M; \mathbf{Z}_2).$$

The normal bundle of N in M is given by τ so $j^*(w(M)) = w(N)c(\tau)$. Let

$$s(1 \otimes w_i) = \Sigma_{2j+k=i} c_j(\tau) \otimes w_k \quad \text{and} \quad s(x \otimes 1) = x \otimes 1$$

define an algebra isomorphism of $W^*(BG)$. Then $j^*(x) = (s(x))(N)$ so

$$(s(x))(N) = (x \cdot c_k(\tau))(M) = 0$$

by (a). Since s is an isomorphism, all the equivariant Stiefel-Whitney numbers of N vanish. Since N is a torsion class, all the ordinary Chern/Pontrjagin numbers of N vanish so N bounds in $\Omega_*^{\mathrm{Spin}^C}$ by Theorem 1.1 and hence

$$N \in \tilde{\Omega}_{*-2k}^{\mathrm{Spin}^C}(BG).$$

Let $\rho:G \mapsto \mathbf{Z}_2$ be a real representation. If $M \in \tilde{\Omega}_k^{\mathrm{Spin}^C}(BG)$, we use ρ to give M a $\mathbf{Z}_2$ structure. Let $f_\rho:M \mapsto RP^j$ be the classifying map. Make f_ρ transverse to RP^{j-2} and let

$$\Delta_\rho(M) = f_\rho^{-1}(RP^{j-2})$$

with the inherited G-structure and Spin^C structure to define an auxiliary Smith homomorphism

$$\Delta_\rho:\tilde{\Omega}_k^{\mathrm{Spin}^C}(BG) \mapsto \tilde{\Omega}_{k-2}^{\mathrm{Spin}^C}(BG) \oplus \mathrm{Tor}(\Omega_{k-2}^{\mathrm{Spin}^C}).$$

LEMMA 2.4. *Let* $\tau:G \mapsto U(k)$ *be fixed point free and let* $\rho:G \mapsto \mathbf{Z}_2$.

(a) Δ *and* Δ_ρ *commute.*

(b) *If* $M \in \ker_k(SW, G)$, *then*

$$\Delta_\rho(M) \in \ker_{k-2}(SW, G) \subseteq \tilde{\Omega}_{k-2}^{\mathrm{Spin}^C}(BG).$$

Proof. Let $f:M \mapsto N(G, j\tau)$ be the classifying map. Let

$$g:N(G, j\tau) \mapsto RP^\nu$$

be the classifying map for the $\mathbf{Z}_2$ structure defined by ρ. Make g transverse to $RP^{\nu-2}$ and let

$$X = g^{-1}(RP^{\nu-2}).$$

Choose the embedding of $N(G, (j - 1)\tau)$ in $N(G, j\tau)$ transverse to X and let

$$Y = X \cap N(G, (j - 1)\tau).$$

Make f transverse to X, Y, and $N(G, (j - 1)\tau)$. Then

$$\Delta(M) = f^{-1}(N(G, (j - 1)\tau)).$$

$gf:\Delta(M) \mapsto RP^{\nu}$ is transverse to $RP^{\nu-2}$ so

$$\Delta_{\rho}\Delta(M) = (gf)^{-1}(RP^{\nu-2}) \cap \Delta(M) = f^{-1}(Y).$$

Similarly

$$\Delta_{\rho}(M) = f^{-1}(X)$$

and $f(\Delta_{\rho}(M))$ is transverse to $N(G, (j-1)\tau)$ so

$$\Delta\Delta_{\rho}(M) = f^{-1}(Y)$$

which proves (a). The proof of (b) is the same as that given for Lemma 2.3 and is omitted.

We now specialize to 2-groups. Identify

$$\mathbf{Z}_n = \{\lambda \in \mathbf{C}:\lambda^n = 1\}$$

and let $\rho_s(\lambda) = \lambda^n$ be the irreducible representations of $\mathbf{Z}_n$ where s is defined mod n. If n is even, $\rho_{n/2}:\mathbf{Z}_n \mapsto \mathbf{Z}_2$ is a real representation. The following is well known.

LEMMA 2.5. *Let* $n = 2^w$.

(a) $\tilde{H}_{2j}(B\mathbf{Z}_n; \mathbf{Z}) = 0$. *If* $\tau:\mathbf{Z}_n \mapsto U(k)$ *is fixed point free, then* $\mu(N(\mathbf{Z}_n; \tau))$ *spans* $\tilde{H}_{2k-1}(B\mathbf{Z}_n; \mathbf{Z}) = \mathbf{Z}_n$.

(b) $w_1(\rho_{n/2})$ *spans* $H^1(B\mathbf{Z}_n; \mathbf{Z}_2) = \mathbf{Z}_2$ *and* $c_1(\rho_1)$ *spans* $H^2(B\mathbf{Z}_n; \mathbf{Z}_2) = \mathbf{Z}_2$.

(c) *Cup product by* $c_1(\rho_1)$ *is an isomorphism from* $H^j(B\mathbf{Z}_n; \mathbf{Z}_2)$ *to* $H^{j+2}(B\mathbf{Z}_n; \mathbf{Z}_2)$.

We summarize the properties of $\tilde{\Omega}_*^{\mathrm{Spin}^C}(B\mathbf{Z}_n)$ which we need and refer to [**3**, Lemmas 3.3 and 3.4] for the proof.

LEMMA 2.6. *Let* $n = 2^w$ *and let* Δ_1 *correspond to* ρ_1. *Let* $N = \Delta_1(M)$.

(a) *Define an algebra isomorphism* t *of* $R_0(\mathbf{Z}_n) \otimes R(\mathrm{Spin}^C)$ *by*

$$t(\theta)(N) = \theta(M)|_N.$$

Then

$$\eta(t(\theta), N) = \eta(\theta \otimes \alpha(\rho_1), M).$$

(b) *If* $X \in \mathrm{Tor}(\Omega_{*-2k}^{\mathrm{Spin}^C})$ *and* $k > 0$, *then*

$$\exists\, M \in \ker_*(\eta, \mathbf{Z}_n) \subset \tilde{\Omega}_*^{\mathrm{Spin}^C}(B\mathbf{Z}_n)$$

so $2M = 0$ *and* $\Delta^k(M) = X$.

We define the generalized quaternionic groups following [**12**]:

$$Q_v = \langle x, y : x^{2u} = y^2, yxy^{-1} = x^{-1}, y^4 = 1 \rangle$$

for $u = 2^{v-2}$ and $v \geqq 2$.

Q_v is a finite group with $8u = 2^{v+1}$ elements; for example $Q_2 = \{\pm 1, \pm i, \pm j, \pm k\}$. Q_v has $2u + 3$ conjugacy classes represented by $\{1, x, \ldots, x^{2u}, y, xy\}$. There are four 1-dimensional representations of Q_v defined by:

$$\rho_0(x) = 1, \rho_x(x) = 1, \rho_y(x) = -1, \rho_{xy}(x) = -1$$

$$\rho_0(y) = 1, \rho_x(y) = -1, \rho_y(y) = 1, \rho_{xy}(y) = -1.$$

The 2-dimensional representations of Q_v are given by:

$$\tau_j(x) = \begin{bmatrix} e^{2\pi ij/4u} & 0 \\ 0 & e^{-2\pi ij/4u} \end{bmatrix} \quad \text{and} \quad \tau_j(y) = \begin{bmatrix} 0 & 1 \\ (-1)^j & 0 \end{bmatrix}$$

$\tau_0 = \rho_0 \oplus \rho_x$ and $\tau_{2u} = \rho_y \oplus \rho_{xy}$. The irreducible inequivalent unitary representations of Q_v are the $2u + 3$ representations $\{\rho_0, \rho_x, \rho_y, \rho_{xy}, \tau_1, \ldots, \tau_{2u-1}\}$. If j is odd, τ_j is fixed point free. Let $\tau = \tau_1$ and let Δ correspond to τ. If $z \in Q_v$, let H_z be the cyclic subgroup generated by z. The 3 maximal Abelian subgroups of Q_v up to conjugation are H_x, H_y, and H_{xy} and have orders $4u$, 4, and 4.

LEMMA 2.7. (a) $\tilde{H}_{2j}(BQ_v; \mathbf{Z}) = 0$. *Let*

$$\tau_z : H_z \mapsto U(2k - 1) \quad \textit{and} \quad \tau_q : Q_v \mapsto U(2k)$$

be fixed point free. $\{\mu(i_z N(H_z; \tau_z))\}_{z=y,xy}$ *spans*

$$\tilde{H}_{4k-3}(BQ_v; \mathbf{Z}) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and $\mu(N(G, \tau_q))$ *spans*

$$\tilde{H}_{4k-1}(BQ_v; \mathbf{Z}) = \mathbf{Z}_{|Q_v|}.$$

(b) $\{w_1(\rho_y), w_1(\rho_{xy})\}$ *spans*

$$H^1(BQ_v; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

$\{c_1(\rho_y), c_1(\rho_{xy})\}$ *spans*

$$H^2(BQ_v; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

and $c_2(\tau)$ *spans*

$$H^4(BQ_v; \mathbf{Z}_2) = \mathbf{Z}_2.$$

$H^3(BQ_v; \mathbf{Z}_2) = \mathbf{Z}_2$ *and* $i_z H^3(BQ_v; \mathbf{Z}_2) = 0$ *in* $H^3(BH_z; \mathbf{Z}_2)$.

(c) *Cup product by* $c_2(\tau_1)$ *is an isomorphism from* $H^j(BQ_v; \mathbf{Z}_2)$ *to* $H^{j+4}(BG; \mathbf{Z}_2)$.

Proof. The structure of the cohomology and homology groups is well known and is an easy exercise in characteristic classes and in the Gysin sequence so we omit details. To show

$$i_z: H^3(BQ_v; \mathbf{Z}_2) \mapsto H^3(BH_z; \mathbf{Z}_2)$$

is the zero map, we may assume H_z is maximal Abelian so $z = x, y$, or xy. We use Poincaré duality with $\mathbf{Z}_2$ coefficients on the 3-dimensional skeleton $N(Q_v, \tau)$ to choose

$$\theta_2 \in H^2(BQ_v; \mathbf{Z}_2)$$

so $w_1(\rho_z)\theta_2$ is the generator of $H^3(BQ_v; \mathbf{Z}_2)$. Since $i_z^*(w_1(\rho_z)) = 0$,

$$i_z^*(H^3(BQ_v; \mathbf{Z}_2)) = 0.$$

We will use the following lemma to reduce questions about representations of Q_v to similar questions about the cyclic groups. Let ind_z be induction from H_z to Q_v.

LEMMA 2.8. $R_0(Q_v) = \text{ind}_x R_0(H_x) + \text{ind}_y R_0(H_y) + \text{ind}_{xy} R_0(H_{xy})$.

Proof. Let

$$\tau_{\text{even}} = \Sigma_{0<s<2u, s\equiv 0(2)} \tau_s \quad \text{and} \quad \tau_{\text{odd}} = \Sigma_{0<s<2u, s\equiv 1(2)} \tau_s.$$

Then

$$\text{ind}_x \rho_s = \tau_s$$

$$\text{ind}_y \rho_0 = \tau_{\text{even}} + \rho_0 + \rho_y \qquad \text{ind}_{xy} \rho_0 = \tau_{\text{even}} + \rho_0 + \rho_{xy}$$

$$\text{ind}_y \rho_1 = \tau_{\text{odd}} \qquad \text{ind}_{xy} \rho_1 = \tau_{\text{odd}}$$

$$\text{ind}_y \rho_2 = \tau_{\text{even}} + \rho_x + \rho_{xy} \qquad \text{ind}_{xy} \rho_2 = \tau_{\text{even}} + \rho_x + \rho_Y$$

$$\rho_0 - \rho_x = \text{ind}_{xy}(\rho_0 - \rho_1) - \text{ind}_y(\rho_2 - \rho_1)$$

$$\rho_y - \rho_x = \text{ind}_y(\rho_0 - \rho_1) - \Sigma_{0\leqq j<2u, j\equiv 0(2)} \text{ind}_x(\rho_j - \rho_{j+1})$$

$$\rho_{xy} - \rho_x = \text{ind}_{xy}(\rho_0 - \rho_1) - \Sigma_{0\leqq j<2u, j\equiv 0(2)} \text{ind}_x(\rho_j - \rho_{j+1})$$

$$\tau_j - 2\rho_x = \text{ind}_x(\rho_j - \rho_0) + (\rho_0 - \rho_x).$$

We conclude this section by studying the behavior of the eta invariant with respect to the Smith homomorphism.

LEMMA 2.9. *Let G be a 2-group. Let $\tau: G \mapsto U(k)$ be fixed point free.*

(a) $\Delta: \ker_*(\eta, G) \mapsto \ker_*(\eta, G) \oplus \text{Tor}(\Omega_*^{\text{Spin}^C})$.

(b) *If $M \in \Omega_*^{\text{Spin}^C}$ and if $2M = 0$, then*

$$\eta(\theta, \Delta^{|G|}M) = 0 \ \forall \ \theta \in R_0(G) \otimes R(\text{Spin}^C).$$

Proof. If suffices to prove (a) if τ is irreducible. By composing with an outer automorphism, we can convert any τ to the standard choice. If G is cyclic, then (a) follows from Lemma 2.6 since t is an isomorphism. Let $M \in \ker_*(\eta, Q_v)$. By Lemma 1.4,

$$\eta(\,(\mathrm{ind}_z \otimes 1)(\theta), M) = \eta(\theta, t_z(M)\,) = 0 \quad \forall\, \theta \in R_0(H_z) \otimes R(\mathrm{Spin}^C)$$

for $z = x, y, xy$. Since

$$(t_z \circ \Delta)(M) = (\Delta \circ t_z)(M) \in \ker_*(\eta, H_z) \oplus \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C}),$$

we use the cyclic case to compute

$$\eta(\,(\mathrm{ind}_z \otimes 1)\theta, \Delta(M)\,) = \eta(\theta, t_z\Delta M) = \eta(\theta, \Delta t_z M) = 0 \quad \forall\, \theta \in R_0(H_z) \otimes R(\mathrm{Spin}^C).$$

We apply Lemma 2.8 to see

$$\eta(\theta, \Delta(M)\,) = 0 \;\forall\, \theta \in R_0(Q_v) \otimes R(\mathrm{Spin}^C)$$

which proves (a). This argument also shows that to prove (b), it suffices to consider $G = \mathbf{Z}_n$ and $\tau = \rho_1$. We compute

$$\alpha(\rho_1)^n = (\rho_0 - \rho_{-1})^n = \Sigma_{0 \leqq i \leqq n} \binom{n}{i} \rho_{-i} \in 2R_0(\mathbf{Z}_n)$$

so (b) follows by Lemma 2.6.

3. The bordism spectral sequence. We first study the Sylow 2-subgroups. If $G = \mathbf{Z}_n$, let

$$N_*(\mathbf{Z}_n) = N(\mathbf{Z}_n, \rho_1) \times \Omega_*^{\mathrm{Spin}^C} \subseteq \tilde{\Omega}_*^{\mathrm{Spin}^C}(B\mathbf{Z}_n).$$

The proof of the following lemma is the same as the proof we shall give shortly for the corresponding lemma concerning Q_v so we omit details.

LEMMA 3.1. *Let* $n = 2^w$.

(a) $N(\mathbf{Z}_n, \rho_1)$ *spans* $\tilde{\Omega}_1^{\mathrm{Spin}^C}(B\mathbf{Z}_n) = \mathbf{Z}_n$.

(b) *If* $M \in \tilde{\Omega}_1^{\mathrm{Spin}^C}(B\mathbf{Z}_n)$ *and* $\eta(\theta, M) = 0 \quad \forall\, \theta \in R_0(\mathbf{Z}_n)$, *then* $M = 0$.

(c) $N_m(\mathbf{Z}_n) \approx H_1(B\mathbf{Z}_n; \Omega_{m-1}^{\mathrm{Spin}^C})$,

$N_m(\mathbf{Z}_n) \cap \ker_m(\eta, \mathbf{Z}_n) \subseteq N(\mathbf{Z}_n, \rho_1) \times \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})$, *and*

$N_m(\mathbf{Z}_n) \cap \ker_m(SW, \mathbf{Z}_n) \cap \ker_m(\eta, \mathbf{Z}_n) = 0.$

The homology of Q_v is more complicated. Let

$$\rho_1(y) = \rho_1(xy) = e^{2\pi i/4}$$

be a fixed point free representation of H_y and H_{xy} not extending to Q_v. Let

$$M_y = i_y N(H_y, \rho_1),\ M_{xy} = i_{xy} N(H_{xy}, \rho_1),\ M_q = N(Q_v, \tau)$$

$$N_*(Q_v) = M_y \times \Omega_*^{\mathrm{Spin}^C} + M_{xy} \times \Omega_*^{\mathrm{Spin}^C} + M_q \times \Omega_*^{\mathrm{Spin}^C} \subseteq \tilde{\Omega}_*^{\mathrm{Spin}^C}(BQ_v).$$

LEMMA 3.2. (a) $\{M_y, M_{xy}\}$ *generates* $\tilde{\Omega}_1^{\mathrm{Spin}^C}(BQ_\nu) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and*

$$\{M_y \times CP^1, M_{xy} \times CP^1, M_q\}$$

generates

$$\tilde{\Omega}_3^{\mathrm{Spin}^C}(BQ_\nu) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{|Q_\nu|}.$$

(b) *If* $M \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BQ_\nu)$ *for* $* = 1, 3$ *and* $\eta(\rho, M) = 0 \ \forall \ \rho \in R_0(Q_\nu)$, $M = 0$.

(c) $$N_m(Q_\nu) \approx H_1(BQ_\nu; \Omega_{m-1}^{\mathrm{Spin}^C}) \oplus H_3(BQ_\nu; \Omega_{m-3}^{\mathrm{Spin}^C}),$$
$$N_m(Q_\nu) \cap \ker_m(\eta, Q_\nu) \subseteq \sum M_z \times \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C}), \quad \text{and}$$
$$N_m(Q_\nu) \cap \ker_m(SW, Q_\nu) \cap \ker_m(\eta, Q_\nu) = 0.$$

Proof. We use the Anderson-Brown-Peterson computations to see

$$\mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C}) = 0 \quad \text{for } * < 9$$

so

$$|\tilde{\Omega}_1^{\mathrm{Spin}^C}(BQ_\nu)| \leqq 2^2 \quad \text{and} \quad |\tilde{\Omega}_3^{\mathrm{Spin}^C}(BQ_\nu)| \leqq 2^2|Q_\nu|$$

by Theorem 1.1 and Lemmas 2.1 and 2.7. By Lemmas 1.3, 1.4, and 1.5:

$$\eta(\rho_{xy} - \rho_0, M_y) = 1/2, \quad \eta(\rho_{xy} - \rho_0, M_{xy}) = 0$$
$$\eta(\rho_y - \rho_0, M_y) = 0, \quad \eta(\rho_y - \rho_0, M_{xy}) = 1/2$$
$$\eta(\rho_{xy} - \rho_0, M_y \times CP^1) = 1/2, \quad \eta(\rho_{xy} - \rho_0, M_{xy} \times CP^1) = 0,$$
$$\eta(\rho_y - \rho_0, M_y \times CP^1) = 0, \quad \eta(\rho_y - \rho_0, M_{xy} \times CP^1) = 1/2,$$
$$\eta(\tau - 2\rho_0, M_y \times CP^1) = 0, \quad \eta(\tau - 2\rho_0, M_{xy} \times CP^1) = 0,$$
$$\eta(\rho_{xy} - \rho_0, M_q) = *$$
$$\eta(\rho_{xy} - \rho_0, M_q) = *$$
$$\eta(\tau - 2\rho_0, M_q) = -2^{-|Q_\nu|},$$

which proves (a, b).

Let

$$M = \Sigma_{z=y,xy,q} M_z \times N_z \in \ker(\eta, Q_\nu).$$

Since $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BQ_\nu)$ is a 2-group, we can assume

$$N_z = S_z + T_z \text{ for } S_z \in P_* = \mathbf{Z}[CP^1, CP^2, \dots, CP^{2k}, \dots] \text{ and}$$
$$T_z \in \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C}).$$

The comultiplication in Lemma 1.3 was defined so that if

$$(1 \otimes s)(\theta) = \Sigma a_i \otimes b_i,$$

then

$$\theta(M_1 \times N_1) = \Sigma a_i(M_1) \otimes b_i(N_1).$$

We observe:

$$(\Lambda^i - \Lambda^{i-1})(M_q) = \Lambda^i(\,(T(M_q) \oplus 1) \otimes \mathbf{C})$$
$$= \Lambda^i(\tau \oplus \tau)(M_q) \text{ and } \gamma(M_q) = 1.$$
$$(\Lambda^i - \Lambda^{i-1})(M_z) = \Lambda^i(\,(T(M_z) \oplus 1) \otimes \mathbf{C})$$
$$= \Lambda^i(\rho_1 + \rho_1^*)(M_z) = \Lambda^i(\tau)(M_z) \text{ and}$$
$$\tau(M_z) = \rho_x \otimes 1 \text{ for } z = y,\ xy.$$

We define algebra isomorphisms t_z of $R_0(Q_v) \otimes R(\mathrm{Spin}^C)$ by

$$t_q(1 \otimes (\Lambda^k - \Lambda^{k-1})\,) = \Sigma_{i+j=k}\Lambda^i(2\tau) \otimes \Lambda^j,$$
$$t_q(1 \otimes \tau) = 1 \otimes \tau, \text{ and } t_q(\rho \otimes 1) = \rho \otimes 1.$$
$$t_z(1 \otimes (\Lambda^k - \Lambda^{k-1})\,) = \Sigma_{i+j=k}\Lambda^i(\tau) \otimes \Lambda^j,$$
$$t_z(1 \otimes \tau) = \rho_x \otimes \tau, \text{ and } t_z(\rho \otimes 1) = r_z(\rho \otimes 1) \quad \text{for } z = y,\ xy.$$

Let $\theta \in R_0(Q_v) \otimes R(\mathrm{Spin}^C)$, and decompose

$$t_z(\theta) = \Sigma_i a_{i,z} \otimes b_{i,z}.$$

Then

$$\theta(M_z \times N_z) = \Sigma_i a_{i,z}(M_z) \otimes b_{i,z}(N_z)$$

so Lemma 1.3 implies

$$0 = \eta(\theta, M) = \Sigma_{i,z}\eta(a_{i,z}, M_z) \cdot \mathrm{index}(b_{i,z}, N_z).$$

Since T_z is a torsion class, $\mathrm{index}(-, T_z) = 0$ so

$$\mathrm{index}(b_{i,z}N_z) = \mathrm{index}(b_{i,z}S_z).$$

Let $\psi \in R(\mathrm{Spin}^C)$. Choose θ_1 so $t_q(\theta_1) = 1 \otimes \psi$ and let $\theta = (\tau - 2)\theta_1$ so $t_q(\theta) = (\tau - 2) \otimes \psi$. Then $t_z(\theta)$ is divisible by $(\tau - 2)$ for all z. Therefore

$$r_z(t_z(\theta)\,) \in \alpha(\tau) \cdot R(H_z) = R_0(H_z)^2 \quad \text{for } z = y,\ xy$$

so θ has zero eta invariant on M_y and M_{xy} by Lemma 1.6. This shows

$$0 = \eta(\tau - 2, M_q) \cdot \mathrm{index}(\psi, S_q)\ \forall\ \psi \in R(\mathrm{Spin}^C)$$

so

$$\mathrm{index}(\psi, S_q) \equiv 0 \bmod |Q_v|\ \forall\ \psi \in R(\mathrm{Spin}^C).$$

Therefore $S_q \in |Q_v|P_*$ by Theorem 1.1 and $M_q \times S_q = 0$ by Lemma 3.1. We argue similarly using $(\rho_y - \rho_0) \otimes \psi$ and $(\rho_{xy} - \rho_0) \otimes \psi$ to show $S_z \in 2 \cdot \Omega_*^{\mathrm{Spin}^C}$ for $z = y,\ xy$. This shows

$$M = \Sigma_z M_z \times T_z \quad \text{for } T_z \in \operatorname{Tor}(\Omega_*^{\mathrm{Spin}^C}).$$

Suppose in addition $M \in \ker_*(SW, Q_v)$. Let $x \in W^*$ and let a be the generator of $H^3(BQ_v; \mathbf{Z}_2) = \mathbf{Z}_2$. Then

$$0 = (ax)(\Sigma_z M_z \times T_z) = (ax)(M_q \times T_Q) = x(T_q)$$

so $T_q \in \ker(SW)$ and $T_q = 0$ by Theorem 1.1. Similarly by evaluating $w_1(\rho_y)x$ or $w_1(\rho_{xy})x$ we show $T_y = T_{xy} = 0$ so $M = 0$. If $M = 0$, then

$$M \in \ker(\eta, Q_v) \cap \ker(SW, Q_v)$$

so $S_y \in 2P_{m-1}$, $S_{xy} \in 2P_{m-1}$, $S_q \in |Q_v|P_{m-3}$, and $T_y = T_{xy} = T_q = 0$. This gives the additive structure of $N_*(Q_v)$.

We now study the manifolds related to

$$\tilde{H}_{\mathrm{even}}(BQ_v; \operatorname{Tor}(\Omega_*^{\mathrm{Spin}^C}))$$

in the bordism spectral sequence. Let

$$\Delta_z : \tilde{\Omega}_m^{\mathrm{Spin}^C}(BQ_v) \mapsto \tilde{\Omega}_{m-2}^{\mathrm{Spin}^C}(BQ_v) \oplus \operatorname{Tor}(\Omega_*^{\mathrm{Spin}^C})$$

correspond to the representation ρ_z. We generalize Lemma 2.6:

LEMMA 3.3. *Let* $v \geqq 2$ *and* $N \in \operatorname{Tor}(\Omega_*^{\mathrm{Spin}^C})$. *For* $k \geqq 0$, $\exists\, M_1(k) \in \Omega_{*+4k}^{\mathrm{Spin}^C}(BQ_v)$ *and* $M_2(k)$, $M_3(k) \in \Omega_{*+4k+2}^{\mathrm{Spin}^C}(BQ_v)$ *so that*

(a) $\Delta^k M_1(k) = \Delta_y \Delta^k M_2(k) = \Delta_x \Delta^k M_3(k) = N$,

$\Delta_x \Delta^k M_2(k) = \Delta_y \Delta^k M_3(k) = 0$, *and* $2M_i(k) = 0$.

(b) $\Delta^k M_i(k) = M_i(k - j)$ *for* $k \geqq j$ *and* $i = 1, 2, 3$.

(c) $\eta(\theta, M_i(k)) = 0\ \forall\, \theta \in R_0(Q_v) \otimes R(\mathrm{Spin}^C)$.

Remark. We will construct $M_i(k)$ satisfying (a). Δ_z and Δ commute. For fixed k and $j < k$, $\Delta^{k-j}M_i(k)$ also satisfies (a). Since the bordism groups are finite, the pigeon hole principal shows we may do this consistently for all k which proves (b). We can then use Lemma 2.9 to see the $M_i(k)$ satisfy (c); since the $M_i(k)$ need not belong to the reduced bordism groups, they need not belong to

$$\ker_*(\eta, G) \subseteq \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG).$$

Proof. If $k = 0$, let $A = N$. Otherwise use Lemma 2.6(b) to choose $A \in \Omega_{*+4k}^{\mathrm{Spin}^C}(B\mathbf{Z}_2)$ so that $2A = 0$ and $\Delta^k A = N$. Fix k and set $M_1 = i(A)$ with the induced Q_v structure; $2M_1 = 0$ and $\Delta^k A = i(N)$. Since the $\mathbf{Z}_2$ structure on N is trivial, $i(N) = N$ and M_1 satisfies (a).

The construction of M_2 is more complicated and lies at the heart of our investigation of $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BQ_v)$. Identify $\mathbf{R}^4 = \mathbf{C}^2 = \mathbf{H}$ and let

$$(x_1, x_2, x_3, x_4) = (z, w) = P$$

in this representation. Let $u = 2^{\nu-2}$. Then $\tau = \tau_1$ is given by:

$$\tau(x)(P) = (e^{\pi i/2u}z, e^{-\pi i/2u}w) = (\cos(\pi/2u) + \sin(\pi/2u)\mathbf{i}) \cdot P$$

$$\tau(y)(P) = (-w, z) = \mathbf{j} \cdot P.$$

Since $2A = 0$, let B be a compact $\mathbf{Z}_2$-Spin^C manifold so $dB = 2A$. Let C be the non-orientable manifold obtained by glueing $A \times [0, 2\pi]$ to B along the boundary. Let L_A be the real line bundle over A corresponding to the given $\mathbf{Z}_2$ structure; L_A extends to a bundle L_C over C by hypothesis. Let ν be the orientation line bundle of C. ν is trivial over $A \times [0, 2\pi]$ and over B; we choose local sections to agree over $A \times \{0\}$ and differ by sign over $A \times \{2\pi\}$. Let M_2 be the unit circle bundle of the 2-plane bundle $\nu \oplus 1$. Since ν is the orientation bundle of C, $\nu \oplus 1$ is orientable and inherits a Spin^C structure from the Pin^C structure on C. Let $\varphi_i \in [0, 2\pi]$ be local periodic angular parameters over $A \times [0, 2\pi]$ and B which are identified so

$$\varphi_1 = \varphi_2 \quad \text{over } X \times \{0\} \quad \text{and}$$

$$\varphi_1 = 2\pi - \varphi_2 \quad \text{over } X \times \{2\pi\}.$$

We give M_2 a Q_ν structure as follows. For r large, let

$$p{:}S^{4r-1} \mapsto S^{4r-1}/Q_\nu = N(Q_\nu, r\tau)$$

be the covering projection and let $g{:}C \mapsto RP^{r-1}$ be the classifying map for L_C with $g(x, \theta) = g(x)$ independent of θ over $X \times [0, 2\pi]$. Embed $\mathbf{R}^r \subseteq \mathbf{H}^r$ as the totally real subspace. Let

$$d(t) = \cos(t/4u) + \sin(t/4u) \cdot \mathbf{i} \quad \text{and}$$

$$e(t) = \cos(t/4) + \sin(t/4) \cdot \mathbf{j}$$

define representations from $\mathbf{R} \mapsto SU(2) = SP(1)$ with $d(2\pi) = \tau(x)$ and $e(2\pi) = \tau(y)$. Let G be a local lift of g from RP^{r-1} to $S^{r-1} \subseteq \mathbf{R}^r \subseteq \mathbf{H}^r$ and define:

$$h(a, \theta, \varphi_1) = p\{d(\varphi_1)e(\theta) \cdot G(a)\} \quad \text{and}$$

$$h(b, \varphi_2) = p\{d(\varphi_2) \cdot G(b)\}.$$

Since $\mathbf{Z}_2$ is in the center of $SP(1)$, h is independent of the local choice of G. We check h extends to M_2 by verifying:

$$h(a, \theta, \varphi_1 + 2\pi) = p\{\tau(x)d(\varphi_1)e(\theta) \cdot G(a)\}$$

$$= p\{d(\varphi_1)e(\theta) \cdot G(a)\} = h(a, \theta, \varphi_1)$$

$$h(b, \varphi_2 + 2\pi) = p\{\tau(x)d(\varphi_2) \cdot G(b)\} = p\{d(\varphi_2) \cdot G(b)\}$$

$$= h(b, \varphi_2)$$

$$\begin{aligned} h(a, 2\pi, 2\pi - \varphi_1) &= p\{d(2\pi - \varphi_1) \cdot \mathbf{j} \cdot G(a)\} \\ &= p\{\mathbf{j} \cdot d(\varphi_1 - 2\pi) \cdot G(a)\} \\ &= p\{\mathbf{j} \cdot \tau(x)^{-1} d(\varphi_1) \cdot G(a)\} = h(a, 0, \varphi_1). \end{aligned}$$

The boundary of $M_2 \times [0, 1]$ is $M_2 \cup -M_2$ where $-M_2$ is given the reversed orientation and inherited Spin^C and Q_v structures. Let $\Psi:\varphi \mapsto 2\pi - \varphi$ define a Spin^C isometry from M_2 to $-M_2$. We show Ψ preserves the Q_v structure by showing $h(c, 2\pi - \varphi)$ and $h(c, \varphi)$ are homotopic maps from M_2 to $N(Q_v, r\tau)$:

$$\begin{aligned} h(a, \theta, 2\pi - \varphi_1) &= p\{d(2\pi - \varphi_1)e(\theta) \cdot G(a)\} \\ &= p\{\mathbf{j}^{-1} d(\varphi_1 - 2\pi)e(\theta)\mathbf{j} \cdot G(a)\} \\ &= p\{d(\varphi_1)e(\theta)\mathbf{j} \cdot G(a)\} \\ h(b, 2\pi - \varphi_2) &= p\{d(2\pi - \varphi_2) \cdot G(b)\} \\ &= p\{\mathbf{j}^{-1} d(\varphi_2 - 2\pi)\mathbf{j} \cdot G(b)\} \\ &= p\{d(\varphi_2)\mathbf{j} \cdot G(b)\}. \end{aligned}$$

Since $SU(2)$ is connected, $g(y)$ and $\mathbf{j} \cdot g(y)$ are homotopic maps from $C \to RP^{4r-1}$ so $h(c, \varphi)$ and $h(c, 2\pi - \varphi)$ are homotopic. Thus $\Psi:M_2 \mapsto -M_2$ is a $Q_v - \mathrm{Spin}^C$ isomorphism and $2M_2 = 0$.

We complete the construction of M_2 by computing Δ_z. Define:

$$\delta_x(P) = \Sigma_a \operatorname{Re}(z_a w_z, |z_a|^2 - |w_a|^2) \in \mathbf{R}^2$$

$$\delta_y(P) = \Sigma_a (z_a)^{2u} + (w_a)^{2u} \in \mathbf{C} = R^2$$

and check

$$\delta_z(\tau(q)P) = \rho_z(q)\delta_z(P).$$

Since δ_z is equivariant with respect to ρ_z, we can let δ_z be the first two coordinates of the classifying map. The transversality condition will be satisfied so $\Delta_z(M_2)$ is given by setting $\delta_z(m_2) = 0$. Because g was chosen to be totally real,

$$\Sigma_a(|z_a|^2 - |w_a|^2) = 0$$

implies $\theta = \pi$ so

$$\Sigma_a z_a w_a = 1/2.$$

This shows $\delta_x h(m_2) \neq 0$ so $\Delta_x(M_2) = 0$. Similarly $\delta_y(h(m_2)) = 0$ corresponds to $\theta = \pi$ and $\varphi = \pi$ so

$$\Delta_y(M_2) = A \times \{\pi\} \times \{\pi\}.$$

Since the normal bundle of A in M_2 is trivial, A inherits the original Spin^C structure. The Q_v structure on A is given by $p(d(\pi)e(\pi) \cdot G)$ which is homotopic to pG. Thus $\Delta_y(M_2) = i(A)$ has the induced Q_v structure. Since $\Delta^k(A) = N$, M_2 has the desired properties. If $v = 2$, there is an outer automorphism of Q_v interchanging the roles of x and y we use to construct M_3. For $v > 2$, we include Q_2 in Q_v to give M_3 a Q_v structure. Since the restriction of ρ_x from Q_v to Q_2 is ρ_x and since the restriction of ρ_y from Q_v to Q_2 is ρ_0, M_3 has the desired properties.

Let G be a 2-group so $G = \mathbf{Z}_n$ for $n = 2^w$ or $G = Q_v$. The $E^2_{p,q}$ term in the bordism spectral sequence is $\widetilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})$. Let

$$E_{\mathrm{even}}(m, G) = \oplus_{p+q=m, p\equiv 0(2)}\widetilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})$$
$$E_{\mathrm{odd}}(m, G) = \oplus_{p+q=m, p\equiv 1(2)}\widetilde{H}_p(BG; \Omega_q^{\mathrm{Spin}^C})$$

decompose the bordism spectral sequence. By Lemma 2.7 and the Universal coefficient theorem,

$$E_{\mathrm{even}}(*, \mathbf{Z}_n) = \oplus_{k\geqq 0}\{\mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_{*-2k-2})\} \quad \text{and}$$
$$E_{\mathrm{even}}(*, Q_v) = \oplus_{k\geqq 0}\{\mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_{*-4k-2}) \oplus \mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_{*-4k-2}) \oplus \mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_{*-4k-4})\}$$

is the direct sum $\mathbf{Z}_2$ factors. Let

$$\pi:\widetilde{\Omega}^{\mathrm{Spin}^C}_*(BG) \oplus \mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_*) \mapsto \mathrm{Tor}(\Omega^{\mathrm{Spin}^C}_*)$$

be projection on the second factor. Let Δ_1 correspond to ρ_1 for $\mathbf{Z}_n$ and let Δ correspond to τ for Q_v. Define $\Omega^{\mathrm{Spin}^C}_*$ module morphisms

$$\Pi = \oplus_{k\geqq 0}\{\pi\Delta_1^{k+1}\}:\widetilde{\Omega}^{\mathrm{Spin}^C}_m(B\mathbf{Z}_n) \mapsto E_{\mathrm{even}}(m, \mathbf{Z}_n) \quad \text{and}$$
$$\Pi = \oplus_{k\geqq 0}\{\pi\Delta^k\Delta_x \oplus \pi\Delta^k\Delta_y \oplus \pi\Delta^{k+1}\}:\widetilde{\Omega}^{\mathrm{Spin}^C}_m(BQ_v) \mapsto E_{\mathrm{even}}(m, Q_v).$$

LEMMA 3.4. *Let G be a 2-group.*

(a) $\mathrm{Ker}_*(SW, G) \subseteq \ker_*(\Pi)$.

(b) $\Pi:\widetilde{\Omega}^{\mathrm{Spin}^C}_m(BG) \mapsto E_{\mathrm{even}}(m, G)$ *is surjective and split. We may choose the splitting*

$$\Pi^{-1}(E_{\mathrm{even}}(m, G)) \subseteq \ker_*(\eta, G).$$

Proof. First let $G = Q_v$. Since Δ, Δ_z, and π are $\Omega^{\mathrm{Spin}^C}_*$ morphisms, Π is an $\Omega^{\mathrm{Spin}^C}_*$ morphism. If $M \in \ker_*(SW, Q_v)$, then

$$\{\Delta^k(M), \Delta^k\Delta_z(M)\} \subseteq \ker_*(SW, Q_v) \subseteq \widetilde{\Omega}^{\mathrm{Spin}^C}_*(BQ_v)$$

by Lemmas 2.3 and 2.4 so $\Pi(M) = 0$ which proves (a). Since Δ and Δ_z are zero on $\Omega^{\mathrm{Spin}^C}_*$, we can extend the domain of Π to the full bordism group and then project to $\widetilde{\Omega}^{\mathrm{Spin}^C}_*(BQ_v)$ at the end if necessary in proving (b). The fact Π is split surjective follows from Lemma 3.3 if we use induction on the grading of $E_{\mathrm{even}}(m, Q_v)$ defined by the dimension. If we fix m, then $\pi \circ \Delta^j \circ \Pi^{-1}$ is a splitting for Π in dimension $m - 4j$; we use the pigeon hole principal to choose

$$\Pi^{-1}E_{\text{even}}(m - 4j, Q_v) = \pi \circ \Delta^j \circ \Pi^{-1}E_{\text{even}}(m, Q_v) \quad \forall\, m, j.$$

If $2M = 0$, then

$$\pi\Delta^{|Q_v|}M \in \ker(\eta, Q_v)$$

by Lemma 2.9 so

$$\Pi^{-1}E_{\text{even}}(m, Q_v) \subseteq \ker_*(\eta, Q_v).$$

The argument is the same for $\mathbf{Z}_n$ if we use Lemma 2.6 (b) instead of Lemma 3.3 so we omit details.

Let G be a 2-group and let $\{iN(H_j, \sigma_j)\}$ be the collection of spherical space forms described in Lemmas 2.5 and 2.7 which is invariant with respect to the appropriate Smith homomorphism and which is a basis for $H_{\text{odd}}(BG; \mathbf{Z}_2)$. Let

$$S_*(G) = \Sigma_j iN(H_j, \sigma_j) \times \Omega_*^{\text{Spin}^C} \subseteq \tilde{\Omega}_*^{\text{Spin}^C}(BG).$$

Since $\tilde{\Omega}_*^{\text{Spin}^C}(BG)$ is a 2-group,

$$S_*(G) = \Sigma_j iN(H_j, \sigma_j) \times (P_* + \text{Tor}(\Omega_*^{\text{Spin}^C})).$$

Let F be the forgetful functor from unitary to Spin^C bordism. Since $\{iN(H_j, \sigma_j)\}$ spans $\tilde{\Omega}_*^U(BG)$ as an Ω_*^U module (see for example [**6**]),

$$S_*(G) = F(\tilde{\Omega}_*^U(BG)) \times \Omega_*^{\text{Spin}^C}$$

which gives an invariant description. Since the collection is invariant under the Smith homomorphism,

$$\Delta_1: S_*(\mathbf{Z}_n) \mapsto S_{*-2}(\mathbf{Z}_n) \mapsto 0 \quad \text{and} \quad \Delta: S_*(Q_v) \mapsto S_{*-4}(Q_v) \mapsto 0.$$

If $G = Q_v$, the Δ_z can also be defined in unitary bordism so $F\Delta_z = \Delta_z F$. Since Ω_*^U is without torsion,

$$\pi\Delta_z\tilde{\Omega}_*^U(BG) = 0$$

so

$$\pi\Delta_z F(\tilde{\Omega}_*^U(BG) \times \text{Tor}(\Omega_*^{\text{Spin}^C})) = 0.$$

Therefore $S_*(G) \subseteq \ker(\Pi)$.

LEMMA 3.5. *Let G be a 2-group.*

(a) *The bordism spectral sequence for $\tilde{\Omega}_*^{\text{Spin}^C}(BG)$ collapses; i.e., all the differentials are zero.*

(b) Π defines a split short exact sequence:

$$0 \mapsto S_m(G) \mapsto \tilde{\Omega}_m^{\text{Spin}^C}(BG) \overset{\pi}{\mapsto} E_{\text{even}}(m, G) \mapsto 0,$$

(c) Δ *and* Δ_1 *define short exact sequences*:

$$0 \mapsto N_m(Q_v) \mapsto S_m(Q_v) \overset{\Delta}{\mapsto} S_{m-4}(Q_v) \mapsto 0.$$

$$0 \mapsto N_m(\mathbf{Z}_n) \mapsto S_m(\mathbf{Z}_n) \overset{\Delta}{\mapsto} S_{m-2}(\mathbf{Z}_n) \mapsto 0.$$

Proof. We give the argument for Q_v as the argument for $\mathbf{Z}_n$ is the same. Since $N_*(Q_v) \subseteq S_*(Q_v)$, we have a short exact sequence:

$$0 \mapsto \ker(\Delta) \cap S_m(Q_v) \mapsto S_m(Q_v) \overset{\Delta}{\mapsto} S_{m-4}(Q_v) \mapsto 0.$$

We use Lemma 3.2 to estimate:

$$\begin{aligned} |S_m(Q_v)| &= |S_{m-4}(Q_v)| \cdot |\ker(\Delta) \cap S_m(Q_v)| \\ &\geqq |S_{m-4}(Q_v)| \cdot |N_m(Q_v)| \\ &\geqq |S_{m-4}(Q_v)| \cdot |H_1(BQ_v; \Omega_{m-1}^{\mathrm{Spin}^C})| \cdot |H_3(BQ_v; \Omega_{m-3}^{\mathrm{Spin}^C})| \end{aligned}$$

which implies $|S_m(Q_v)| \geqq |E_{\mathrm{odd}}(m, Q_v)|$. Since $S_m(Q_v) \subseteq \ker(\Pi)$, this implies

$$\begin{aligned} |\tilde{\Omega}_m^{\mathrm{Spin}^C}(BQ_v)| &= |\ker(\Pi_m)| \cdot |E_{\mathrm{even}}(m, Q_v)| \\ &\geqq |S_m(Q_v)| \cdot |E_{\mathrm{even}}(m, Q_v)| \\ &\geqq |E_{\mathrm{odd}}(m, Q_v)| \cdot |E_{\mathrm{even}}(m, Q_v)|. \end{aligned}$$

Since the reverse inequality is provided by Lemma 2.1, all these inequalities must be equalities.

We derive the following

COROLLARY 3.6. *Let G be a spherical space form group.*

(a) *The bordism spectral sequence of G collapses.*

(b) *Let H be a p-Sylow subgroup of G. Then $i \circ t$ is an isomorphism of* $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(p)}$.

Proof. At odd primes, this was checked in [**6**] so we may assume $p = 2$ and $H = \mathbf{Z}_n$ or Q_v. Induction and transfer are maps of spectral sequences (see [**10**]). The composition $i \circ t$ is multiplication by $|G{:}H|$ on $\tilde{H}(BG_v; \Omega_q^{\mathrm{Spin}^C})$ and induces a map which is an isomorphism of $E_{p,q}^2(BG_v)_{(2)}$. The bordism spectral sequence for $\mathbf{Z}_n$ and Q_v collapses by Lemma 3.5 and consequently all the differentials of the bordism spectral sequence for BG at the prime 2 collapse as well. This proves (a). We use the 5-Lemma and (a) to derive (b) $i \circ t$ is an isomorphism on $H_*(BG; -)_{(p)}$.

We can now complete the proof of Theorem 0.1. Let G be a spherical space form group. The odd primary part of Theorem 0.1 was proved in [**6**] so it suffices to study the 2-primary part. Let H be the Sylow 2-subgroup of G. By Lemma 1.4,

$$t{:}\ker_*(SW, G) \cap \ker_*(\eta, G) \mapsto \ker_*(SW, H) \cap \ker_*(\eta, H).$$

Since t is injective on bordism (2) by Corollary 3.6, it suffices to prove Theorem 0.1 for H. We suppose $H = Q_v$ as the cyclic case is similar.

Let

$$M \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BH) \cap \ker_*(\eta, H) \cap \ker_*(SW, H).$$

We proceed by induction on $*$ to show $M = 0$; the cases $* = 1, 3$ follow from Lemma 3.2. We use Lemma 2.1 to see

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BH) = 0 \quad \text{for } * = 0.2.$$

By Lemma 3.4, $M \in \ker_*(SW, H)$ implies $M \in \ker(\Pi) = S_*(H)$. By Lemmas 2.4 and 2.9,

$$\Delta(M) \in \ker_{*-4}(\eta, H) \cap \ker_{*-4}(SW, H) = 0.$$

Consequently by Lemma 3.3,

$$\Delta(M) \in N_*(H) \cap \ker_*(\eta, H) \cap \ker_*(SW, H)$$

and this is zero by Lemma 3.2.

4. The additive structure of $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)}$ and $bu_*(BG)_{(2)}$. In this section, we will construct an analytic splitting

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} \cong A_*(G) \otimes Z[X^4, \dots, X^{4k}, \dots] \oplus \ker_*(\eta, G)$$

$$\ker_*(\eta, G) \cong \tilde{H}_*(BG; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

We will identify the groups $A_*(G)$ with $bu_*(G)_{(2)}$ later and this will lead to the proof of Theorems 0.2 and 0.3.

We must describe one additional piece of structure on G. A τ-structure for G is an assignment $H \mapsto \tau_H$ to each 2-subgroup H of G of a fixed point free representation $\tau_H:H \mapsto SU(2)$ so the assignment is invariant under restriction and conjugation; i.e., up to unitary equivalence

(i) If $E \subseteq H$, then $\tau_E(e) = \tau_H(e)\ \forall\ e \in E$.

(ii) If $E = gHg^{-1}$, then $\tau_H(h) = \tau_E(ghg^{-1})\ \forall\ h \in H$.

LEMMA 4.1. (a) *Let H be a 2-subgroup of G and let $\tau:H \mapsto SU(2)$ be fixed point free. If $g \in E$, let $E = H \cap g^{-1}Hg$ and let $\tau_E(e) = \tau(geg^{-1})$. Then*

$$\mathrm{Tr}(\tau_E(e)) = \mathrm{Tr}(\tau(e)) \quad \forall\ e \in E.$$

(b) *If $\tau:G_2 \mapsto SU(2)$ is a fixed point free representation of a Sylow 2-subgroup of G, then $\exists\ !\tau$-structure on G so $\tau' = \tau_{G_2}$.*

Proof. We suppose (a) is false and argue for a contradiction. τ and τ_E both define fixed point free $SU(2)$ representations of E. Choose $e \in E$ so $\tau_E(e)$ and $\tau(e)$ have eigenvalues $\{\mu, \mu^{-1}\}$ and $\{\lambda, \lambda^{-1}\}$ where $\mu \neq \lambda^{\pm 1}$. If e has order 1, 2 or 4,

$$\{\mu, \mu^{-1}\} = \{\lambda, \lambda^{-1}\} \in \{\{1, 1\}, \{-1, -1\}, \{i, -i\}\}.$$

Therefore order$(e) > 4$. Let N be the subgroup of H generated by all elements of order >4. If $H \cong \mathbf{Z}_n$, then $N \subseteq \mathbf{Z}_n$. If $H \cong Q_v$, then $H \cong H_x$. Therefore N is a cyclic normal subgroup of H. Since

$$\text{ord}(e) = \text{ord}(geg^{-1}) > 4,$$

$e \in N$ and $geg^{-1} \in N$. Since N is cyclic, $geg^{-1} = e^k$ for k odd. Since $\mu \neq \lambda^{\pm 1}$, $geg^{-1} \neq e^{\pm 1}$. By replacing g by an odd power of g if necessary, we can preserve these relations and assume in addition the order of g is a power of 2. Let $\sigma:G \mapsto U(*)$ be fixed point free and let $a = \sigma^{-1}(-1)$; a is the unique element of G which has order 2; a is in the centre of G since σ is faithful. If $g^2 \in \{a, 1\}$ then $geg^{-1} = e^{\pm 1}$. Since this is false, order$(g) > 4$. Let $X = \langle g, e\rangle$ be a 2-subgroup of G. X contains two elements which don't commute and which have order greater than 4. Thus X is neither cyclic nor generalized quaternionic. This contradiction proves (a).

We use (a) to prove (b). Let G_2 be a Sylow 2-subgroup of G and let $\tau:G \mapsto SU(2)$ be fixed point free. If H is a 2-group, $\exists\, g \in G$ so $gHg^{-1} \subseteq G_2$. If G has a τ-structure, then

$$\text{Tr}(\tau_H(h)) = \text{Tr}(\tau'(ghg^{-1}))$$

is uniquely specified by τ'. To show the existence of a τ-structure, we must show $\tau_H = \tau'(ghg^{-1})$ is independent of the choice of g up to unitary equivalence since then the assignment $H \mapsto \tau_H$ will be invariant under restriction and conjugation. Since this question is invariant under conjugation, we suppose without loss of generality that $H \subseteq G_2$. Let

$$E = G_2 \cap g^{-1}G_2 g.$$

Since $gHg^{-1} \subseteq G_2$, $H \subseteq E$. Thus by (a), $\text{Tr}(\tau(ghg^{-1})) = \text{Tr}(\tau'(h))$ is independent of g.

If $E \subseteq G$, the restriction of a τ-structure on G to E defines a τ-structure on E. We fix henceforth a τ-structure on G and hence one on all the subgroups. We say (H, σ) is admissible if H is a 2-subgroup of G and if $\sigma + \sigma^* = k\tau_H$ for $k = \dim(\sigma)$. This implies σ is fixed point free. If k is odd, then H must be cyclic.

Inequivalent Spin^C structures on $N(H, \sigma)$ differ by a complex line bundle or equivalently by a representation $\rho:H \mapsto U(1)$; let $N(H, \sigma, \rho)$ be $N(H, \sigma)$ where the Spin^C structure has been twisted by ρ. Since the rational Chern/Pontrjagin numbers and Stiefel-Whitney numbers of $N(H, \sigma)$ and $N(H, \sigma, \rho)$ agree, $N(H, \sigma, \rho) = 0$ in $\Omega_*^{\text{Spin}^C}$ so

$$N(H, \sigma\, \rho) \in \tilde{\Omega}_*^{\text{Spin}^C}(BG).$$

Let

$$A_*(G) = \text{span}_{\mathbf{Z}}\{i(N(H, \sigma, \rho))\} \subseteq \tilde{\Omega}_*^{\text{Spin}^C}(BG)$$

for (H, σ) admissible and $\rho:H \mapsto U(1)$.

LEMMA 4.2. (a) *If* $E \subseteq G$, *then*

$$i(A_*(E)) \subseteq A_*(G) \quad \textit{and} \quad t(A_*(G)) \subseteq A_*(E).$$

(b) *If E is a Sylow 2-subgroup of G, $i \circ t$ is an isomorphism of $A_*(G)$.*

Proof. If $C \subseteq D$, let $i(C, D)$ and $t(C, D)$ be the induction and transfer maps. Let (H, σ) be admissible with respect to E, then (H, σ) is also admissible with respect to G. Since induction is functorial,

$$i(H, G)i(E, H)N(H, \sigma, \rho) = i(E, G)N(H, \sigma, \rho)$$

so $i(A_*(E)) \subseteq A_*(G)$. To study transfer, let $E \subseteq G$ and $H \subseteq G$ and let

$$X = t(E, G)i(H, G)N(H, \sigma, \rho)$$

for admissible (H, σ). We must show $X \in A_*(E)$ to complete the proof. Since we can change the orientation and Spin^C structure arbitrarily, we ignore ρ for the moment. We compute the induction and transfer maps as follows. Let $m = 2\dim(\sigma) - 1$ and let $H \mapsto S^m \mapsto N(H, \sigma)$ be the principal left H bundle defining

$$N(H, \sigma) \in \tilde{\Omega}_*^{\mathrm{Spin}^C}(BH).$$

Then

$$G \mapsto G \times {}_H S^m \mapsto N(H, \sigma)$$

is the principal left G bundle defining the induced G-structure on $i(H, G)N(H, \sigma)$. Let G/H be the right coset space and let $\{g_i\}$ be coset representatives. If $g \in G$, let

$$g \cdot g_i = g_{i(g)} \cdot h_i(g)$$

give the left action of G on G/H. Define a left action of G on $G/H \times S^m$ by

$$g(g_i \times z) = g_{i(g)} \times \sigma(h_i(g)) \cdot z.$$

Then $G/H \times S^m$ and $G \times {}_H S^m$ are isomorphic left principal G-bundles. The transfer homomorphism is defined by restricting this bundle to E so

$$E \mapsto G/H \times S^m \mapsto E \backslash \{(G/H \times S^m)\} = X$$

defines the E structure on X. Decompose the left action of E on G/H into orbits Y_j, then $X_j = E\backslash(Y_j \times S^m)$ are the connected components of X. Choose $g_j \in Y_j$ and let

$$E_j = E \cap g_j H g_j^{-1} = \{e \in E : e g_j H = g_j H\}$$

be the isotropy subgroup. Let

$$\sigma_j(e) = \sigma(g_j^{-1} e g_j) \quad \text{and} \quad \tau_j(e) = \tau(g_j^{-1} e g_j).$$

Then τ_j is the representation corresponding to E_j. Since

$$\sigma_j + \sigma_j^* = \dim(\sigma_j)\tau_j,$$

(E_j, σ_j) is admissible. Then $E_j \mapsto S_m \mapsto N(E_j, \sigma_j)$ is isomorphic to $g_j \times S^m$ as a principal E_j bundle. Furthermore $(E/E_j) \times S^m$ is isomorphic to $Y_j \times S^m$ as a principal E bundle so

$$E \mapsto Y_j \times S^m \mapsto E\backslash(Y_j \times S^m)$$

is $i(E_j, E)N(E_j, \sigma_j)$ which proves (a). Since $A_*(G)$ is a 2-group, Corollary 3.6 shows $i \circ t$ is an isomorphism.

We restrict for the moment to 2-groups. Let Δ correspond to $\tau:G \mapsto SU(2)$. Let (H, σ) be admissible. If H is not cyclic, then $\tau|_H = \tau^*|_H$ is irreducible so

$$\sigma = k \cdot \tau_H = k \cdot \tau|_H \quad \text{for } k = \dim(\sigma)/2.$$

If H is cyclic, then $\tau|_H = \rho_1 \oplus \rho_1^*$ decomposes as the sum of two 1-dimensional representations. Consequently σ and $\dim(\sigma) \cdot \rho_1$ are equivalent as representations of H to $O(2j)$. This shows $N(H, \sigma)$ and $N(H, \dim(\sigma) \cdot \rho_1)$ differ at most by the orientation chosen and by the Spin^C structure chosen. Since we allow arbitrary changes of orientation and Spin^C structure in defining $A_*(G)$, we may choose $\sigma = k\tau$ or $\sigma = k\tau + \rho_1$ depending on the parity. Therefore:

$$A_{4k-1}(G) = \mathrm{span}_{\mathbf{Z}}\{iN(H, k\tau, \rho)\} \quad \text{and}$$
$$A_{4k+1}(G) = \mathrm{span}_{\mathbf{Z}}\{iN(H, k\tau \oplus \rho_1, \rho)\}$$

where H ranges over all subgroups of G in dimension $4k - 1$ and the cyclic subgroups of G in dimension $4k+1$. Since

$$\Delta iN(H, \sigma + \tau, \rho) = iN(H, \sigma, \rho),$$

this collection is Δ invariant so

$$\Delta:A_*(G) \mapsto A_{*-4}(G) \mapsto 0.$$

If G is cyclic, $A_*(G) = \mathrm{span}_{\mathbf{Z}}\{iN(H, k\rho_1, \rho)\}$ is Δ_1 invariant so

$$\Delta_1:A_*(\mathbf{Z}_n) \mapsto A_{*-2}(\mathbf{Z}_n) \mapsto 0.$$

Let $\alpha = \alpha(\tau) = \tau - 2$ and let $I = \alpha R(G)$. If $G = \mathbf{Z}_n$, then $I = R_0(\mathbf{Z}_n)^2$. Let

$$\delta:R_0(G)/I^{k+1} \mapsto R_0(G)/I^k \quad \text{and} \quad \delta:I/I^{k+1} \mapsto I/I^k$$

be the natural projections. We use Lemma 1.6 to relate $A_*(G)$ to the representation theory of G as follows:

LEMMA 4.3. *Let G be a 2-group.*

(a) $\exists\, !f_{4k-1}:A_{4k-1}(G) \mapsto I/I^{k+1} \mapsto 0$ *and*

$\exists\, !f_{4k+1}:A_{4k+1}(G) \mapsto R_0(G)/I^{k+1} \mapsto 0$

so

$$\eta(\theta, M) = \eta(\theta \cdot f_{4k\pm 1}(M), N(G, (k + 1)\tau)) \ \forall\ M \in A_{4k\pm 1}(G)\ \forall\ \theta \in R_0(G).$$

(b) $\{f_{4k+1}(M) = 0\} \Leftrightarrow \{\eta(\theta, M) = 0\ \forall\ \theta \in R_0(G)\}$.

(c) $f_{4k\pm 1-4} \circ \Delta = \delta \circ f_{4k\pm 1}$.

Proof. The tangential operator of the Spin^C complex for $N(H, \sigma, \rho)$ is the tangential operator of the Spin^C complex for $N(H, \sigma)$ with coefficients in ρ so by Lemma 1.5,

$$\eta(\theta, N(H, \sigma, \rho)) = (\theta \cdot \rho, \beta(\sigma))_H.$$

Let $A'_{4k\pm 1}(G)$ be the free $\mathbf{Z}$-group on symbols $\{iN(H, \sigma, \rho)\}$ for $\sigma = k\tau$ or $k\tau \oplus \rho_1$ and let

$$\pi : A'_{4k\pm 1}(G) \mapsto A_{4k\pm 1}(G) \mapsto 0$$

be the natural projection. By Lemmas 1.3, 1.4, and 1.5:

$$\begin{aligned}\eta(\theta, iN(H, k\tau, \rho)) &= (r(\theta)\rho, \beta(k\tau))_H = (r(\theta)\rho\alpha, \beta((k + 1)\tau))_H \\ &= (\theta\alpha \cdot \mathrm{ind}(\rho), \beta((k + 1)\tau))_G \\ &= \eta(\theta\alpha \cdot \mathrm{ind}(\rho), N(G, (k + 1)\tau)).\end{aligned}$$

$$\begin{aligned}\eta(\theta, iN(H, k\tau \oplus \rho_1, \rho)) &= (r(\theta)\rho, \beta(k\tau \oplus \rho_1))_H \\ &= (r(\theta)\rho\alpha(\rho_{-1}), \beta((k + 1)\tau))_H \\ &= (\theta\ \mathrm{ind}(\rho\alpha(\rho_{-1})), \beta((k + 1)\tau))_G \\ &= \eta(\theta\ \mathrm{ind}(\rho\alpha(\rho_{-1})), N(G, (k + 1)\tau)).\end{aligned}$$

Let

$$\tilde{f}_{4k-1}(iN(H, k\tau, \rho)) = \alpha \cdot \mathrm{ind}(\rho) : A'_{4k-1}(G) \mapsto I \quad \text{and}$$
$$\tilde{f}_{4k+1}(iN(H, k\tau \oplus \rho_1, \rho)) = \mathrm{ind}(\alpha(\rho_1)\rho) : A'_{4k+1}(G) \mapsto R_0(G)$$

so

$$\eta(\theta, \pi(M)) = \eta(\theta \cdot \tilde{f}_{4k\pm 1}(M), N(G(k + 1)\tau)\ \forall\ \theta \in R_0(Q_\nu).$$

By Lemma 1.6,

$$\{\theta_1 \in I^{k+1}\} \Leftrightarrow \{\eta(\theta\theta_1, N(G(k + 1)\tau)) = 0\ \forall\ \theta \in R_0(Q_\nu)\}$$

so $\tilde{f}_{4k\pm 1}$ extends uniquely to a map

$$f_{4k\pm 1} : A_{4k\pm 1}(G) \mapsto R_0(G)/I^{k+1}$$

with the desired properties. If $G = Q_\nu$, let

$$\rho_s(z) = e^{2\pi i s/\mathrm{order}(z)} : H_z \mapsto U(1).$$

Then using Lemma 2.8,

$$\begin{aligned}\text{range}(\tilde{f}_{4k-1}(A_{4k-1}(\mathbf{Z}_n))) &\supseteq \text{span}_{\mathbf{Z}}\{\tilde{f}_{4k-1}(N(G, 2k\tau, \rho))\} \\ &= \text{span}_{\mathbf{Z}}\{\alpha(\tau)\rho\} = I \\ \text{range}(\tilde{f}_{4k+1}(A_{4k-1}(\mathbf{Z}_n))) &\supseteq \text{span}_{\mathbf{Z}}\{\tilde{f}_{4k+1}(N(G, 2k\tau + \rho_1, \rho))\} \\ &= \text{span}_{\mathbf{Z}}\{\alpha(\rho_1)\rho\} = R_0(\mathbf{Z}_n)\end{aligned}$$

$$\text{range}(\tilde{f}_{4k-1}(A_{4k-1}(Q_v)))$$

$$\supseteq \text{span}_{\mathbf{Z}}\{\tilde{f}_{4k-1}(i(N(H_x, k\tau, \rho_s))), \tilde{f}_{4k-1}(N(Q_v, k\tau, \rho_z))\}$$

$$= \text{span}_{\mathbf{Z}}\{\alpha \cdot \tau_s, \alpha \cdot \rho_z\} = I$$

$$\text{range}(\tilde{f}_{4k+1}(A_{4k+1}(Q_v)))$$

$$\supseteq \text{span}_{\mathbf{Z}}\{\tilde{f}_{4k+1}(i(N(H_z, k\tau \oplus \rho_1, \rho_s)))\}$$

$$= \text{span}_{\mathbf{Z}}\{\text{ind}_z(\rho_s\alpha(\rho_1))\} = \text{span}_{\mathbf{Z}}\{\text{ind}_z(R_0(H_z))\} = R_0(Q_v).$$

(c) follows from the definition of $\tilde{f}_{4k\pm 1}$ since

$$\Delta(i(N(H, \sigma \oplus \tau, \rho))) = iN(H, \sigma, \rho).$$

We use Theorem 0.1 to show f_* is an isomorphism.

LEMMA 4.4. *Let G be a 2-group.*
(a) *If $M \in A_*(G)$ and $\eta(\theta, M) = 0\ \forall\ \theta \in R_0(G)$, then $M = 0$.*
(b) *$f_{4k\pm 1}$ defines isomorphisms*

$$f_{4k-1}(G):A_{4k-1}(G) \cong I/I^{k+1} \quad \textit{and} \quad f_{4k+1}(G) \cong R_0(G)/I^{k+1}.$$

Proof. Since f_* is surjective and

$$\{f_*(M) = 0\} \Leftrightarrow \{\eta(\theta, M) = 0\ \forall\ \theta \in R_0(G)\},$$

(a) and (b) are equivalent. Suppose first G is cyclic. Then only the parity mod 2 is relevant since

$$A_{2j-1}(G) = \text{span}_{\mathbf{Z}}\{i(N(H, j\rho_1, \rho))\}.$$

Let

$$\tilde{F}_{2j-1}(i(N(H, j\rho_1, \rho))) = \alpha(\rho_1)\text{ind}(\rho)$$

define $\tilde{F}_{2j-1}:A'_{2j-1}(G) \mapsto R_0(G) \mapsto 0$ so

$$\eta(\theta, \pi(M)) = \eta(\theta \cdot \tilde{F}_{2j-1}(M), N(G(j + 1)\rho_1)$$

$$\forall\ \theta \in R_0(G)\ \forall\ M \in A'_{4j-1}(G).$$

Extend

$$F_{2j-1}:A_{2j-1}(G) \mapsto R_0(G)/R_0(G)^j \mapsto 0.$$

Then

$$\{F_{2j-1}(M) = 0\} \Leftrightarrow \{\eta(\theta, M) = 0 \ \forall\, \theta \in R_0(G)\}$$

so we must show F_{2j-1} is injective. We proceed by induction on j; the case $j = 1$ follows from Lemma 3.1. Let $F_{2j-1}(M) = 0$. We use Theorem 0.1 to show $M = 0$ by showing

$$M \in \ker_{2j-1}(SW, G) \cap \ker_{2j-1}(\eta, G).$$

Let Δ_1 be the Smith homomorphism corresponding to ρ_1. If

$$\delta_1 : R_0(G)/R_0(G)^{j+1} \mapsto R_0(G)/R_0(G)^j$$

is the natural projection, then

$$\delta_1 \circ F_{2j-1} = F_{2j-3} \circ \Delta_1$$

so $\Delta_1(M) = 0$ by induction. Since $w(M) = c(\rho_1)^j(M)$, the equivariant Stiefel-Whitney classes on $A_{2j-1}(G)$ are given by the representation theory. If

$$x \in H^{2j-1}(BG; \mathbf{Z}_2) \quad \text{for } j > 1,$$

then

$$x = c_1(\rho_1)y \quad \text{for } y \in H^{2j-3}(BG; \mathbf{Z}_2).$$

Since $\Delta(M)$ is the Poincaré dual of $c_1(\rho_1)$ in M,

$$x(M) = y(\Delta(M)) = 0 \quad \text{and} \quad M \in \ker_{2j-1}(SW, G).$$

Since

$$\Lambda^i((T(M) \oplus 1) \otimes \mathbf{C}) = \Lambda^i(j\rho_1 \oplus j\rho_{-1}),$$

we can express $\theta(M)$ in terms of the representation theory for $\theta \in R_0(G) \otimes \mathbf{Z}[\Lambda^i]$, so the difficulty arises from γ; we must show

$$\eta(\theta \otimes \gamma^w, M) = 0 \ \forall\, w \in \mathbf{Z}, \forall\, \theta \in R_0(G).$$

The determinant representation γ involves a square when the Spin^c structure is changed so

$$\gamma(i(N(H, j\rho_1, \rho))) = \rho_j \rho^2.$$

Define $\widetilde{F}_{2j-1,w}$ on $A'_{2j-1}(G)$ by

$$\widetilde{F}_{2j-1,w}(i(N(H, k\rho_1, \rho))) = \mathrm{ind}(\rho^{2w+1}) \cdot \alpha(\rho_{2w+1}) \in R_0(G);$$

$\widetilde{F}_{2j-1,0} = \widetilde{F}_{2j-1}$. By Lemmas 1.3, 1.4, and 1.5,

$$\eta(\theta \cdot \gamma^w, \pi(X)) = \eta(\theta \cdot \rho_j \cdot \widetilde{F}_{2j-1,w}(X), N(G, k\rho_1 \oplus \rho_{2w+1}))$$

$\forall\, X \in A'_{2j-1}(G)$. Let $\pi(M') = M$ so

$$\widetilde{F}_{2j-1}(M') \in R_0(G)^{j+1}.$$

We show $\eta(\theta \cdot \gamma^w, M) = 0$ by showing

$$\widetilde{F}_{2j-1,w}(M') \in R_0(G)^{j+1}.$$

Let $T_w(g) = g^{2w+1}$ define an outer automorphism of $H \subseteq G$. Dually, T_w defines an algebra isomorphism of $R(H)$ preserving $R_0(H)$ commuting with transfer and induction. Since

$$T_w\alpha(\rho_1) = \alpha(\rho_{2w+1}) \quad \text{and} \quad T_w(\text{ind}(\rho)) = \text{ind}(\rho^{2w+1}),$$
$$\widetilde{F}_{2j-1,w}(M') = T_w(\widetilde{F}_{2j-1}(M)) \in R_0(G)^{j+1}$$

which proves the lemma if G is cyclic.

Next let $G = Q_v$; the case $* = 1, 3$ follows from Lemma 3.2. We proceed by induction on the dimension. Let $f_{4k\pm 1}(M) = 0$ for $k > 1$; we must show

$$M \in \ker_{4k\pm 1}(SW, G) \cap \ker_{4k\pm 1}(\eta, G).$$

Since $\delta f = f\Delta$,

$$f_{4k\pm 1-4}(\Delta(M)) = 0$$

so $\Delta(M) = 0$ by induction. In dimension $4k-1$,

$$T(M) \oplus 1 = k\tau(M)$$

so $w(M) = c(\tau)^k$ and the equivariant Stiefel-Whitney numbers are given by the representation theory on $A_{4k-1}(G)$. If $x \in H^{4k-1}(BG; \mathbf{Z}_2)$, decompose

$$x = y \cdot c_2(\tau) \quad \text{for } y \in H^{4k-5}(BG; \mathbf{Z}_2).$$

Then $x(M) = y(\Delta(M)) = 0$ so $M \in \ker_{4k-1}(SW, G)$. In dimension $4k + 1$,

$$T(M) \oplus 1 = (k\tau \oplus \rho_1)(M)$$

so $w(M) = c(\tau)^k(1 + c_1(\rho_1))$. Since

$$c_1(\rho_1)^2 = c_2(\tau) \bmod 2,$$

we can express $x(M)$ for $x \in W^{4k+1}(BG)$ in the form

$$x(M) = (x_1 + x_2c_1(\rho_1))(M)$$

for the $x_i \in H^*(BG; \mathbf{Z}_2)$. We decompose

$$x_1 = c_2(\tau)^{2k}y_1 \quad \text{and} \quad x_2 = c_2(\tau)^{2k-1}y_3$$

for $y_1 \in H^1(BG; \mathbf{Z}_2)$ and $y_3 \in H^3(BG; \mathbf{Z}_2)$. By Lemma 2.7, y_3 vanishes on cyclic subgroups and hence on $A_{4k+1}(G)$ so

$$x(M) = c_2(\tau)^{2k}y_1(M) = c_2(\tau)^{2k-1}y_1(\Delta M) = 0 \quad \text{and}$$
$$M \in \ker_{4k+1}(SW, G).$$

Let $z = x$, y, or xy. By Lemma 4.2, $t_z(M) \in A_{4k\pm 1}(H)$. By Lemma 1.4,

$$\eta(\theta, t_z(M)) = \eta(\text{ind}_z\, \theta, M) = 0\ \forall\ \theta \in R_0(H_z)$$

so $f_{4k\pm1}(t_z(M)) = 0$ and $t_z(M) = 0$ since we have proved Lemma 4.3 in the cyclic case. Therefore

$$\eta(\theta, M) = 0\ \forall\ \theta \in \text{ind}_z\, R_0(H_z) \otimes R(\text{Spin}^C).$$

Since these span $R_0(G) \otimes R(\text{Spin}^C)$ by Lemma 2.8, this shows

$$M \in \ker_{4k\pm1}(\eta, G).$$

It is convenient to introduce different generators for

$$P_* = \mathbf{Z}[CP^1, CP^2, \dots, CP^{2k}, \dots]$$

analogous to the Hazewinkle generators Ω_*^U. Let $X^{4i} = CP^{2i} - (CP^1)^i$ have arithmetic genus zero and let $Q_* = \mathbf{Z}[X^{4i}]$ so $P_* = Q_*[CP^1]$. If $N = \Sigma_i N_i \times (CP^1)^i$ for $N_i \in Q_*$ having positive degree, then

$$\text{index}(1, N) = 0.$$

Let $M_i \in \tilde{\Omega}_*^{\text{Spin}^C}(BG)$ be the spherical space forms discussed in Lemmas 2.5 and 2.7 so $\{\mu(M_i)\}$ is a basis for $H_{\text{odd}}(BG; \mathbf{Z}_2)$ and so $\{M_i\}$ is Δ or Δ_1 invariant. We assume $\{M_i\}$ includes the manifolds of Lemmas 3.1 and 3.2. Let

$$U_*(G) = A_*(G) \cdot Q_* \subseteq \tilde{\Omega}_*^{\text{Spin}^C}(BG) \quad \text{and}$$

$$V_*(G) = \Sigma_i M_i \cdot \text{Tor}(\Omega_*^{\text{Spin}^C}) \subseteq \tilde{\Omega}_*^{\text{Spin}^C}(BG).$$

LEMMA 4.5. *Let G be a 2-group.*

(a) $A_*(G) = \tilde{\Omega}_*^{\text{Spin}^C}(BG)$ *for* $* = 1, 3$.

(b) $|(A_*(G) \otimes Q_*)_m| \leqq |\oplus_{p+q=m, p\equiv 1(2)} H_p(BG; \Omega_*^{\text{Spin}^C}/\textit{torsion})|$.

(c) $|V_m(G)| \leqq |\oplus_{p+q=m, p\equiv 1(2)} H_p(BG; \text{Tor}(\Omega_*^{\text{Spin}^C})|$.

Proof. (c) is immediate. In [7, see 4.2 and 5.4], we studied the K-theory groups of spherical space forms and showed:

$$|R_0(\mathbf{Z}_n)/R_0(\mathbf{Z}_n)^{k+1}| = n^k \quad \text{and} \quad |R_0(Q_\nu)/I^{k+1}| = 4 \cdot (4|Q_\nu|)^k.$$

This implies

$$|I(Q_\nu)/I(Q_\nu)^{k+1}| = (4|Q_\nu|)^k.$$

We use Lemma 4.4 to see

$$|A_{2j-1}(\mathbf{Z}_n)| = n^k, \quad |A_{4k-1}(Q_\nu)| = (4|Q_\nu|)^k,$$

$$|A_{4k+1}(Q_\nu)| = 4(4|Q_\nu|)^k$$

or equivalently by Lemmas 2.5 and 2.7 that

$$|A_{2j-1}(G)| = |\oplus_{p\leqq j} H_{2p-1}(BG; \mathbf{Z})|.$$

This implies

$$|A_*(G)| = |\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)| \quad \text{for } * = 1, 3$$

and proves (a). Finally,

$$\begin{aligned}
&|\oplus_{a+b=j} A_{2a-1}(G) \otimes Q_{2b}| \\
&= |\oplus_{c+b\leqq j} H_{2c-1}(BG; \mathbf{Z}) \otimes Q_{2b}| \\
&= |\oplus_{c+b\leqq j} H_{2c-1}(BG; Q_{2b})| \\
&= |\oplus_c H_{2c-1}(BG; \oplus_{b\leqq j-c} Q_{2b})| \\
&= |\oplus_c H_{2c-1}(BG; P_{2(j-c)})| \\
&= |\oplus_c H_{2c-1}(BG; \Omega_{2j-2c}^{\mathrm{Spin}^C}/\text{torsion})|.
\end{aligned}$$

We can now begin to construct the splitting of Theorem 0.2:

LEMMA 4.6. *Let G be a 2-group. Cartesian product gives an injective map*

$$A_*(G) \otimes Q_* \mapsto \tilde{\Omega}_*^{\mathrm{Spin}^C}(BG).$$

Then

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG) \cong A_*(G) \otimes Q_* \oplus \ker_*(\eta, G) \quad \textit{and}$$

$$\ker_*(\eta, G) = V_*(G) \oplus \Pi^{-1}E_{\mathrm{even}}(*, G) \cong \tilde{H}_*(BG; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

Proof. We first assume $G = Q_v$. The first step is to show

$$N_*(Q_v) \subseteq U_*(Q_v) + V_*(Q_v)$$

where

$$N_*(Q_v) = \Sigma_z M_z \times \Omega_*^{\mathrm{Spin}^C}$$

is as defined in Lemma 3.2. Since

$$\Sigma_z M_z \times \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C}) \subseteq V_*(Q_v)$$

by construction and since $U_*(Q_v)$ is a Q_* module, we must show

$$\Sigma_z M_z \times (CP^1)^j \subseteq U_*(Q_v) + V_*(Q_v).$$

Since $M_z \in A_*(Q_v)$ by construction, we must show

$$\tilde{\Omega}_3^{\mathrm{Spin}^C}(BQ_v) \times (CP_1)^j \in U_{2j+3}(Q_v) + V_{2j+3}(Q_v).$$

Since $U_*(Q_v) = \tilde{\Omega}_*^{\mathrm{Spin}^C}(BQ_v)$ for $* = 1, 3$ by Lemma 4.5, we proceed by induction and take $j > 0$. Let

$$M \in \tilde{\Omega}_3^{\mathrm{Spin}^C}(BQ_v)$$

and let $x = f_3(M) \in I/I^2$. Decompose $3 + 2j = 4k \pm 1$ for $k > 1$ and choose $M_1 \in A_{3+2j}(Q_v)$ so

$$f_{3+2j}(M_1) = x \cdot \alpha^{k-1} \in I^k/I^{k+1}.$$

If $\theta \in R_0(Q_v)$, then:

$$\eta(\theta, M_1) = \eta(\theta \cdot x \cdot \alpha^{k-1}, N(Q_v, (k+1)\tau))$$
$$= \eta(\theta \cdot x, N(Q_v, 2\tau)) = \eta(\theta, M).$$

Since

$$f_{2j-1}(\Delta M_1) = x \cdot \alpha^{k-1} \in I^k,$$

$f_{2j-1}(\Delta M_1) = 0$ so $\Delta M_1 = 0$ by Lemma 4.4. Since

$$M_1 \in A_{2j+3}(Q_v) \subseteq S_{2j+3}(Q_v),$$

$M_1 \in N_{2j+3}(Q_v)$ by Lemma 3.2. Let

$$M_1 = M_y \times (a_y(CP^1)^{j+1} + B_y + T_y)$$
$$+ M_{xy} \times (a_{xy}(CP^1)^{j+1} + B_{xy} + T_{xy})$$
$$+ M_q \times (a_q(CP^1)^j + B_q + T_q)$$

where $T_z \in \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})$ and where B_z are polynomials of lower degree in (CP^1) with coefficients of positive degree from Q_*. Since the generators of Q_* all have degree at least 4, (CP^1) appears to a power at most $j - 1$ in B_z. Therefore by induction,

$$\Sigma_z M_z \times (B_z + T_z) \in U_{2j+3}(Q_v) + V_{2j+3}(Q_v).$$

By Lemma 1.3 and by the choice of the generators for Q_*,

$$\eta(\theta, \Sigma_z M_z \times (B_z + T_z))$$
$$= \Sigma_z \eta(\theta, M_z) \cdot \mathrm{index}(1, B_z + T_z) = 0\ \forall\ \theta \in R_0(Q_v).$$

If

$$M_2 = a_y M_y \times CP^1 + a_{xy} M_{xy} \times CP^1 + M_q,$$

then

$$M_2 \times (CP^1)^j = M - \Sigma_z M_z \times (B_z + T_z) \in U_{2j+3}(Q_v)$$
$$+ V_{2j+3}(Q_v) \quad \text{and}$$
$$\eta(\theta, M) = \eta(\theta, M_1) = \eta(\theta, M_2 \times (CP^1)^j$$
$$= \eta(\theta, M_2)\ \forall\ \theta \in R_0(Q_v).$$

Thus $f_3(M - M_2) = 0$ and $M = M_2$ so

$$M \times (CP^1)^j \in U_{3+2j}(Q_v) + V_{3+2j}(Q_v).$$

This shows

$$N_*(Q_v) \subseteq U_*(Q_v) + V_*(Q_v).$$

We note

$$\Delta: U_*(Q_v) \mapsto U_{*-4}(Q_v) \mapsto 0 \quad \text{and} \quad \Delta: V_*(Q_v) \mapsto V_{*-4}(Q_v) \mapsto 0.$$

For $* = 1, 3$,

$$U_*(G) + V_*(G) = \widetilde{\Omega}_*^{\text{Spin}^C}(BQ_v)$$

so $S_*(G) = U_*(G) + V_*(G)$. Since

$$N_*(Q_v) \subseteq U_*(Q_v) + V_*(Q_v) \subseteq S_*(Q_v),$$

the exact sequence of Lemma 3.5

$$0 \mapsto N_*(Q_v) \mapsto S_*(Q_v) \mapsto S_{*-4}(Q_v) \mapsto 0$$

shows $U_*(Q_v) + V_*(Q_v) = S_*(Q_v)$. We count orders:

$$\begin{aligned} &|E_{\text{odd}}(m, Q_v)| \\ &= |S_m(Q_v)| = |U_m(Q_v) + V_m(Q_v)| \\ &\leqq |U_m(Q_v)| \cdot |V_m(Q_v)| \leqq |(A_* \otimes Q_*)_m| \cdot |V_m(Q_v)| \\ &\leqq |\oplus_{p+q=m, p\equiv 1(2)} H_p(BQ_v; P_q \oplus \text{Tor}(\Omega_q^{\text{Spin}^C}))| \\ &= |E_{\text{odd}}(m, Q_v)|. \end{aligned}$$

Since the inequalities are inequalities,

$$S_m(Q_v) = U_m(Q_v) \oplus V_m(Q_v), \quad U_*(Q_v) = A_*(Q_v) \otimes Q_* \quad \text{and}$$

$$\widetilde{\Omega}_m^{\text{Spin}^C}(BQ_v) = U_m(Q_v) \oplus V_m(Q_v) \oplus \Pi^{-1}E_{\text{even}}(m, Q_v).$$

$$V_m(Q_v) \oplus \Pi^{-1}E_{\text{even}}(m, Q_v) \cong \{\widetilde{H}_*(BQ_v, \text{Tor}(\Omega_*^{\text{Spin}^C}))\}_m.$$

By Lemma 1.3,

$$V_m(Q_v) \subseteq \ker_m(\eta, Q_v)$$

and by construction

$$\Pi^{-1}E_{\text{even}}(m, Q_v) \subseteq \ker_m(\eta, Q_v).$$

Let

$$M \in \ker_m(\eta, Q_v) \cap U_m(Q_v).$$

We complete the proof by showing $M = 0$. We proceed by induction so

$$\Delta(M) \in \ker_{m-4}(\eta, Q_v) \cap U_{m-4}(Q_v) = 0.$$

By Lemma 3.1,

$$M \in \Sigma M_z \times \text{Tor}(\Omega_*^{\text{Spin}^C}) \subseteq V_m(Q_v).$$

Since $U_m(Q_v) \cap V_m(Q_v) = 0$, $M = 0$. The argument is the same if G is cyclic so we omit details.

It is useful to have a more concrete isomorphism

$$\ker_*(\eta, G) \cong \tilde{H}_*(G; \Omega_*^{\mathrm{Spin}^C}).$$

Let $\{N_{a,m}\}$ be a basis for the $\mathbf{Z}_2$ vector space $\mathrm{Tor}(\Omega_m^{\mathrm{Spin}^C})$ and let $\{w_{a,m}\} \in W^m$ be a dual basis with respect to the pairing

$$W^m(BG) \otimes \mathrm{Tor}(\Omega_m^{\mathrm{Spin}^C}) \mapsto \mathbf{Z}_2.$$

Let $\{\theta_{i,m}\}$ be a basis for $\tilde{H}^m(BG; \mathbf{Z}_2)$.

LEMMA 4.7. *Let G be a 2-group. The* $\{\theta_{i,p}w_{a,q}\}_{p+q=m}$ *give a dual basis for* $\ker_m(\eta, G)$ *in the pairing*

$$W^m(BG) \otimes \tilde{\Omega}_m^{\mathrm{Spin}^C}(BG) \mapsto \mathbf{Z}_2.$$

Proof. Suppose G is cyclic. Fix m and let

$$\{\tilde{N}(a, 2p) = \Pi^{-1}(N_{a,m-2p})\}$$

be a $\mathbf{Z}_2$ basis for $\Pi^{-1}(E_{\mathrm{even}}(m, G))$ and

$$\{\tilde{N}(a, 2p - 1) = N(G, p \cdot \rho_1) \times N_{a,m+1-2p}\}$$

be a $\mathbf{Z}_2$ basis for $V_m(G)$.

The $\{\tilde{N}(a, p)\}$ form a $\mathbf{Z}_2$ basis for $\ker_m(\eta, G)$. Let

$$\theta_1 = w_1(\rho_{n/2}) \quad \text{and} \quad \theta_2 = c_1(\rho_1)$$

so $\{\theta_1\theta_2^k, \theta_2^{k+1}\}_{k\geqq 0}$ is a basis for $\tilde{H}^*(BG; \mathbf{Z}_2)$. We compute:

$$w \cdot \theta_1\theta_2^k(\tilde{N}(a, 2p)) = w \cdot \theta_1\theta_2^{k-p}(N_{a,m-2p}) = 0 \text{ for } 2k + 1 > 2p$$

$$w \cdot \theta_2^k(\tilde{N}(a, 2p)) = w \cdot \theta_2^{kp}(N_{a,m-2p}) = 0 \text{ for } 2k > 2p$$

$$w \cdot \theta_2^k(\tilde{N}(a, 2p)) = w(N_{a,m-2p}) \text{ for } 2k = 2p$$

$$w \cdot \theta_1\theta_2^{k-1}(\tilde{N}(a, 2p-1)) = w \cdot \theta_2{}^{k-p}(N_{a,m+1-2p}) = 0 \quad \text{for } 2k - 1 > 2p - 1$$

$$w \cdot \theta_1\theta_2^{k-1}(\tilde{N}(a, 2p-1)) = w(N_{a,m+1-2p}) \text{ for } 2k - 1 = 2p - 1$$

$$w \cdot \theta_2^k(\tilde{N}(a, 2p-1)) = w \cdot \theta_2^{k-p}(N_{a,m+1-2p}) = 0 \quad \text{for } 2k > 2p - 1.$$

We use the index p of $\{\theta_{i,p}w_{a,m-1}\}$ and the index q of $\{\tilde{N}(a, q)\}$ to define a partial ordering; the pairing yields an upper triangular matrix with non-zero entries on the diagonal and so is non-singular. The proof if $G = Q_v$ is slightly more messy owing to the more complicated structure of the cohomology ring but essentially the same. We omit details in the interests of brevity.

We can now complete the proof of Theorem 0.2. Let i and t denote induction and transfer corresponding to $G_2 \subseteq G$. Since i is surjective by Corollary 3.6,

$$\begin{aligned}\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} &= i\{A_*(G_2) \cdot Q_* \oplus \ker_*(\eta, G_2)\} \\ &= A_*(G) \cdot Q_* + \ker_*(\eta, G).\end{aligned}$$

If there is a non-trivial relation $\sum_i A_i Q_i + N = 0$ in this decomposition, we apply the injective map t to get a non-trivial relation in $\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG_2)$ which would contradict Lemma 4.6. Therefore

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} = A_*(G) \otimes Q_* \oplus \ker_*(\eta, G).$$

Since $|G:G_2|$ is odd, $i_* \circ t_* = 1$ on $H_*(BG; \mathbf{Z}_2)$ so

$$(t^* \otimes 1) \circ (i^* \otimes 1) = 1 \quad \text{on } W^*(BG).$$

Since $W^*(BG)$ completely detects $\ker_*(\eta, G)$, $i \circ t = 1$ on $\ker_*(\eta, G)$. Let $\{\theta_{i,m,G}\}$ be a basis for $\tilde{H}^*(BG; \mathbf{Z}_2)$ and let $\{\theta_{j,m,\perp}\}$ be a basis for $\ker(t^*)$ in $\tilde{H}^*(BG_2; \mathbf{Z}_2)$. Then $\{i^*\theta_{i,m,G}, \theta_{j,m,\perp}\}$ is a basis for $\tilde{H}^*(BG_2; \mathbf{Z}_2)$. Let $X \in \ker_m(\eta, G)$, then:

$$\begin{aligned}\{X = 0\} &\Leftrightarrow \{i^*\theta_{i,p,G} \cdot w_{a,q}(tX) = \theta_{j,p,\perp} \cdot w_{a,q}(tX) \\ &\qquad\qquad = 0 \ \forall\, i, j, a, p + q = m\} \\ &\Leftrightarrow \{t^*i^*\theta_{i,p,g} \cdot w_{a,q}(X) = 0 \ \forall\, i, j, a, p + q = m\} \\ &\Leftrightarrow \{\theta_{i,p,g} \cdot w_{a,q}(X) = 0 \ \forall\, i, j, a, p + q = m\}\end{aligned}$$

so these cohomology classes form a dual basis to $\ker_*(\eta, G)$ and show

$$\ker_*(\eta, G) \cong \tilde{H}_*(BG, \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

This proves

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} \cong A_*(G) \otimes Q_* \oplus \tilde{H}_*(BG); \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

In [**4**] we showed

$$\tilde{\Omega}_*^{\mathrm{Spin}^C}(BG)_{(2)} \cong bu_*(BG)_{(2)} \otimes Q_* \oplus \tilde{H}_*(BG; \mathrm{Tor}(\Omega_*^{\mathrm{Spin}^C})).$$

We compare these two isomorphisms to see

$$A_*(G) \cong bu_*(BG)_{(2)}$$

which proves Theorem 0.2. Since all the isomorphisms are preserved by transfer and induction, this isomorphism is functorial with respect to transfer and induction. Theorem 0.3 follows from the isomorphism of Lemma 4.4.

We conclude this paper by expressing $bu_*(BG)$ in terms of the representation theory. We clear the previous notation. Let $\tau(G):G \mapsto U(\nu)$ be a fixed point free representation of G; if τ is irreducible, then ν is independent of the particular $\tau(G)$ chosen (see [**12**]). Suppose first the Sylow subgroup G_2 is cyclic. Since

$$\alpha(\nu \cdot \rho_1)R(G_2) = \alpha(\tau)R(G_2),$$

we can choose $\omega \in R(G_2)$ so

$$\alpha(\nu \cdot \rho_1) = \omega \cdot \alpha(\tau(G_2))$$

and so that multiplication by ω is an isomorphism on $R_0(G_2)$. Let $j = \nu \cdot l + a$ for $0 \leqq a < \nu$ and let

$$\pi_l: R_0(G) \mapsto R_0(G)/\alpha(\tau)^{l+1}R(G).$$

Define

$$\tilde{F}_{2j-1}(N(G_2, j \cdot \rho_1, \rho)) = \mathrm{ind}(\omega \cdot \rho \cdot \alpha((\nu - a) \cdot \rho_1))$$

so

$$\eta(\theta, \pi(M)) = \eta(\theta \cdot \tilde{F}_{2j-1}(M), N(G,(l + 1)\tau)) \ \forall \ \theta \in R_0(G) \text{ and}$$
$$M \in A'_{2j-1}(G).$$
$$\pi_l\tilde{F}_{2j-1} = F_{2j-1}: A_{2j-1}(G) \mapsto R_0(G)/\alpha(\tau)^{l+1}R(G)$$

so

$$\{F_{2j-1}(M) = 0\} \Leftrightarrow \{\eta(\theta, M) = 0 \ \forall \ \theta \in R_0(G)\}.$$

Let $M \in A_{2j-1}(G)$ so $t(M) \in A_{2j-1}(G_2)$. Then

$$\{F_{2j-1}(M) = 0\}$$
$$\Rightarrow \{\eta(\theta, M) = 0 \ \forall \ \theta \in R_0(G)\}$$
$$\Rightarrow \{\eta(\theta, M) = 0 \ \forall \ \theta \in \mathrm{ind}\ R_0(G)\}$$
$$\Rightarrow \{\eta(\theta, tM) = 0 \ \forall \ \theta \in \mathrm{ind}\ R_0(G_2)\}$$
$$\Rightarrow \{t(M) = 0\} \Rightarrow \{M = 0\}$$
$$\Rightarrow \{\eta(\theta, M) = 0 \ \forall \ \theta \in R_0(G)\} \Rightarrow \{F_{2j-1}(M) = 0\}.$$

Thus F_{2j-1} is an isomorphism from $A_*(G)$ to its image

$$\mathrm{ind}(R_0(G_2)^{\nu-a}) \in R_0(G)/\alpha(\tau)^l R(G)).$$

The analysis is similar if $G_2 = Q_\nu$ which proves

THEOREM 4.8. *Adopt the notation given above.*
(a) *If G_2 is cyclic, decompose $j = l \cdot \nu + a$ for $0 \leqq a < \nu$. Then*

$$bu_{2j-1}(BG)_{(2)} \cong \pi_l\{\mathrm{ind}\ R_0(G_2)^{\nu-a}\}.$$

(b) *If G_2 is generalized quaternonic, $\nu = 2\mu$ is even. Decompose $k = \mu \cdot l + a$ for $0 \leqq a < \mu$. Let $\tau_1: G_2 \mapsto SU(2)$ be fixed point free.*

$$bu_{4k-1}(BG)_{(2)} \cong \pi_l\{\text{ind}\{\alpha(\tau_1)^{\nu-a}R(G_2)\}\}$$

$$bu_{4k+1}(BG)_{(2)} \cong \pi_l\{\text{ind}\{\alpha(\tau_1)^{\nu-a-1}R_0(G_2)\}\}.$$

Remark. This isomorphism is functorial with respect to transfer and induction;

$$bu_{2j\nu-3}(BG)_{(2)} \cong \tilde{K}(S^{2j\nu-1}/G)_{(2)}.$$

m The p-Sylow subgroup at odd primes is cyclic and the formula given in (a) for $p = 2$ is true at odd primes as well. The bordism spectral sequence for $BP_*(BG)$ (see **[9]**) collapses. Since this sequence factors through a bordism spectral sequence for bu_*, the bordism spectral sequence for $bu_*(BG)$ degenerates so $i \circ t = 1$ on $bu_*(BG)$. We omit details since the techniques are not analytic in nature.

References

1. D. W. Anderson, E. H. Brown and F. P. Peterson, *Spin cobordism*, Bulletin of the American Mathematics Society *72* (1966), 257-260.

2. M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Camb. Phil. Soc. *77* (1975), 43-69. *II*, Math. Proc. Camb. Phil. Soc. *78* (1975), 405-432. *III*, Math. Proc. Camb. Phil. Soc. *79* (1976), 71-99.

3. A. Bahri and P. Gilkey, *The eta invariant,* Pin^C *bordism, and equivariant* Spin^C *bordism for cyclic* 2-*groups*, Pacific Journal Math *128* (1987), 1-24.

4. ——— Pin^C *bordism and equivariant* Spin^C *bordism for cyclic* 2-*groups*, Proceedings of AMS *99* (1987), 380-382.

5. P. E. Conner and E. E. Floyd, *Differentiable periodic maps* (Springer-Verlag, 1964).

6. P. Gilkey, *The eta invariant and equivariant unitary bordism for spherical space form groups* (to appear in Compositio Math).

7. ——— *The eta invariant and the K-theory of spherical space forms*, Inventiones Math *76* (1984), 421-453.

8. ——— *Invariance theory, the heat equation, and the Atiyah-Singer index theorem* (Publish or Perish Press, 1985).

9. D. Johnson and S. Wilson, *Projective dimension and Brown-Peterson homology*, Topology *12* (1973), 327-353.

10. P. Landweber, *Complex bordism of classifying spaces*, Proceedings of the American Mathematical Society *27* (1971), 175-179.

11. R. E. Stong, *Notes on cobordism* (Princeton University Press, 1968).

12. J. A. Wolf, *Spaces of constant curvature*, 5th ed (Publish or Perish Press, 1984).

University of Oregon,
Eugene, Oregon