CHARACTER DEGREES AND DERIVED LENGTH IN p-GROUPS

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1. Introduction. There are a number of theorems which bound $d.l.(G)$, the derived length of a group G, in terms of the size of the set $c.d.(G)$ of irreducible character degrees of G assuming that G is in some particular class of solvable groups ([**1**], [**3**], [**4**], [**7**]). For instance, Gluck [**4**] shows that $d.l.(G) \leq 2\,|c.d.(G)|$ for any solvable group, whereas Berger [**1**] shows that $d.l.(G) \leq |c.d.(G)|$ if G has odd order. One of the oldest (and smallest) such bounds is a theorem of Taketa [**7**] which says that $d.l.(G) \leq |c.d.(G)|$ if G is an M-group. Most of the existing theorems are an attempt to extend Taketa's bound to all solvable groups. However, it is not even known for M-groups whether or not this is the best possible bound. This suggests that given a class of solvable groups one might try to find the maximum derived length of a group with n character degrees (i.e. the best possible bound). In fact there might be bounds related to properties of $c.d.(G)$ other than size hence it seems reasonable to pose the following general goal:

PROBLEM. *Given a finite set X of positive integers which occurs as $c.d.(G)$ for some solvable group G, compute the maximum derived length among solvable groups which have X as their set of character degrees.*

Based on our experience one might also wish to consider this question for particular classes of solvable groups.

In this paper we will show that within the class of p-groups there are some bounds which are better than those implied by Taketa's theorem.

THEOREM A. *Let $n \geq 3$ be an integer and G a finite p-group with irreducible character degrees less than or equal to p^n. Then $d.l.(G) \leq n$.*

The only case of real interest here is that of $c.d.(G) = \{1, p, p^2, \ldots, p^n\}$ for which Taketa's theorem tells us $d.l.(G) \leq n + 1$.

While results on co-prime group actions are an important tool in the study of solvable groups, those techniques are not available within p-groups. Consequently it has been necessary to develop some theorems about actions of p-groups on p-groups. Theorem A will be obtained as an immediate corollary of the following theorem on orbit sizes in p-group actions.

THEOREM B. *Let p be an odd prime and $n \geq 3$ an integer and let G be a p-group acting on an abelian p-group A such that every G-orbit in A has size at most p^n. Then $G^{(n-1)}$ acts trivially on A.*

As we note in Proposition 2.1, it is easy to show that $G^{(n)}$ acts trivially. The body of the paper is devoted to the proof of Theorem B which is of interest apart from the character theory. We also present some examples which indicate some of the boundaries around these two theorems.

This paper grew out of an unsuccessful attempt to construct a p-group of derived length 4 with 4 irreducible character degrees. Based on previous experience, it seemed most likely that such an example would be found with degrees $1, p, p^2$, and p^3. Since the example does not exist in this case, the author believes that any p-group with 4 irreducible character degrees has derived length at most 3. However, as of this writing, this has not been proved.

As far as the best bound for derived length of p-groups, clearly any group with 2 degrees has derived length 2, and 3 is the best bound for $\{1, p, p^2\}$ and $\{1, p, p^2, p^3\}$ (as proved below). If we let Y_n denote the set $\{1, p, p^2, \ldots, p^n\}$ then we note that the argument in Corollary 2.13 shows that any bound on p-groups with degrees Y_n implies a bound for p-groups with degrees Y_m, $m > n$. In particular, if $c.d.(G) = Y_n$ implies $d.l.(G) \leq b$, then $c.d.(G) = Y_{n+i}$ implies $d.l.(G) \leq b + i$ for all $i > 0$. Since the methods in this paper can surely be pushed harder, this is not the end of the story even for the sets Y_n.

The author wishes to mention that the group theory computer system CAYLEY [**2**] was instrumental in the discovery and proof of the results in this paper. (This involvement is partially documented in [**6**].) Special thanks also go to M. Isaacs and S. Larson for helpful discussions.

2. p-Group actions. The first proposition in this section is primarily intended to highlight some simple facts which we will build on.

PROPOSITION 2.1. *Let G be a p-group acting faithfully on a set Ω such that every G-orbit in Ω has size at most p^n. Then G has derived length at most n.*

Proof. Let $\mathcal{O}$ be a G-orbit in Ω and consider the action of G on $\mathcal{O}$. This action defines a homomorphism from G into the symmetric group S_{p^n}. Since any Sylow p-subgroup of S_{p^n}, has derived length n, we see that $G^{(n)}$ must be in the kernel of the action on Ω. The facts that $\mathcal{O}$ was arbitrary and G acts faithfully on Ω give us $G^{(n)} = 1$ as claimed.

As a Sylow p-subgroup of S_{p^n} demonstrates, there is no hope of improving the conclusion of this proposition. However, our main theorem will show that if Ω is replaced by an abelian p-group (with G acting via automorphisms) then the derived length of G is more tightly controlled.

The following definition provides our basic tool for studying the action of G.

DEFINITION 2.2. *Let a p-group G act on an abelian p-group A. If $\mathcal{O}$ is a G-orbit in A such that the elements of $\mathcal{O}$ generate A, then we will say that $\mathcal{O}$ is a generating orbit.*

We note that the action of G on a generating orbit induces a homomorphism into an appropriate symmetric group. In particular, if G acts faithfully on A, then this map must be an embedding, consequently our first lemma lists some properties of p-subgroups of symmetric groups.

LEMMA 2.3. *Let Γ be a Sylow p-subgroup of the symmetric group on p^2 symbols. Let $B_1, \ldots, B_p$ be blocks of imprimitivity for the action of Γ and K be the kernel of the action on these blocks. If $P \subseteq \Gamma$ is a non-abelian subgroup, then*

(a) $P \nsubseteq K$ and

(b) $Z_2(\Gamma) \subseteq P$ (*where* $Z_2(\Gamma)/Z(\Gamma) = Z(\Gamma/Z(\Gamma))$).

Proof. First note that if $p = 2$, we must have $P = \Gamma$ and so we may assume that p is odd. Also K is an elementary abelian group of index p in Γ and so $P \nsubseteq K$.

Let $P_1 = P \cap K$. Then $P_1 \lhd P$ and $P_1 \lhd K$, hence $P_1 \lhd \Gamma$. Since $|Z(\Gamma)| = p$, it follows that $Z(\Gamma) \subseteq P_1$. Furthermore, since $|P : P_1| = p$ and P is non-abelian, we must have $P_1 > Z(\Gamma)$. Finally, the fact that $|Z_2(\Gamma) : Z(\Gamma)| = p$ implies that $Z_2(\Gamma) \subseteq P_1 \subseteq P$.

We now wish to restrict our attention to $p \geq 3$ and establish some notation for the next few results. Let p be an odd prime and P be a non-abelian p-group acting faithfully on an abelian p-group U (written additively) with generating orbit $\mathcal{O}$ of size p^2. The orbits of P' on $\mathcal{O}$ form p blocks of imprimitivity for P which we label $B_1, \ldots, B_p$. Since $\mathcal{O}$ is a single P-orbit, we know that P does not fix these blocks and so we can choose the labelling such that some element of P cyclically permutes the blocks $B_1 \rightarrow B_2 \rightarrow \ldots \rightarrow B_p \rightarrow B_1$. If we choose an element x of P which maps into $Z(\Gamma)$, then we can order the elements of $\mathcal{O}$ so that x performs a cyclic shift within each block. We will write u^+ to denote u^x and $u^{(i)}$ to denote u^{x^i} for all $u \in \mathcal{O}$. Lemma 2.3 gives us the following computational facts:

(CF1) There is an element of P which cyclically permutes the blocks B_i.

(CF2) For each i, $0 \leq i < p$, there is an element of P which maps $u \rightarrow u^{(i)}$ for all $u \in \mathcal{O}$.

(CF3) Given m, n, i there is an element of P which fixes B_m element-wise and maps $u \rightarrow u^{(i)}$ for all $u \in B_n$.

The last two facts follow from the facts that $Z(\Gamma) \subseteq P$ and $Z_2(\Gamma) \subseteq P$ respectively.

LEMMA 2.4. *Let P, U, $\mathcal{O}$, and B_i be as above and let $a \in B_1$ and $b \in B_2$. The P-orbit of $a + b$ is exactly the set*

$$\{a' + b' \mid a' \in B_i,\, b' \in B_{i+1} \text{ (subscripts read mod } p)\}.$$

Proof. Let $a' \in B_i$ and $b' \in B_{i+1}$ for some i. By CF1, some element takes B_1 to B_i and B_2 to B_{i+1}. By CF2, we can then map a to a' and finally CF3 allows us to move b to b' while leaving a' fixed. Thus the above set is contained in a single P-orbit. To see that the P-orbit is no larger, note that P simply acts on the blocks B_i like a cyclic group of order p. Hence, if $a^g \in B_j$ for some j, then we must have $B_1^g = B_j$ and so $b^g \in B_2^g = B_{j+1}$. Thus $(a + b)^g = a^g + b^g$ is in the above set for all $g \in G$.

We now wish to study the case in which these sums are not all distinct.

LEMMA 2.5. *Let P, U, $\mathcal{O}$, and B_i be as above. Suppose we have subscripts r and s, and distinct elements a_r, $b_r \in B_r$ and a_s, $b_s \in B_s$ such that*

$$a_r + a_s = b_r + b_s.$$

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Then the element $a_r - b_r$ is fixed under the action of P.

Proof. We can write

$$b_r = a_r^{(i)} \quad \text{and} \quad b_s = a_s^{(j)}$$

for some i, j with $0 < i, j < p$. Hence we have

$$a_r + a_s = a_r^{(i)} + a_s^{(j)}$$

or

$$a_s^{(j)} = a_s + (a_r - a_r^{(i)}).$$

Now by CF3, we can spin B_s while fixing B_r which yields

$$\begin{aligned} a_s^{(2j)} &= a_s^{(j)(j)} = a_s^{(j)} + (a_r - a_r^{(i)}) = a_s + 2(a_r - a_r^{(i)}) \\ a_s^{(3j)} &= a_s + 3(a_r - a_r^{(i)}) \\ &\vdots \end{aligned}$$

and so on. Likewise we can rotate B_r while fixing B_s. Note that if we fix B_s, we fix a_s and $a_s^{(j)}$ and so fix $(a_r - a_r^{(i)})$. Consequently starting with

$$a_r^{(i)} = a_r - (a_r - a_r^{(i)})$$

we obtain

$$\begin{aligned} &a_r^{(2i)} = a_r - 2(a_r - a_r^{(i)}) \\ &\vdots \\ &\textit{etc.} \end{aligned}$$

Note that since $a_r^{(p)} = a_r$, the element $a_r - a_r^{(i)}$ must have order p.

Now consider a shift in P (by CF1) taking B_r to B_s. Then

$$a_r \to a_s^{(mj)} \quad \text{and} \quad a_r^{(i)} \to a_s^{(nj)}$$

for some integers m and n with $0 \leq m,\ n < p$ and $m \neq n$. Thus

$$a_r - a_r^{(i)} \to a_s^{(mj)} - a_s^{(nj)} = (m-n)(a_r - a_r^{(i)}).$$

This tells us that the shift map normalizes, and so centralizes, the subgroup $(a_r - a_r^{(i)})$ of order p.

Similarly, any element of P which stabilizes the blocks maps

$$a_r \to a_r^{(ki)} \quad \text{and} \quad a_r^{(i)} \to a_r^{(li)}$$

hence

$$a_r - a_r^{(i)} \to (l-k)(a_r - a_r^{(i)})$$

and so $a_r - a_r^{(i)} = a_r - b_r$ is fixed by P as claimed.

This provides enough information that we can now describe an action with generating orbit of size p^2.

PROPOSITION 2.6. *Let p be an odd prime and P be a non-abelian p-group acting faithfully on an abelian p-group U (written additively) with generating orbit $\mathcal{O}$ of size p^2. Then either*

(1) *P has an orbit of size p^3 on U consisting of sums of pairs of elements in $\mathcal{O}$.*

or

(2) *P fixes an element $u \in U$ such that some of the elements in $\mathcal{O}$ are identified in $U/\langle u\rangle$.*

Proof. Let $B_1, \ldots, B_p$ be blocks of imprimitivity for the action of P on $\mathcal{O}$ as above and let

$$S = \{a_1 + a_2 \mid a_1 \in B_1 \text{ and } a_2 \in B_2\}.$$

If $|S| < p^2$, then we have

$$a_1 + a_2 = b_1 + b_2$$

for some distinct elements $a_1, b_1 \in B_1$ and $a_2, b_2 \in B_2$ and so by Lemma 2.5, the element $u = a_1 - b_1$ is P-fixed and clearly suffices for conclusion 2 in the proposition. Thus we may assume that $|S| = p^2$.

On the other hand, Lemma 2.4 tells us that S lies in a single P-orbit. If this orbit is strictly larger than S, then it must have size p^3 and we are done, hence we may assume that S is a complete P-orbit.

Now choose $a_2 \in B_2$ and $a_3 \in B_3$. By the preceding discussion and Lemma 2.4, we must have $a_2 + a_3 \in S$, say

$$a_2 + a_3 = a_1 + a_2^{(i)}$$

for some $a_1 \in B_1$ and some i, $0 < i < p$. By CF3, we can act on this equation by an element of P which fixes B_2 and spins B_3 giving

$$a_2 + a_3^+ = a_1^* + a_2^{(i)}$$

where $a_1^* \in B_1$. Subtracting these two equations, we get

$$a_3^+ - a_3 = a_1^* - a_1$$

or

$$a_3^+ + a_1 = a_3 + a_1^*.$$

It follows from Lemma 2.5 that $u = a_3^+ - a_3$ is fixed by P, which completes the proof.

We will need the following lemma whose proof is an easy application of the Three Subgroups Lemma.

LEMMA 2.7. *Let G act faithfully on a group H and $K \lhd H$ admit G. If G acts trivially on K and on H/K, then G is abelian.*

The next lemma relates the existence of fixed elements to the derived length of the acting group.

LEMMA 2.8. *Let G be an odd p-group acting faithfully on an abelian p-group A with generating orbit $\mathcal{O}$ of size p^3. Let $a \in A$ be G-fixed such that the image of $\mathcal{O}$ in $\bar{A} = A/\langle a \rangle$ has size p^2 and let $\bar{G}$ be the image of G acting faithfully on $\bar{A}$ (so that $\bar{\mathcal{O}}$ is a generating orbit for $\bar{A}$ under the action of $\bar{G}$). Suppose that $\bar{G}$ fixes an element $\bar{b} \in \bar{A}$ such that some of the elements in $\bar{\mathcal{O}}$ are identified in $A/\langle \bar{b} \rangle$. Then G is metabelian.*

Proof. Let $K_1 \subseteq G$ be the kernel of the action of G on $\bar{A}/\langle \bar{b} \rangle = A/\langle a, b \rangle$ and $K_2 \subseteq G$ be the kernel of the action of G on $\langle a, b \rangle$ (i.e. $K_2 = C_G(\langle a, b \rangle)$). If we let $N = K_1 \cap K_2$, then N acts trivially on each of $A/\langle a, b \rangle$ and $\langle a, b \rangle$ and so by Lemma 2.7, N is abelian.

To see that G/N is abelian, first note that G/K_1 acts faithfully on $A/\langle a, b \rangle$ which by hypothesis has a generating orbit of size p. Since this means that G/K_1 is isomorphic to a p-subgroup of S_p, we see that $G' \subseteq K_1$. On the other hand, G/K_2 acts faithfully on $\langle a, b \rangle$, yet by the choice of a and b, acts trivially on $\langle a, b \rangle / \langle a \rangle$ and $\langle a \rangle$. By Lemma 2.7 again we see that $G' \subseteq K_2$. Hence $G' \subseteq K_1 \cap K_2 = N$ and so G is metabelian as claimed.

Our method of proof actually proves the following somewhat technical sounding proposition which has the main theorem as an immediate corollary. The notation $\mathcal{O} + \mathcal{O}$ below is used as shorthand for the set of all elements $x + y$ with $x, y \in \mathcal{O}$.

PROPOSITION 2.9. *Let p be an odd prime and $n \geq 3$. Suppose G is a p-group acting faithfully on an abelian p-group A with generating orbit $\mathcal{O}$ such that $|\mathcal{O}| \leq p^n$ and every G-orbit in $\mathcal{O} + \mathcal{O}$ has size $\leq p^n$. Then G has derived length at most $n - 1$.*

Proof. We will prove the proposition by induction on n, however most of the proof will be the same for all cases $n \geq 3$. Fix $n \geq 3$.

Let G acting on A be a counterexample with $|A|$ as small as possible. Note that if $|\mathcal{O}| = p^k$ then G is isomorphic to a p-subgroup of S_{p^k} and so G has derived length $\leq k$. Consequently, in any counterexample we must have $|\mathcal{O}| = p^n$ and $d.l.(G) = n$. Let $w \in A$ be a G-fixed element of order p. Then by the minimality of A, we see that $G^{(n-1)}$ must act trivially on $\bar{A} = A/\langle w \rangle$ and so fixes each coset of $\langle w \rangle$ in A setwise. If we choose $x \in G^{(n-1)} \cap Z(G)$ with $x \neq 1$, then x must move some element of $\mathcal{O}$, however it can at most add a multiple of w and we may assume that x adds exactly w to some element in $\mathcal{O}$. Since x is central and G is transitive on $\mathcal{O}$ the action of x must add w to every element in $\mathcal{O}$ (we will need this later).

Now let K be the kernel of the action of G on $\bar{A}$ and $\bar{G} = G/K$. Since K acts trivially on $A/\langle w \rangle$ and on $\langle w \rangle$, we see that K is abelian (by Lemma 2.7) and so $d.l.(\bar{G}) \geq n - 1$. Furthermore $\bar{G}$ acts faithfully on $\bar{A}$ with generating orbit $\bar{\mathcal{O}}$ and we see by the action of x above that some elements of $\mathcal{O}$ differ only by w. Thus $|\bar{\mathcal{O}}| < |\mathcal{O}|$ and so $|\bar{\mathcal{O}}| \leq p^{n-1}$. We claim that $\bar{\mathcal{O}} + \bar{\mathcal{O}}$ must contain a $\bar{G}$-orbit of size $\geq p^n$. If $n > 3$, this follows from our inductive hypothesis.

If $n = 3$, we have shown that $\bar{G}$ is a non-abelian p-group acting faithfully on the abelian p-group $\bar{A}$ with generating orbit of size p^2. By Proposition 2.6, we have the claim unless $\bar{G}$ fixes some element $\bar{u} \in \bar{A}$ such that elements of $\bar{\mathcal{O}}$ are identified in $\bar{A}/\langle \bar{u} \rangle$. But in this case we are in the setting of Lemma 2.8 which implies that G has derived length 2 contrary to our choice of a counterexample.

Thus in either case, we see that $\bar{\mathcal{O}} + \bar{\mathcal{O}}$ contains a $\bar{G}$-orbit of size $\geq p^n$. However, the element x above adds $2w$ to each element of the pre-image of this orbit assuring that the complete pre-image forms a single G-orbit. Since this pre-image necessarily has size $\geq p^{n+1}$ we have shown that no counter-example to the proposition can exist.

THEOREM 2.10. *Let p be an odd prime and $n \geq 3$ an integer and let G be a p-group acting on an abelian p-group A such that every G-orbit in A has size at most p^n. Then $G^{(n-1)}$ acts trivially on A.*

Proof. If the theorem is not true, choose a counterexample G acting on A such that $|G| + |A|$ as small as possible. Choose an orbit $\mathcal{O}$ such that $G^{(n-1)}$ acts nontrivially on $\mathcal{O}$. Then by the minimality, $\mathcal{O}$ is a generating orbit for A and G acts faithfully on A. But the hypotheses of Proposition 2.9 are clearly satisfied implying that $G^{(n-1)} = 1$ contrary to our choice. This contradiction completes the proof.

It is now a simple matter to settle the character theoretic question which motivated this work.

THEOREM 2.11. *Let p be an odd prime, $n \geq 3$ an integer, and G a finite p-group with irreducible character degrees less than or equal to p^n. Then the derived length of G is at most n.*

Proof. Choose $A \subseteq G$ to be self-centralizing abelian normal. Now if $\lambda \in \mathrm{Irr}(A)$ lies under $\chi \in \mathrm{Irr}(G)$, then since $\chi(1) \leq p^n$, Clifford's theorem implies that the G-orbit of λ has size at most p^n. Hence in the action of G on $\hat{A} = \mathrm{Irr}(A)$, we see that every G-orbit has size at most p^n. By Theorem 2.10, $G^{(n-1)}$ acts trivially on $\hat{A}$ and so on A. But A is self-centralizing and so $G^{(n-1)} \subseteq A$ is abelian, telling us that $d.l.(G) \leq n$ as claimed.

Our final topic in this section will be extending Theorem 2.11 to include the case $p = 2$. The major obstacle to this extension is the fact that Theorem 2.10 is *not* true for $p = 2$ (as shown in Example 3.3), however, the character theoretic fact can be approached more directly for $p = 2$.

PROPOSITION 2.12. *Let G be a finite 2-group with irreducible character degrees less than or equal to* 8. *Then the derived length of G is at most* 3.

Proof. Let $A \subseteq G$ be self-centralizing abelian normal and $\lambda \in \mathrm{Irr}(A)$. As above, if λ lies under $\chi \in \mathrm{Irr}(G)$, then since $X(1) \leq 8$, the G-orbit of λ has size at most 8. Thus the action of G on the orbit of λ provides a homomorphism from G into the symmetric group S_8. Suppose that the image of G is a full Sylow 2-subgroup of S_8 and let $T = I_G(\lambda)$. In this case, $|G:T| = 8$ and the image of T in S_8 is isomorphic to the direct product of a dihedral group of order 8 with a cyclic group of order 2. Consequently, T/A is non-abelian and so we can choose $\theta \in \mathrm{Irr}(T \mid \lambda)$ with $\theta(1) > 1$. But now θ^g is irreducible and has degree > 8

contrary to hypothesis and so the image of G in S_8 must be properly contained in a Sylow subgroup. However, any such proper subgroup is metabelian, and so G'' fixes λ. Since λ was arbitrary, G'' acts trivially on $\mathrm{Irr}(A)$ and so on A, hence $G'' \subseteq A$ is abelian.

COROLLARY 2.13. *Let $n \geq 3$ be an integer and G a finite 2-group with irreducible character degrees less than or equal to 2^n. Then the derived length of G is at most n.*

Proof. We will proceed by induction on n, with the case $n = 3$ handled by the proposition. For $n > 3$, we note that no nonlinear character of a p-group restricts irreducibly to the commutator subgroup. Consequently, G' has irreducible characters with degree at most $2^{(n-1)}$ and by our inductive hypothesis, G' has derived length at most $n - 1$.

We note that Theorem A is simply a combination of Theorem 2.11 and Corollary 2.13 in a single statement.

3. Examples. In this section we wish to note a few examples related to the preceding theorems. We begin by showing that $n = 2$ will not work in Theorem 2.10.

EXAMPLE 3.1. *Given any prime p there is a p-group G acting on an abelian p-group A such that every G-orbit in A has size at most p^2 and yet G' acts non-trivially on A.*

Proof. Let G be the Sylow p-subgroup of $GL(3, p)$ consisting of upper triangular matrices (with 1's on the diagonal) and let A be an elementary abelian group of order p^3 (row vectors) under the natural action of G. It is easy to see that elements in a single G-orbit differ only in the last two coordinates and so G-orbits have size $\leq p^2$. On the other hand, G is non-abelian, but acts faithfully on A.

It also follows from 3.1 that the conclusion of Theorem 2.10 cannot be strengthened for $n = 3$. The next example shows that 2.10 gives the best bound for $n = 4$ as well.

EXAMPLE 3.2. *Given an odd prime p there is a p-group G acting on an abelian p-group A such that every G-orbit in A has size at most p^4 and G'' acts non-trivially on A.*

Proof. Let A be a vector space of dimension $p^2 + 1$ over $GF(p)$ and denote the basis of A by the set:

$$\{1, 2, 3, \ldots, p^2, \omega\}.$$

We will define G as a group of linear transformations on A. Let Γ be the Sylow p-subgroup of S_{p^2} generated by the permutations:

$$\Gamma = \langle (1\ 2\ 3 \ldots p), (1\ p+1\ 2p+1 \ldots)(2\ p+2\ 2p+2 \ldots) \ldots (p\ 2p\ 3p \ldots) \rangle$$

and let H be a group of transformations on A which fix ω and act on the rest of the basis like a non-abelian subgroup of order p^3 in Γ. Finally, let G be generated by H and the transformation σ which maps 1 to $1 + \omega$ and fixes all other basis vectors. This group G has order p^{p^2+3}.

Clearly the action of G on $A/\langle \omega \rangle$ is the same as the action of H and so G-orbits in

$A/\langle\omega\rangle$ can have size at most p^3. However, the full action of G only adds multiples of ω to the vectors in any orbit and so G-orbits in A have size $\leq p^4$.

To show that $G'' \neq 1$, we will find two commutators which don't commute. Let $z \neq 1$ be an element in the center of H. Since H is non-abelian, order p^3, z is a commutator in H and so in G. Furthermore, the element $[z, \sigma]$ adds ω to 1 and subtracts ω from some other basis element (say 2), while fixing all other basis vectors and so $[z, \sigma]$ doesn't commute with z.

We also note that Theorem 2.10 is not true for $p = 2$.

EXAMPLE 3.3. *There is a* 2*-group G acting on an abelian* 2*-group A such that every G-orbit in A has size at most* 8 *and yet G'' acts non-trivially on A.*

Proof. Let G and A be the corresponding groups in the proof of Example 3.2 with $p = 2$ (i.e. A is a vector space of dimension 5 and G turns out to be isomorphic to a full Sylow 2-subgroup of S_8). It is easily checked in this case that in fact no G-orbit in $A/\langle\omega\rangle$ has size greater than 4 and so the G-orbits in A are at most 8. Since G is isomorphic to a full Sylow 2-subgroup of S_8, it has derived length 3 as claimed.

Finally, we wish to present two examples related to Theorem 2.11. Example 3.4 was first mentioned to me by M. Isaacs and was the motivation for the current work.

EXAMPLE 3.4. *Given any prime there is a p-group G with irreducible character degrees* $\{1, p, p^2\}$ *and derived length* 3.

Proof. Since the squares of the degrees of the irreducible characters sum to the group order and any group with only two character degrees has derived length 2, it suffices to find a p-group of order p^6 with derived length 3. For $p \geq 5$, such groups exist and can be found in various classifications of small p-groups such as [**5**].

For $p = 2$, let G be a Sylow 2-subgroup of S_8.

For $p = 3$, the simplest example seems to be the following group of order 3^7 constructed by M. Isaacs:

$$\begin{aligned} G = \langle u, v, w, x, y, z \mid\ & u^3 = v^3 = w^3 = x^9 = y^3 = z = 1, \\ & [y, x] = zu,\ [z, x] = u,\ [u, x] = v,\ [v, y] = w,\ [u, z] = w^{-1}, \\ & \text{and all other commutators are trivial}\rangle. \end{aligned}$$

EXAMPLE 3.5. *Given any prime p there is a p-group G with irreducible character degrees* $\{1, p, p^2, p^3\}$ *and derived length* 3.

Proof. Let H be a group as in Example 3.4 with degrees $\{1, p, p^2\}$ and derived length 3. Then if P is any non-abelian group of order p^3, the direct product $G = H \times P$ has the desired properties.

The same trick will produce groups with character degrees $\{1, p, p^2, p^3, p^4\}$ and derived length 3, however, it is not known if these degrees can occur with derived length 4.

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