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ON A CHARACTERIZATION OF REAL HYPERSURFACES OF TYPE *A* IN A COMPLEX SPACE FORM

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ABSTRACT. In this paper, under certain conditions on the orthogonal distribution T_0, we give a characterization of real hypersurfaces of type A in a complex space form $M_n(c)$, $c \neq 0$.

0. **Introduction.** A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

There exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space P_nC by Takagi [13], who showed that these hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E, and in [3] Cecil-Ryan and [6] Kimura proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Also Berndt [1], [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in H_nC they are said to be of type A_0, A_1, A_2, and B.

Now, let us consider the following conditions that the second fundamental tensor A of M in $M_n(c)$, $c \neq 0$ may satisfy

$$(\nabla_X A)Y = -\frac{c}{4}\{\eta(Y)\phi X + g(\phi X, Y)\xi\}, \tag{0.1}$$

$$g\big((A\phi - \phi A)X, Y\big) = 0, \tag{0.2}$$

for any tangent vector fields X and Y of M.

Maeda [8] investigated the condition (0.1) and used it to find a lower bound of $||\nabla A||$ for real hypersurfaces in P_nC. In fact, it was shown that $||\nabla A||^2 \geq \frac{c^2}{4}(n-1)$ for such hypersurfaces and the equality holds if and only if the condition (0. 1) holds. Moreover, in this case it was known that M is locally congruent to real hypersurfaces of type A_1, and A_2. Also Chen, Ludden and Montiel [4] generalized this inequality to real hypersurfaces

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in H_nC and showed that the equality holds if and only if M is congruent to of type A_0, A_1, and A_2.

On the other hand, the condition (0.2) was considered by Okumura [11] for $c > 0$, Montiel and Romero [10] for $c < 0$ respectively. Also it was known that a real hypersurface satisfying (0.2) is locally congruent to one of type A_0, A_1, and A_2. Now let us define a distribution T_0 by $T_0(x) = \{X \in T_xM \mid X \perp \xi_{(x)}\}$ of a real hypersurface M of $M_n(c)$, $c \neq 0$, which is orthogonal to the structure vector field ξ and holomorphic with respect to the structure tensor ϕ. If we restrict the properties (0.1) and (0.2) to the orthogonal distribution T_0, then for any vector fields X, Y, and Z in T_0 the shape operator A of M satisfies the following conditions

$$(\nabla_X A)Y = -\frac{c}{4}g(\phi X, Y)\xi \tag{0.3}$$

and

$$(A\phi - \phi A)X = \theta(X)\xi \tag{0.4}$$

for a 1-form θ defined on T_0. Thus the above conditions (0.3) and (0.4) are weaker than the conditions (0.1) and (0.2) respectively. Then it is natural that real hypersurfaces of type A in $M_n(c)$, $c \neq 0$, should satisfy the conditions (0.3) and (0.4). From this point of view we give a characterization of real hypersurfaces of type A in $M_n(c)$ as the following

THEOREM. *Let M be a connected real hypersurface of $M_n(c)$, $c \neq 0$, and $n \geq 3$. If it satisfies (0.3) and (0.4), then M is locally congruent to one of the following spaces:*

(1) In case $M_n(c) = P_nC$

(A_1) a tube of radius r over a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$,

(A_2) a tube of radius r over a totally geodesic P_kC ($1 \leq k \leq n-2$), where $0 < r < \frac{\pi}{2}$,

(2) In case $M_n(c) = H_nC$

(A_0) a horosphere in H_nC, i.e., a Montiel tube,

(A_1) a tube of a totally geodesic hyperplane H_kC ($k = 0$ or $n-1$),

(A_2) a tube of a totally geodesic H_kC ($1 \leq k \leq n-2$).

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1. **Preliminaries.** We begin with recalling fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $\big(M_n(c), \bar{g}\big)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field defined on a neighborhood of a point x in M. We denote by J the almost complex structure of $M_n(c)$.

For a local vector field X on the neighbourhood of x in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighbourhood of x in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric on M induced from the metric $\bar{g}$ on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called *an almost contact metric structure* on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X, where I denotes the identity transformation. Furthermore the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{1.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of g and A denotes the shape operator in the direction of C on M.

Since the ambient space is of constant holomorphic sectional curvature c the equations of Gauss and Codazzi are respectively obtained:

(1.2)
$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{1.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The second fundamental form is said to be *η-parallel* if the shape operator A satisfies $g\big((\nabla_X A)Y, Z\big) = 0$ for any vector fields X, Y and Z in T_0.

2. **Proof of the Theorem.** Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, and let T_0 be a distribution defined by $T_0(x) = \{X \in T_xM \mid X \perp \xi_{(x)}\}$. Now we prove the theorem in the introduction. In order to prove this Theorem we should verify that ξ is principal from the conditions (0.3) and (0.4). If we acquire this fact, from the condition (0.4) we can see that the structure tensor ϕ and the shape operator A of a real hypersurface M in $M_n(c)$, $c \neq 0$, commute with each other. Then by using theorems of Okumura [11] for $c > 0$ and of Montiel and Romero [10] for $c < 0$ we get that a real hypersurface M satisfying (0.3) and (0.4) is locally congruent to one of type A_1, and A_2 in P_nC and A_0, A_1, and A_2 in H_nC respectively. Namely we can obtain another new characterization of real hypersurfaces of type A in $M_n(c)$, $c \neq 0$. For this purpose we need a lemma obtained from the restricted condition (0.4) as the following

LEMMA 2.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If there is a 1-form θ satisfying the condition (0.4), then we have*

$$g\big((\nabla_X A)Y, Z\big) = \mathcal{S}g(AX, Y)g(Z, V), \tag{2.1}$$

where $\mathcal{S}$ denotes the cyclic sum with respect to X, Y and Z in T_0 and V stands for the vector field defined by $\nabla_\xi \xi$.

PROOF. For any vector fields X, Y and Z orthogonal to ξ, the condition (0.4) implies that $g\big((A\phi - \phi A)Y, Z\big) = 0$. Differentiating this equation covariantly in the direction of X, we get

$$g\big((\nabla_X A)\phi Y + A(\nabla_X \phi)Y + A\phi \nabla_X Y - (\nabla_X \phi)AY - \phi(\nabla_X A)Y - \phi A \nabla_X Y, Z\big) + g\big((A\phi - \phi A)Y, \nabla_X Z\big) = 0.$$

By taking account of (1.1), the above equation is reformed as

$$g\big((\nabla_X A)Y, \phi Z\big) + g\big((\nabla_X A)Z, \phi Y\big) = \eta(AY)g(X, AZ) + \eta(AZ)g(Y, AX) + g(X, A\phi Y)g(Z, V) + g(X, A\phi Z)g(Y, V). \tag{2.2}$$

In this equation we shall replace X, Y and Z cyclically and we shall then add the second equation to (2.2), from which we subtract the third one. Consequently by means of the Codazzi equation we get

$$2g\big((\nabla_X A)Y, \phi Z\big) = 2\eta(AZ)g(AX, Y) + g(X, V)\{g(Y, A\phi Z) - g(Z, A\phi Y)\} + g(Y, V)\{g(X, A\phi Z) - g(Z, A\phi X)\},$$

from which together with the condition (0.4) we can get the equation (2.1).

Next, from this lemma it remains only to show the following.

LEMMA 2.2. *Let M be a real hypersurface of $M_n(c)$, $n \geq 3$, $c \neq 0$. If M satisfies (0.3) and (0.4), then the structure vector field ξ is principal.*

PROOF. The purpose of this lemma is to show that the structure vector field ξ is principal. In order to prove this, let us suppose that there is a point where ξ is not principal. Then there exists a neighborhood $\mathcal{U}$ of this point, on which we can define a unit vector field U orthogonal to ξ in such a way that

$$\beta U = A\xi - g(A\xi, \xi)\xi = A\xi - \alpha\xi, \tag{2.3}$$

where β denotes the length of vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point x in $\mathcal{U}$. Hereafter, unless otherwise stated, let us continue our discussion on this neighborhood $\mathcal{U}$.

A vector field V is defined by $\nabla_\xi \xi$. Then, from this definition together with (1.1) it follows

$$V = \beta\phi U. \tag{2.4}$$

On the other hand, (0.3) implies that the shape operator A of M becomes to η-parallel, that is, $g\big((\nabla_X A)Y, Z\big) = 0$ for any vector fields X, Y and Z in T_0. From this and (2.1) it follows that

$$\mathcal{S} g(AX, Y)g(Z, V) = 0, \tag{2.5}$$

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for any vector fields X, Y and Z in T_0, where $\mathcal{S}$ denotes the cyclic sum with respect to X, Y and Z. When we put $Z = V$ in (2.5), it reduces to

$$g(AX, Y)g(V, V) + g(AY, V)g(X, V) + g(AV, X)g(Y, V) = 0. \tag{2.6}$$

Furthermore, we put $Y = V$ in (2.6) and then $X = V$ in the obtained equation. Then the following equations

$$2g(AX, V)g(V, V) + g(AV, V)g(X, V) = 0, \tag{2.7}$$

$$g(AV, V)g(V, V) = 0 \tag{2.8}$$

are obtained. Then from (2.7) and (2.8) we know that $g(AV, V) = 0$ and $g(AX, V) = 0$ for any vector field X in T_0 orthogonal to V, which implies $AV = g(AV, \xi)\xi$. Substituting (2.3) into this equation and noticing that U and V are mutually orthogonal by (2.4), we get $AV = 0$. From this, together with (2.6), it follows

$$g(AX, Y) = 0 \tag{2.9}$$

for any vector fields X and Y belonging to T_0. So, it follows from this and (2.3) that $AX = g(AX, \xi)\xi = \beta g(X, U)\xi$ for any $X \in T_0$, which means that

$$AX = 0, \quad AU = \beta\xi \tag{2.10}$$

for any $X \in T_0$ orthogonal to U.

Now let us keep on our discussion on the open set $\mathcal{U}$. Then the condition (0.3) implies that

$$(\nabla_X A)Y = \lambda(X, Y)\xi, \tag{2.11}$$

where the function $\lambda(X, Y)$ is given by

$$\begin{aligned}\lambda(X, Y) &= g\big((\nabla_X A)Y, \xi\big) = g\big(Y, (\nabla_X A)\xi\big)\\ &= g\big(Y, (X\alpha)\xi + \alpha\nabla_X\xi + (X\beta)U + \beta\nabla_X U - A\phi AX\big)\\ &= \alpha g(\phi AX, Y) + (X\beta)g(Y, U) + \beta g(Y, \nabla_X U) - g(Y, A\phi AX)\end{aligned}$$

for any vector fields X and Y in T_0. When we put $X = U$ and $Y = \phi U$ in (2.11), then by (2.10)

$$(\nabla_U A)\phi U = \beta g(\phi U, \nabla_U U)\xi. \tag{2.12}$$

On the other hand, from the equation of Codazzi (1.3) and (2.3), together with (2.10) it follows that

$$\begin{aligned}\frac{c}{4}\phi U &= (\nabla_\xi A)U - (\nabla_U A)\xi = \nabla_\xi(AU) - A\nabla_\xi U - \nabla_U(A\xi) + A\nabla_U\xi\\ &= \nabla_\xi(\beta\xi) - A\nabla_\xi U - \nabla_U(\beta U + \alpha\xi)\\ &= (\xi\beta)\xi + \beta\phi A\xi - A\nabla_\xi U - (U\beta)U - \beta\nabla_U U - (U\alpha)\xi\\ &= (\xi\beta - U\alpha)\xi + \beta^2\phi U - A\nabla_\xi U - (U\beta)U - \beta\nabla_U U.\end{aligned}$$

Then taking the inner product of the last formula with ϕU, we obtain

$$\beta g(\phi U, \nabla_U U) = \beta^2 - \frac{c}{4},$$

where we have used $g(A\nabla_\xi U, \phi U) = 0$ which can be obtained by the first formula of (2.10). Substituting this equation into (2.12), we get

$$(\nabla_U A)\phi U = (\beta^2 - \frac{c}{4})\xi. \tag{2.13}$$

On the other hand, by the assumption (0.3) we have

$$(\nabla_U A)\phi U = -\frac{c}{4} g(\phi U, \phi U)\xi = -\frac{c}{4}\xi.$$

From this, comparing with (2.13), we have $\beta = 0$. This makes a contradiction. The set $\mathcal{U}$ should be empty. Thus there does not exist such an open neighborhood $\mathcal{U}$ in M, which means that the structure vector field ξ is principal.

REFERENCES

1. J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395**(1989), 132–141.

2. ———, *Real hypersurfaces with constant principal curvatures in complex space forms*, Geometry and Topology of Submanifolds II, Avignon, 1988, 10–19, World Scientific, 1990.

3. T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269**(1982), 481–499.

4. B. Y. Chen, G. D. Ludden and S. Montiel, *Real submanifold of a Kaehler manifold*, Algebras Groups Geom. **1**(1984), 176–212.

5. U-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32**(1990), 207–221.

6. M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296**(1986), 137–149.

7. ———, *Sectional curvatures of holomorphic planes on a real hypersurface in $P_n(C)$*, Math. Ann. **276** (1987), 487–497.

8. Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28**(1976), 529–540.

9. S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37**(1985), 515–535.

10. S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20**(1986), 245–261.

11. M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

12. Y. J. Suh and R. Takagi, *A rigidity for real hypersurfaces in a complex projective space*, Tohoku Math. J. **43**(1991), 501–507.

13. R. Takagi, *On homogeneous real hypersurfaces of a complex projective space*, Osaka J. Math. **10**(1973), 495–506.

14. ———, *Real hypersurfaces in a complex projective space with constant principal curvature I, II*, J. Math. Soc. Japan **27**(1975), 43–53, 507–516.

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