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GENERAL HEREDITY AND STRENGTH FOR RADICAL CLASSES

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1. Introduction. An H-relation, as introduced by Rossa and Tangeman [**4**], is a relation σ on the class of associative rings with their subrings satisfying the following conditions:

(1) $I\sigma R$ implies that I is a subring of R;

(2) if $I\sigma R$ and f is a homomorphism of R, then $(If)\sigma(Rf)$;

(3) if $I\sigma R$ and J is an ideal of R, then $(I \cap J)\sigma J$.

Puczylowski [**3**] imposes also the condition

(4) if J is an ideal of R, then $J\sigma R$.

A further condition satisfied by many familiar H-relations is the following:

(5) if f is a homomorphism from a ring R onto a ring S and $B\sigma S$, then there exists a subring A of R such that $A\sigma R$ and $Af = B$.

Conditions (4) and (5) are both satisfied by the standard H-relations 'ideal', 'left ideal', 'right ideal', 'accessible subring', 'left accessible subring', 'right accessible subring', and 'subring'.

We refer to [**1**] and [**10**] for the standard definitions and results of radical theory. For a given class C of rings, LC denotes the lower radical class generated by C and HC denotes the homomorphic closure of C. The notation $I \triangleleft R$ means that I is an ideal of R, and $GF(q)$ denotes the finite field of order q. Finally, if α is a radical class, S_α denotes the class of all α semisimple rings.

Let α be a radical class and let σ be an H-relation. In Section 2 we consider constructions for the largest σ-hereditary radical class contained in α and the smallest α-hereditary radical class which contains α. A construction of the smallest σ-strong radical class containing α is considered in Section 3 (an example of Sands [**6**] shows that the largest σ-strong radical class contained in α need not exist). Much of this work has been inspired by earlier results of E. R. Puczylowski.

2. σ-hereditary classes. A class C of rings is *σ-hereditary* if $A\sigma R \in C$ implies $A \in C$.

THEOREM 1. *The following are equivalent for any H-relation σ and any class of rings C:*

(i) $A\sigma R \in HC$ implies $A \in LC$,

(ii) LC is σ-hereditary.

Proof. One implication is obvious. The other is only a slight generalization

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of Theorem 4 is Rossa and Tangeman [**4**] and the proof, using the Kreiling-Tangeman construction [**9**], is similar to their proof.

Let C be a class of rings such that $A\sigma R \in C$ implies $A \in LC$. If C is homomorphically closed, or if σ satisfies (5), then (i) of the theorem is satisfied and so LC is σ-hereditary. However, the condition that $A\sigma R \in C$ implies $A \in LC$ is not in itself enough to guarantee that LC is σ-hereditary even when σ is an H-relation which satisfies (4), as the following example shows.

Define σ_1 as follows: $A\sigma_1 B$ if and only if A is a nonzero subring of B and $B \cong GF(4)$. It is straightforward to check that $\sigma = \sigma_1 \cup \sigma_2$ is an H-relation which satisfies (4), where σ_2 denotes 'ideal'. Let C be the class of all accessible subrings of $(GF(4)[x])/(x^2)$. Since no ring in C is isomorphic to $GF(4), A\sigma B \in C$ implies that $A \triangleleft B$ and so $A \in C$. However, $GF(4) \in HC$ and $GF(2)\sigma GF(4)$, but $GF(2) \notin LC$.

Let σ be an H-relation and let α be a radical class. Define $\alpha_0(\sigma) = \alpha$ and for integers $n \geqq 0, \alpha_{n+1}(\sigma) = \{A$: if B is a homomorphic image of A and $S\sigma B$, then $S \in \alpha_n(\sigma)\}$. We shall denote the intersection $\cap\{\alpha_n(\sigma) : n \geqq 0\}$ by $\alpha_\omega(\sigma)$.

Notice that if σ satisfies (5), then the description of $\alpha_{n+1}(\sigma)$ can be simplified:

$$\alpha_{n+1}(\sigma) = \{A : S\sigma A \text{ implies } S \in \alpha_n(\sigma)\}.$$

We also note that if σ is reflexive, $\alpha_0(\sigma) \supseteq \alpha_1(\sigma) \supseteq \dots$ is a descending chain.

In the next theorem we show that $\alpha_\omega(\sigma)$ is the largest σ-hereditary radical class contained in α, and following that we give examples to show that when $\alpha_0(\sigma) \supseteq \alpha_1(\sigma) \supseteq \dots$ is a descending chain it may stabilize at any point or not stabilize at all. When σ denotes 'ideal', 'left ideal' or 'right ideal' this construction agrees with the one considered by Puczylowski [**3**] who shows that for these three H-relations $\alpha_\omega(\sigma) = \alpha_2(\sigma)$ for all radical classes α and that in general $\alpha_1(\sigma) \neq \alpha_2(\sigma)$.

THEOREM 2. *If α is a radical class and σ an H-relation, then $\alpha_\omega(\sigma)$ is the largest σ-hereditary radical class contained in α.*

Proof. One checks, by induction, that each $\alpha_n(\sigma)$ is a radical class and then the result follows easily.

By Proposition 3 of [**4**], given any H-relation σ and positive integer n, the relation α^n is also an H-relation where $A\sigma^n R$ if and only if there exists an ascending chain of subrings

$$A = A_1\sigma A_2\sigma \dots \sigma A_n\sigma R.$$

We use σ^0 to mean equality; that is, $A\sigma^0 R$ if and only if $A = R$. Also $\bar{\sigma}$ denotes the union

$$\bigcup_{n \geqq 0} \sigma^n.$$

It is straightforward to check that if σ is an H-relation and α is a radical class, then α is σ-hereditary if and only if α is $\bar{\sigma}$-hereditary. For H-relations satisfying (5) we have the following more general result.

THEOREM 3. *If σ is an H-relation satisfying* (5), *then for any radical class α we have $\alpha_n(\sigma) = \alpha_1(\sigma^n)$ and $\alpha_\omega(\sigma) = \alpha_1(\bar{\sigma})$.*

Proof. It is easy to see that σ^n satisfies (5) since σ satisfies (5) and so we may use the simplified descriptions of both $\alpha_n(\sigma)$ and $\alpha_1(\sigma^n)$. Clearly $\alpha_0(\sigma) = \alpha_1(\alpha^0)$ and $\alpha_1(\sigma) = \alpha_1(\sigma^1)$. Assume that

$$\alpha_{n-1}(\sigma) = \alpha_1(\sigma^{n-1}) \quad \text{for some } n \geqq 2.$$

Let $R \in \alpha_n(\sigma)$ and suppose that $A\sigma^n R$. Then there is a subring B of R such that $A\sigma^{n-1}B\sigma R$. Since $R \in \alpha_n(\sigma), B \in \alpha_{n-1}(\sigma) = \alpha_1(\sigma^{n-1})$ and so $A \in \alpha$. Thus $R \in \alpha_1(\sigma^n)$.

Now suppose that $R \in \alpha_1(\sigma^n)$ and $A\sigma R$. If $B\sigma^{n-1}A$, then $B\sigma^n R$ and hence $B \in \alpha$. Thus $A \in \alpha_1(\sigma^{n-1}) = \alpha_{n-1}(\sigma)$ and so $R \in \alpha_n(\sigma)$.

It follows that $\alpha_n(\sigma) = \alpha_1(\sigma^n)$.

Finally,

$$\alpha_\omega(\sigma) = \bigcap_{n \geqq 0} \alpha_n(\sigma) = \bigcap_{n \geqq 0} \alpha_1(\sigma^n) = \alpha_1\left(\bigcup_{n \geqq 0} \sigma^n\right) = \alpha_1(\bar{\sigma}).$$

Assume that σ and α are such that the radical classes $\alpha_n(\sigma)$ form a descending chain. If $\alpha_0(\sigma) = \alpha_1(\sigma)$ we say that the construction (of the largest σ-hereditary radical class contained in α) stops at stage 0. If $\alpha_{n-1}(\sigma) \neq \alpha_n(\sigma) = \alpha_{n+1}(\sigma)$ we say that the construction stops at stage n. Of course, if the construction stops at stage n, $\alpha_n(\sigma) = \alpha_\omega(\sigma)$.

If α is σ-hereditary, the construction stops stage 0. If $\sigma = \bar{\sigma}$ and σ satisfies (5), then the construction stops at stage 1 or stage 0. This is the case when σ denotes 'accessible subring', 'left accessible subring', 'right accessible subring', or 'subring'. As we mentioned earlier, if σ denotes 'ideal', 'left ideal' or 'right ideal', Puczylowski has shown that the construction stops at stage 2, 1 or 0 and that all three cases can occur.

We define an H-relation τ by $A\tau B$ if either $A = B$ or A is a maximal subring of B. Conditions (1) and (2) are clear. Suppose that $A\tau B$ and that $K \triangleleft B$. Let C be a subring of K strictly containing $A \cap K$. Then $A \subsetneqq A + C$ and so $A + C = B$. Thus

$$K = (A + C) \cap K = (A \cap K) + C = C.$$

It follows that $(A \cap K)\tau K$ and so τ satisfies (3). Clearly τ is reflexive and satisfies (5). Let $\sigma = \tau \cup \sigma_1$ where σ_1 denotes 'ideal'. Then σ is a reflexive H-relation which satisfies (4) and (5).

Let p be a prime number and

$$\alpha = L\{GF(p^2), GF(p^4), GF(p^8), \dots, GF(p^{2^n})\}.$$

We now show that, for $0 \leqq i < n$,

$$\alpha_i(\sigma) = L\{GF(p^{2^{i+1}}), \dots, GF(p^{2^n})\}$$

and that $\alpha_n(\sigma) = \{0\}$.

The result is true when $i = 0$. Assume that it is true for $i = k$. Let $k+1 < j \leqq n$ and let $C\sigma GF(p^{2^j})$. Since a field has no proper ideals and since a subring of a finite field is a field, either $C = 0$, $C = GF(p^{2^j})$ or $C = GF(p^{2^{j-1}})$. It follows that $C \in \alpha_k(\sigma)$ and so

$$GF(p^{2^j}) \in \alpha_{k+1}(\sigma) \quad \text{for all} \quad k+1 < j \leqq n.$$

Now let $0 \neq A \in \alpha_{k+1}(\sigma)$. Then $A\sigma A$ implies $A \in \alpha_k(\sigma)$ and so A has a nonzero accessible subring which is a homomorphic image of one of the fields $GF(p^{2^j})$, $k+1 \leqq j \leqq n$. Since a nonzero homomorphic image of a field must be the field itself, and an idempotent accessible subring is an ideal, it follows that one of the fields, say K, is an ideal of A. Then K is a direct summand and hence a homomorphic image of A. Therefore $K \in \alpha_{k+1}(\sigma)$. Since

$$GF(p^{2^k})\sigma GF(p^{2^{k+1}}) \quad \text{and} \quad GF(p^{2^k}) \notin \alpha_k(\sigma),$$

$GF(p^{2^{k+1}}) \notin \alpha_{k+1}(\sigma)$. Thus $K = GF(p^{2^j})$ for some j with $k+1 < j \leqq n$. It follows that

$$\alpha_{k+1}(\sigma) = L\{GF(p^{2^{k+2}}), \dots, GF(p^{2^n})\}.$$

In particular, we have $\alpha_{n-1}(\sigma) = L\{GF(p^{2^n})\}$ and from this it is easy to see that $\alpha_n(\sigma) = \{0\}$.

Thus, in this situation, the construction stops at stage n. If we now let

$$\alpha = L\{GF(p^2), GF(p^4), GF(p^8), \dots\},$$

then, as above, we can show that

$$\alpha_k(\sigma) = L\{GF(p^{2^{k+1}}), \dots\}.$$

So in this case $\alpha_n(\sigma) \neq \alpha_{n+1}(\sigma)$ for any n and so the construction does not stop at any finite stage.

We now consider the smallest σ-hereditary radical class containing a given radical class. It is clear that such a radical class exists because an intersection of σ-hereditary radical classes is σ-hereditary.

For each class C of rings let $\sigma^0(C) = C$ and define, for $n \geqq 0$,

$$\sigma^{n+1}(C) = \{A : \text{ there is an } R \in \sigma^n(C) \text{ with } A\sigma R\}.$$

It is routine to check that, for $n, m \geqq 0$,

$$\sigma^{n+m}(C) = \sigma^n(\sigma^m(C)).$$

For a radical class α the class $\sigma(\alpha)$ need not be a radical class: let $\alpha^0 = \alpha, \alpha^1 = L\sigma(\alpha), \alpha^2 = L\sigma(\alpha^1), \ldots, \alpha^{n+1} = L\sigma(\alpha^n), \ldots$. We have the following relation between the various classes defined above.

THEOREM 4. *If α is any radical class and σ is an H-relation which satisfies* (5), *then $\alpha^n = L\sigma^n(\alpha)$ for each $n \geqq 0$.*

Proof. Of course, $L\sigma^0(\alpha) = L\alpha = \alpha = \alpha^0$. Assume that $\alpha^m = L\sigma^m(\alpha)$. Since $\sigma^m(\alpha) \subseteq \alpha^m, \sigma^{m+1}(\alpha) \subseteq \sigma(\alpha^m)$ and so

$$L\sigma^{m+1}(\alpha) \subseteq L\sigma(\alpha^m) = \alpha^{m+1}.$$

Recall from [**4**] that a class C of rings is *σ-transfer hereditary* to a class D of rings if $A\sigma R \in C$ implies $A \in D$. Note that $\sigma^m(\alpha)$ is σ-transfer hereditary to $\sigma^{m+1}(\alpha)$. Let $R \in H\sigma^m(\alpha)$ and suppose $S\sigma R$. By (5) there is a ring $A \in \sigma^m(\alpha)$ with a subring B satisfying $B\sigma A$ and such that there is a homomorphism f with $Af = R$ and $Bf = S$. Now $A \in \sigma^m(\alpha)$ implies $B \in \sigma^{m+1}(\alpha)$ which in turn implies $S \in H\sigma^{m+1}(\alpha)$. Thus $H\sigma^m(\alpha)$ is σ-transfer hereditary to $H\sigma^{m+1}(\alpha)$ and so it follows from [**4**, Theorem 9] that $LH\sigma^m(\alpha) = L\sigma^m(\alpha)$ is σ-transfer hereditary to

$$LH\sigma^{m+1}(\alpha) = L\sigma^{m+1}(\alpha).$$

Let $R \in \sigma(\alpha^m)$. Then there exists a T such that $R\sigma T$ and $T \in \alpha^m = L\sigma^m(\alpha)$. Therefore $R \in L\sigma^{m+1}(\alpha)$. Hence $\sigma(\alpha^m) \subseteq L\sigma^{m+1}(\alpha)$ and so

$$\alpha^{m+1} = L\sigma(\alpha^m) \subseteq L\sigma^{m+1}(\alpha).$$

It follows that $\alpha^{m+1} = L\sigma^{m+1}(\alpha)$, as required.

THEOREM 5. *For any H-relation σ and any radical class α the smallest σ-hereditary radical class containing α is $L(\cup\{\alpha^n : n \geqq 0\})$.*

Proof. Clearly $\cup\{\alpha^n : n \geqq 0\}$ is homomorphically closed and σ-hereditary, so $L(\cup\{\alpha^n : n \geqq 0\})$ is σ-hereditary by Theorem 1. Also, it is clear that any σ-hereditary radical class containing α must contain α^n for all n, so the result follows.

If σ is relexive we have an ascending chain of radical classes $\alpha = \alpha^0 \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^n \subseteq \ldots$. We shall say that the construction (of the smallest σ-hereditary radical class containing α) terminates at stage n if $\alpha^{n-1} \neq \alpha^n = \alpha^{n+1}$. If α is σ-hereditary so that $\alpha^0 = \alpha^1$ we shall say that the construction terminates at stage 0.

We now give results which are dual to those of Puczylowski, showing that for 'ideal', 'left ideal' and 'right ideal' the construction terminates at stage 2.

THEOREM 6. *For any radical class α, the smallest left hereditary radical class containing α is the lower radical class generated by all rings A such A is a left ideal of a ring B which is a left ideal of a ring R in α.*

Proof. The radical class defined here is $L\sigma^2(\alpha)$, where σ denotes 'left ideal', By Theorem 4 this equals α^2. By Theorem 5 it suffices to show that α^2 is left hereditary, and since $\alpha^2 = L\sigma^2(\alpha)$, it suffices to show that $L\sigma^2(\alpha)$ is left hereditary.

Suppose that $D\sigma A\sigma B\sigma R \in \alpha$. Then $(D+BA)\sigma B$ and so $D+BA \in \sigma^2(\alpha)$. Also, $(D\cap BA)\sigma BA\sigma R$, so $D\cap BA \in \sigma^2(\alpha)$. Since $BA \triangleleft A$ it follows that $(D\cap BA)\triangleleft D$. Now

$$D/(D\cap BA) \cong (D+BA)/BA \in L\sigma^2(\alpha),$$

and thus $D \in L\sigma^2(\alpha)$ since it is an extension of $D\cap BA$ (in $\sigma^2(\alpha)$) by a ring in $L\sigma^2(\alpha)$. We have shown that $D \in \sigma(\sigma^2(\alpha))$ implies that $D \in L\sigma^2(\alpha)$ and so, since σ satisfies (5), Theorem 1 implies that $L\sigma^2(\alpha)$ is σ-hereditary.

Clearly the corresponding result holds for right ideals.

THEOREM 7. *For any radical class α the smallest hereditary radical class containing α is the lower radical class generated by all rings A such that A is an ideal of a ring B which is an ideal of a ring R in α.*

Proof. As in the proof of Theorem 6 it is sufficient to show that $L\sigma^2(\alpha)$ is hereditary where σ denotes 'ideal'.

Suppose that $D\triangleleft B\triangleleft A\triangleleft R \in \alpha$. Let $E = D+AD+DA+ADA$ be the ideal of A generated by D. Then $E^3 \subseteq D$. Let $F = E+RE+ER+RER$ be the ideal of R generated by E. Then $F^3 \subseteq E$ and so $F^9 \subseteq D$. Since $F\triangleleft R, F^9\triangleleft R$ and hence

$$F^9 = D\cap F^9 \in \sigma(\alpha) \subseteq \alpha^2.$$

It follows that $F^9 \subseteq \alpha^2(D)$, the α^2 radical of the ring D. Now suppose that $D\cap F^m \subseteq \alpha^2(D)$ for some $m \geqq 2$. Because $D\cap F^m \triangleleft D\cap F^{m-1}$ and

$$(D\cap F^{m-1})/(D\cap F^m) \cong ((D\cap F^{m-1})+F^m)/F^m$$

is an ideal of $F^{m-1}/F^m \triangleleft R/F^m$,

$$(D\cap F^{m-1})/(D\cap F^m) \in \sigma^2(\alpha).$$

Thus

$$(D\cap F^{m-1})/(D\cap F^m) \in \alpha^2$$

by Theorem 4. Since α^2 is closed under extensions, $D \cap F^{m-1} \in \alpha^2$ and so

$$D \cap F^{m-1} \subseteq \alpha^2(D).$$

It follows that $D = D \cap F \subseteq \alpha^2(D)$ and hence $D \in \alpha^2$. We have shown that $D \in \sigma(\sigma^2(\alpha))$ implies $D \in L\sigma^2(\alpha)$ and so, since σ satisfies (5), it follows from Theorem 1 that $L\sigma^2(\alpha)$ is hereditary.

We now give examples to show that in these cases the construction need not terminate at stage 1 (see also [**2**]).

Let Q denote the field of rational numbers, R be the matrix ring

$$\begin{bmatrix} Q & Q \\ 0 & 0 \end{bmatrix}$$

and let $\alpha = L\{R\}$. Since they admit premultiplication by

$$\begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \quad q \in Q,$$

the left ideals and the ideals of R are divisible. Hence, in each of these cases, all rings in $\sigma(\alpha)$ and so all rings in α^1 are divisible. Now

$$\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} Q & Q \\ 0 & 0 \end{bmatrix}.$$

Therefore, in each case,

$$\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}$$

is in α^2, but it is not in α^1 because it is not divisible.

In what follows we give examples to show that, in general, the construction may terminate at any finite stage and an example where it does not terminate in a finite number of steps. We use the same H-relation σ as before; namely, $A\sigma R$ if and only if A is a maximal subring of R or $A \triangleleft R$. Let n be a positive integer and let $\alpha = L\{GF(p^{2^n})\}$. For $m \leqq n$ we have

$$\sigma^m(\{GF(p^{2^n})\}) = \{0, GF(p^{2^{n-m}}), \dots, GF(p^{2^{n-1}}), GF(p^{2^n})\}$$

and, as in the proof of Theorem 4, we can show that α^m is the lower radical class generated by these rings. Hence it follows from Theorem 1 that $\alpha^n = \alpha^{n+1}$. Since the field $GF(p^{2^{n-m}})$ is not an image of any ring in $\sigma^{m-1}(\{GF(p^{2^n})\})$, it does not belong to α^{m-1}. Thus $\alpha^{m-1} \neq \alpha^m$ for $m \leqq n$ and so in this case the construction terminates at stage n.

Let $p_1, p_2, \dots, p_n, \dots$ be an enumeration of the primes. Let

$$C = \{GF(p_n^{2^n}) : n = 1, 2, \dots\}$$

and let $\alpha = LC$. It is clear from the above results that in this case $\alpha^n \neq \alpha^{n-1}$ for any n, and hence the construction does not terminate at any finite stage.

3. σ-strong classes. Wiegandt [**11**] defines a class M to be *σ-regular* if $0 \neq A\sigma R \in M$ implies that A has a nonzero homomorphic image in M. A radical class α is said to be *σ-strong* if the corresponding semisimple class S_α is σ-regular. The proof of the following theorem is straightforward.

THEOREM 8. *Let σ be an H-relation. A radical class α is σ-strong if and only if $A\sigma R, A \in \alpha$, implies $A \subseteq \alpha(R)$.*

An example is given in [**6**] which shows that a radical class α need not contain a largest σ-strong radical class when σ denotes 'left ideal'. On the other hand, the following theorem shows that every radical class is contained in a smallest σ-strong radical class for all H-relations σ.

THEOREM 9. *If σ is an H-relation and $\alpha_i : i \in I$ are σ-strong radical classes, then $\alpha = \cap\{\alpha_i : i \in I\}$ is also σ-strong.*

Proof. Let α_i be σ-strong and let $A\sigma R$ with $A \in \alpha_i$. Let A^* be the ideal of R generated by A. Then $A\sigma A^*$ and so from Theorem 8 we see that $A \subseteq \alpha_i(A^*)$. From the Anderson-Divinsky-Sulinski theorem we know that $\alpha_i(A^*) \triangleleft R$ and so $A \subseteq \alpha_i(A^*) \subseteq A^*$ implies that $A^* = \alpha_i(A^*) \in \alpha_i$.

Now let $\alpha = \cap\{\alpha_i : i \in I\}$ where each α_i is σ-strong. Let $A\sigma R$ with $A \in \alpha$. Then $A \in \alpha_i$ for each $i \in I$ and, as above, $A^* \in \alpha_i$ for each $i \in I$. Thus $A^* \in \alpha$ and since $A^* \triangleleft R$ we have $A^* \subseteq \alpha(R)$. Therefore $A \subseteq \alpha(R)$ and, by Theorem 8, α is σ-strong.

The fact that any radical class is contained in a smallest σ-strong radical class has been previously observed by Puczylowski [**3**], who shows that any class C contains a largest σ-regular class and that if C is a semisimple class then so too is this largest σ-regular class contained in it. Also, he gives a construction for the largest σ-regular class contained in a semisimple class. However, we believe that this construction need not terminate by taking an intersection over just the integers. We now give an example to justify this.

We recall that the construction is as follows. Let $M = M_0$ be a class of rings. Assuming that M_n has been defined, let $M_{n+1} = \{A \in M_n : 0 \neq P\sigma A$ implies that P has a nonzero homomorphic image in $M_n\}$. Let

$$M_\omega = \cap\{M_n : n \geqq 0\}.$$

Let E be the algebraic closure of $GF(p)$ and let F be the subfield of E which is the union of the subfields $GF(p^{2^n}), n = 1, 2, \ldots$. Let T be the ring of all finite rank linear transformations of a vector space V of countable dimension over F. We shall represent T as the ring of bounded row finite matrices with entries from F, indexed by the positive integers. Let U_n be the subring of T consisting of all matrices with entries in position (n, n) taken from $GF(p^{2n})$ and with all other entries zero. Let U be the sum of the subrings U_n. Note that

$$U_n \cong GF(p^{2^n}) \quad \text{and} \quad U \cong \oplus\{GF(p^{2^n}) : n \geqq 1\}.$$

We now define H-relations as follows:

$A\sigma_1 B$ if and only if B is finite and either A is a maximal subring of B or $A = B$;

$A\sigma_2 B$ if and only if either there is an isomorphism f from B onto T such that $Af = U$, or $A = B = 0$;

$A\sigma_3 B$ if and only if $A \lhd B$.

That σ_2 is an H-relation follows because T is a simple ring. Now let $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$. Then σ is an H-relation satisfying (4).

Let M be the semisimple class generated by

$$\{T, GF(p^2), GF(p^4), \ldots, GF(P^{2^n}), \ldots\}.$$

As in previous examples, using the fact that

$$GF(p^{2^{k-1}})\sigma GF(p^{2^k}),$$

one may show that $GF(p^{2^k})$ is in M_n for $n < k$, but is not in M_n for $n \geqq k$. Now $0 \neq A\sigma T$ implies that $A \cong T$ or $A \cong U$. Since $GF(p^{2^n})$ is a homomorphic image of U for each $n \geqq 1$, it follows that $T \in M_n$ for all n and so $T \in M_\omega$. Now $U\sigma T$ and the only nonzero homomorphic images of U are direct sums of subsets of $\{GF(p^{2^n}) : n \geqq 1\}$. Let S be such a homomorphic image and let k be least among integers n such that $GF(p^{2^n})$ occurs in the direct sum decomposition of S. If $S \in M_\omega$, then $S \in M_{k+1}$, in which case $GF(p^{2^k}) \lhd S$ implies that $GF(p^{2^k})$ has a nonzero homomorphic image in M_k, which is false. Thus U has no nonzero homomorphic image in M_ω and so M_ω is not σ-regular.

We can show, using examples similar to those already presented, that this construction may stop at any finite stage, but we omit details since this does not cover all possible situations in this case. More positively, we shall show that if this construction is continued for all ordinals, then the largest σ-regular subclass is obtained.

Let $M_0 = M$ and assume that M_μ has been defined for an ordinal μ. We define $M_{\mu+1} = \{A \in M_\mu : 0 \neq P\sigma A$ implies that P has a nonzero homomorphic image in $M_\mu\}$. If M_μ has been defined for all ordinals μ less than a limit ordinal λ, then we define

$$M_\lambda = \cap\{M_\mu : \mu < \lambda\}.$$

Finally, let $M^* = \cap M_\mu$ where the intersection is taken over all ordinals μ.

THEOREM 11. *If M is any nonempty class of rings and σ is an H-relation, then M^* is the largest σ-regular class contained in M.*

Proof. It is straightforward to check that if C is a σ-regular subclass of M, then $C \subseteq M^*$.

It remains to show that M^* is σ-regular. Let $0 \neq P\sigma A \in M^*$. Since P has only a set of ideals there exists an ordinal δ such that the ideals of P may be indexed by the ordinals less than δ. Let $I_\mu, \mu < \delta$ be the ideals of P other than P itself. For each $\mu < \delta$, let $T(\mu)$ be the class of ordinals υ such that P/I_μ is in M_υ. If $T(\mu)$ is the class of all ordinals for any μ then $P/I_\mu \in M^*$, as required. If not, for each $\mu < \delta$, there exists an ordinal η_μ such that $\upsilon \in T(\mu)$ implies $\upsilon \leqq \eta_\mu$. Since the ordinals $\mu < \delta$ form a set there exists an ordinal η with $\eta > \eta_\mu$ for all $\mu < \delta$. Since $A \in M^* \subseteq M_{\eta+1}$ we have P/I_μ is in M_η for some ideal I_μ of P. This implies that $\eta \in T(\mu)$, which contradicts $\eta > \eta_\mu$. It follows that M^* is σ-regular.

When M is a semisimple class it follows from the result of Puczylowski [**3**] that M^* is a semisimple class. So if α is a radical class with semisimple class M, the radical class corresponding to M^* is the smallest σ-strong radical class containing α.

Of course, every radical class is σ-strong when σ denotes 'ideal' so in this case $M^* = M_0$ for every semisimple class M. When σ denotes 'subring', $M^* = M_1$ and the σ-strong radical classes are precisely the strict radical classes (see [**8**]). Puczylowski [**3**] asks whether this construction stops at some finite stage when σ denotes 'left ideal'. We have not even been able to show that it stops at M_ω in this case, but we do not have a negative answer to his question. Indeed, we shall now show that, in this case, $M^* = M_1$ if a relatively weak hereditary condition is satisfied by the radical class corresponding to M.

THEOREM 12. *Let α be a radical class satisfying $R \in \alpha$ implies $R^2 \in \alpha$, and let $M = S_\alpha$. Then, in the above construction, with σ denoting 'let ideal', we have $M^* = M_1$.*

Proof. Let α_1 be the upper radical class determined by M_1, and let α_2 be the upper radical class determined by M_2. Since $M \supseteq M_1 \supseteq M_2$ we have $\alpha \subseteq \alpha_1 \subseteq \alpha_2$, and as Puczylowski [**3**] has shown that M_1 and M_2 are semisimple classes it suffices to show that $\alpha_1 = \alpha_2$. Now $R \in \alpha_1$ if and only if R has no nonzero image in M_1 and this is so if and only if every nonzero homomorphic image of R has a nonzero left ideal in α. Similarly, $R \in \alpha_2$ if and only if every nonzero homomorphic image of R has a nonzero left ideal in α_1.

Let $0 \neq R \in \alpha_2$. Then there exists a nonzero left ideal A of R with $A \in \alpha_1$ and so a nonzero left ideal B of A with $B \in \alpha$. Then AB is a left ideal of R and $AB \lhd B$. We claim that $AB \in \alpha$. Let $a \in A$ and consider the mapping $x \longrightarrow ax + B^2$ from B to AB/B^2. This mapping is a ring homomorphism and so $B \in \alpha$ implies aB/B^2 is in α. Since AB/B^2 has trivial multiplication, summing these ideals over all $a \in A$ shows that AB/B^2 is in α. By our assumption on α, $B \in \alpha$ implies $B^2 \in \alpha$ and so it follows that $AB \in \alpha$.

If $AB \neq 0$ we have a nonzero left ideal of R belonging to α, as required. Otherwise we may assume that $AB = 0$ for all left ideals B of A with $B \in \alpha$.

Then for each such left ideal B and each $a \in A$ the mapping $x \longrightarrow xa$ from B to Ba is a ring homomorphism and so $Ba \in \alpha$. Summing over all $a \in A$ we have $BA \in \alpha$. Now B and BA are ideals of $B + BA$ which are in α and hence $B + BA$ is in α. Since $B + BA$ is an ideal of A,

$$B \subseteq B + BA \subseteq \alpha(A).$$

Thus we have shown that if B is a left ideal of A with $B \in \alpha$, then $B \subseteq \alpha(A)$. It follows that $A/\alpha(A)$ has no nonzero left ideals in α and so, since $A \in \alpha_1, \alpha(A) = A$. Thus in both cases R has a nonzero left ideal in α. It follows that $R \in \alpha_1$. Thus $\alpha_1 = \alpha_2$ and the proof is complete.

We conclude with a theorem about upper radical classes which is related to results of Sands [**5**] and Wiegandt [**11**]. Recall that if A is a $\triangleleft$-regular class of rings,

$$UA = \{R : R \text{ has no nonzero homomorphic images in } A\}$$

is the upper radical class determined by A.

THEOREM 13. *Let A be a $\triangleleft$-regular class of rings, let σ be any H-relation and let B be any class of rings. Suppose that $0 \neq S\sigma R \in A$ implies that S has a nonzero homomorphic image in B. If $\alpha = UA$, then $0 \neq S\sigma R \in S_\alpha$ implies that S has a nonzero homomorphic image in B.*

Proof. As has been pointed out in [**7**] the upper radical construction of [**5**] still produces the smallest semisimple class S_α containing A when starting from a $\triangleleft$-regular class $A = A_1$. This construction proceeds as follows: if $\mu \geqq 1$ is any ordinal, $A_{\mu+1} = \{R : R \text{ has an ideal } I \in A_\mu \text{ and } R/I \text{ is in } A_\mu\}$; if λ is a limit ordinal, $A_\lambda = \{R : R \text{ contains a descending chain of ideals } I_i \text{ such that } \cap I_i = 0$ and each R/I_i is in some $A_{\mu_i}, \mu_i < \lambda\}$; S_α is then the union of all the A_μ.

We prove the result by transfinite induction. Suppose that $0 \neq A\sigma R \in A_\mu$ implies that A has a nonzero homomorphic image in B. Let $0 \neq S\sigma T \in A_{\mu+1}$. Then there is an ideal K of T such that K and T/K are both in A_μ. If $S \subseteq K$, then $S\sigma K$ and so S has a nonzero homomorphic image in B. If $S \not\subseteq K$, then

$$0 \neq ((S + K)/K)\sigma(T/K)$$

and so $(S + K)/K$, and hence S, has a nonzero homomorphic image in B. Now suppose that λ is a limit ordinal and $0 \neq A\sigma R \in A_\mu$ implies that A has a nonzero homomorphic image in B holds for all ordinals $\mu < \lambda$. Let $T \in A_\lambda$. Then T has a descending chain of ideals I_i such that $\cap I_i = 0$ and $T/I_i \in A_{\mu_i}$ for some $\mu_i < \lambda$. Let $0 \neq S\sigma T$. Since $\cap I_i = 0$ there is some j such that $S \not\subseteq I_j$. As above it follows that S has a nonzero homomorphic image in B.

The result now follows by transfinite induction.

COROLLARY 14. *If A is a $\triangleleft$-regular class of rings, then UA is σ-strong if and only if A is σ-regular.*

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Proof. Taking $B = A$ in the theorem we see that if A is σ-regular, then UA is σ-strong. The converse is due to Wiegandt [**11**, Proposition 3].

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