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GROWTH ESTIMATES FOR SOLUTIONS OF NONLINEAR SECOND-ORDER DIFFERENCE EQUATIONS

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Abstract

We prove several growth theorems for solutions of certain nonlinear second-order difference equations.

1. Introduction

In this paper we study the asymptotic behaviour of the second-order nonlinear difference equation:

$$\Delta(c_{n-1}\Delta x_{n-1}) = f(n, x_n, \Delta x_{n-1}) + g(n, x_n, \Delta x_{n-1}), \quad n \in \mathbb{N}, \tag{1.1}$$

where f and g are real functions satisfying some additional conditions which we explain later. Our results are natural extensions and generalisations of those in [9] and are also closely related to those in [4] and [8], where the growth of the solutions of the corresponding homogeneous difference equation was investigated. This equation models, for example, the amplitude of oscillation of the weights on a discretely weighted vibrating string.

Similar equations have been investigated by many authors, see, for example [1, Chapter 6.15]. Our motivation here stems from [1, Problem 6.24.40]. According to [1] the following result holds.

THEOREM A. *Consider the equation*

$$\Delta(c_{n-1}\Delta x_{n-1}) = d_n f(n, x_n, \Delta x_{n-1}) + g(n, x_n, \Delta x_{n-1}),$$

where

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(a) $c_n > 0, n \geq 0$ *and* $\sum_{i=1}^{\infty} 1/c_{i-1} = \infty$;
(b) $|g(n, u, v)| \leq g_n$, *where* g_n *is a real sequence such that* $\sum_{i=1}^{\infty} g_n < \infty$;
(c) d_n *is a real sequence such that* $\sum_{j=1}^{+\infty} |d_j| \sum_{i=2}^{j} 1/c_{i-1} < \infty$;
(d) f *is a real function such that* $|f(n, u, v)| \leq L|u|, u, v \in \mathbb{R}, n \in \mathbb{N}$ *for some* $L > 0$.

Then $x_n = \mathcal{O}\left(\sum_{i=2}^{n} 1/c_{i-1}\right)$ *as* $n \to +\infty$.

We want to say that the following, slightly different, equation

$$\Delta(c_n \Delta x_n) = d_n f(n, x_n, \Delta x_n) + g(n, x_n, \Delta x_n)$$

is considered in [1]. However, it is easy to see that under the same conditions as in Theorem A the same asymptotic formula holds (see also the proof of Lemma 2.4). Theorems 3.1, 3.5 and 3.6 generalise and extend Theorem A in many directions. In Section 2 we quote three auxiliary results which we apply in the third section. In Section 3 we study the asymptotic behaviour of the solutions of the second-order nonlinear difference equation (1.1).

2. Auxiliary results

In order to investigate the growth of the solution x_n, we first need two auxiliary results which are discrete variants of Bellman-Gronwall's lemma. Applications and further generalisations of this lemma can be found, for example, in [2–10].

LEMMA 2.1 ([6, page 112]). *If* $x_n, b_n, c_n \geq 0$ *and*

$$x_n \leq a_n + b_n \sum_{i=1}^{n-1} c_i x_i, \quad n \in \mathbb{N},$$

then

$$x_n \leq a_n + b_n \sum_{i=1}^{n-1} a_i c_i e^{\sum_{j=i+1}^{n-1} b_j c_j}, \quad n \in \mathbb{N}.$$

From Lemma 2.1 and the inequality $1 + x \leq e^x$, for $x \geq 0$, it is easy to get the following corollary.

COROLLARY 2.2 ([3, page 198]). *If* $x_n, c_n \geq 0$, c *is a positive constant and*

$$x_n \leq c + \sum_{i=1}^{n-1} c_i x_i, \quad n \in \mathbb{N},$$

then $x_n \leq c e^{\sum_{i=1}^{n-1} c_i}$, $n \in \mathbb{N}$.

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The following lemma is a special case of [10, Theorem 4].

LEMMA 2.3. *If* $x_n, c_n \geq 0$, *c is a positive constant*, $p \in [0, 1)$ *and*

$$x_n \leq c + \sum_{i=1}^{n-1} c_i x_i^p, \quad n \in \mathbb{N},$$

then

$$x_n \leq \left(c^{1-p} + (1-p) \sum_{i=1}^{n-1} c_i \right)^{1/(1-p)}, \quad n \in \mathbb{N}.$$

The next result can be considered as the main technical part of this paper.

LEMMA 2.4. *Consider* (1.1) *where*

(a) $c_n > 0$, $n \geq 0$;
(b) f *and* g *are real functions*; *and*
(c) $|g(n, u, v)| \leq g_n$, *where* g_n *is an arbitrary real sequence.*

Then the following inequalities hold:

$$|\Delta x_{n-1}| \leq \frac{1}{c_{n-1}} \left(|y_0| + \sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} |f(i, x_i, \Delta x_{i-1})| \right) \tag{2.1}$$

and

$$\frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \leq c_1 |x_1| + |y_0| + \sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} |f(i, x_i, \Delta x_{i-1})|. \tag{2.2}$$

PROOF. Let $y_n = c_n(x_{n+1} - x_n)$. Then from (1.1) we have

$$y_n - y_{n-1} = f(n, x_n, \Delta x_{n-1}) + g(n, x_n, \Delta x_{n-1}), \quad n \in \mathbb{N}. \tag{2.3}$$

Summing (2.3) from 1 to $n-1$, we obtain

$$x_n - x_{n-1} = \frac{1}{c_{n-1}} \left(y_0 + \sum_{i=1}^{n-1} (f(i, x_i, \Delta x_{i-1}) + g(i, x_i, \Delta x_{i-1})) \right). \tag{2.4}$$

From this and condition (c) we get

$$|\Delta x_{n-1}| \leq \frac{1}{c_{n-1}} \left(|y_0| + \sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} |f(i, x_i, \Delta x_{i-1})| \right).$$

Summing (2.4) from 2 to n, we have

$$x_n = x_1 + y_0 \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(\sum_{j=1}^{i-1} (f(j, x_j, \Delta x_{j-1}) + g(j, x_j, \Delta x_{j-1})) \right),$$

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which yields

$$|x_n| \le |x_1| + |y_0| \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} g_j + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} |f(j, x_j, \Delta x_{j-1})| \tag{2.5}$$

and consequently

$$\frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \le c_1|x_1| + |y_0| + \sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} |f(i, x_i, \Delta x_{i-1})|,$$

as desired.

3. The main results

In this section we study the asymptotic behaviour of the solutions of the second-order nonlinear difference equation (1.1).

THEOREM 3.1. *Consider* (1.1) *where*

(a) $c_n > 0$, $n \ge 0$;

(b) $|g(n, u, v)| \le g_n$, *where* g_n *is an arbitrary real sequence*; *and*

(c) f *is a real function such that* $|f(n, u, v)| \le |d_n||u|^\alpha$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$, *for some* $\alpha \in [0, 1]$ *and a real sequence* d_n.

Then the following asymptotic formula holds:

$$x_n = \mathcal{O}\left(a_n + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{n-1} a_j |d_j| e^{\sum_{k=j+1}^{n-1} |d_k| \sum_{i=2}^{k} 1/c_{i-1}}\right) \quad \text{as } n \to +\infty,$$

where $a_n = \sum_{i=2}^{n} \left(1 + \sum_{j=1}^{i-1} g_j\right)/c_{i-1}$.

PROOF. Let $y_n = c_n(x_{n+1} - x_n)$. From inequality (2.5) in Lemma 2.4 and by condition (c) we have

$$|x_n| \le |x_1| + |y_0| \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} g_j + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} |d_j||x_j|^\alpha.$$

From this and by the well-known inequality $|x|^\alpha \le 1 + |x|$, $x \in \mathbb{R}$, $\alpha \in [0, 1]$, we have

$$\begin{aligned} |x_n| + 1 &\le |x_1| + 1 + |y_0| \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} g_j + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} |d_j|(|x_j| + 1) \\ &\le |x_1| + 1 + |y_0| \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} g_j + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{n-1} |d_j|(|x_j| + 1). \end{aligned}$$

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Let $\hat{a}_n = |x_1| + 1 + |y_0| \sum_{i=2}^{n}(1/c_{i-1}) + \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} g_j \right)/c_{i-1}$, then by Lemma 2.1,

$$|x_n| + 1 \leq \hat{a}_n + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{n-1} \hat{a}_j \, |d_j| \, e^{\sum_{k=j+1}^{n-1} |d_k| \sum_{i=2}^{k} 1/c_{i-1}}. \tag{3.1}$$

Since a_n is increasing we have

$$1 \leq a_1 \left(\left[\frac{1}{a_1} \right] + 1 \right) \leq a_n \left(\left[\frac{1}{a_1} \right] + 1 \right), \quad n \geq 1.$$

On the other hand, we have $|\hat{a}_n| \leq \max\{|x_1| + 1, |y_0|, 1\}(1 + a_n)$, for $n \geq 1$, which implies

$$|\hat{a}_n| \leq \max\{|x_1| + 1, |y_0|, 1\} \left(\left[\frac{1}{a_1} \right] + 2 \right) a_n, \quad n \geq 1.$$

From this and (3.1) the result follows.

COROLLARY 3.2. *Consider* (1.1) *where*

(a) $c_n > 0$, $n \geq 0$ *and* $\sum_{i=1}^{\infty} 1/c_{i-1} = \infty$;
(b) $|g(n, u, v)| \leq g_n$, *where* g_n *is an arbitrary real sequence*;
(c) d_n *is a real sequence such that* $\sum_{j=1}^{+\infty} |d_j| \sum_{i=2}^{j} 1/c_{i-1} < \infty$; *and*
(d) f *is a real function such that* $|f(n, u, v)| \leq |d_n||u|^{\alpha}$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$, *for some* $\alpha \in [0, 1]$.

Then the following asymptotic formula holds:

$$x_n = \mathcal{O}\left(\sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(\sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(1 + \sum_{j=1}^{i-1} g_i \right) \right) \right) \quad \text{as } n \to +\infty.$$

PROOF. Let a_n and $\hat{a}_n$ be as in Theorem 3.1. Dividing (3.1) by $a_n \sum_{i=2}^{n} 1/c_{i-1}$ and using conditions (a) and (c) we obtain that there are positive constants C_1, C_2, C_3 and C_4 such that

$$\frac{|x_n| + 1}{a_n \sum_{i=2}^{n} 1/c_{i-1}} \leq C_1 + C_2 \sum_{j=1}^{n-1} |d_j| \leq C_3 + C_4 \sum_{j=1}^{n-1} |d_j| \sum_{i=2}^{j} \frac{1}{c_{i-1}} < \infty,$$

as desired.

COROLLARY 3.3. *Consider* (1.1) *where*

(a) $c_n > 0$, $n \geq 0$ *and* $\sum_{i=1}^{\infty} 1/c_{i-1} = \infty$;
(b) $|g(n, u, v)| \leq g_n$, *where* g_n *is a real sequence such that* $\sum_{i=1}^{\infty} g_n < \infty$;

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(c) *d_n is a real sequence such that* $\sum_{j=1}^{+\infty} |d_j| \sum_{i=2}^{j} 1/c_{i-1} < \infty$; *and*

(d) *f is a real function such that,* $|f(n, u, v)| \le |d_n||u|^{\alpha}$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$, *for some* $\alpha \in [0, 1]$.

Then $x_n = \mathcal{O}\left(\sum_{i=2}^{n} 1/c_{i-1}\right)$ *as* $n \to +\infty$.

PROOF. Since $\sum_{i=1}^{\infty} g_n < \infty$ we have $a_n \asymp \sum_{i=2}^{n} 1/c_{i-1}$. From this and (3.1) we obtain that there are positive constants C_1 and C_2 such that

$$\begin{aligned} |x_n| + 1 &\le \sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(C_1 + C_2 \sum_{j=1}^{n-1} |d_j| \sum_{i=2}^{j} \frac{1}{c_{i-1}} e^{\sum_{k=j+1}^{n-1} |d_k| \sum_{i=2}^{k} 1/c_{i-1}} \right) \\ &\le \sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(C_1 + C_3 e^{\sum_{j=1}^{n-1} |d_j| \sum_{i=2}^{j} 1/c_{i-1}} \right) \le C_4 \sum_{i=2}^{n} \frac{1}{c_{i-1}}, \end{aligned}$$

as desired.

REMARK. Note that Corollary 3.3 in the case $\alpha = 1$ reduces to Theorem A ([1, Problem 6.24.40]).

EXAMPLE 1. Consider the equation

$$\Delta(\sqrt{n+1}\Delta x_{n-1}) = \frac{\sqrt{|x_n|}}{n^2(1 + (\Delta x_{n-1})^2)} + \frac{1}{n^3 + x_n^2}, \quad n \in \mathbb{N}.$$

Here we have $c_n = \sqrt{n+2}$ and $d_n = 1/n^2$. Since $\sum_{i=2}^{\infty} 1/\sqrt{i+1} = \infty$, condition (a) in Corollary 3.3 is satisfied. Condition (b) follows from the following inequality:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + x_n^2} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$$

Using the asymptotics

$$\sum_{i=2}^{j} \frac{1}{\sqrt{i+1}} \sim \int_2^j \frac{dx}{\sqrt{x+1}} \sim 2\sqrt{j},$$

we get

$$\sum_{j=1}^{+\infty} |d_j| \sum_{i=2}^{j} \frac{1}{c_{i-1}} \asymp \sum_{j=1}^{+\infty} \frac{1}{j^{3/2}} < \infty,$$

that is, condition (c) is satisfied. Condition (d) holds for $\alpha = 1/2$, since

$$|f(n, x_n, \Delta x_{n-1})| = \frac{\sqrt{|x_n|}}{n^2(1 + (\Delta x_{n-1})^2)} \le \frac{1}{n^2} |x_n|^{1/2}.$$

Hence by Corollary 3.3 we have $x_n = \mathcal{O}\left(\sum_{i=2}^{n} 1/\sqrt{i+1}\right) = \mathcal{O}(\sqrt{n})$.

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COROLLARY 3.4. *Consider* (1.1) *where*

(a) $c_n > \delta > 0$, $n \geq 0$;
(b) $|g(n, u, v)| \leq g_n$, *where* g_n *is a real sequence such that* $\sum_{i=1}^{\infty} g_n < \infty$;
(c) d_n *is a real sequence such that* $\sum_{j=1}^{+\infty} |d_j| \sum_{i=2}^{j} 1/c_{i-1} < \infty$; *and*
(d) f *is a real function such that* $|f(n, u, v)| \leq |d_n||u|^\alpha$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$, *for some* $\alpha \in [0, 1]$.

Then $x_n = \mathcal{O}(n)$ *as* $n \to +\infty$.

PROOF. Since in this case $\sum_{i=2}^{n} 1/c_{i-1} < (n-1)/\delta$, we obtain the result by Corollary 3.3.

THEOREM 3.5. *Consider* (1.1) *where*

(a) $c_n > 0$, $n \geq 0$ *and* $\sum_{i=1}^{\infty} 1/c_{i-1} = \infty$;
(b) $|g(n, u, v)| \leq g_n$, *where* g_n *is a real sequence such that* $\sum_{i=1}^{\infty} g_n < \infty$;
(c) d_n *is a real sequence such that, for some* $\alpha \in [0, 1)$,

$$\sum_{i=1}^{+\infty} |d_i| \left(\sum_{j=2}^{i} \frac{1}{c_{j-1}} \right)^\alpha < \infty; \quad \textit{and}$$

(d) f *is a real function such that* $|f(n, u, v)| \leq |d_n||u|^\alpha$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$.

Then the following asymptotic formula holds:

$$x_n = \mathcal{O}\left(\sum_{i=2}^{n} \frac{1}{c_{i-1}} \right) \quad \textit{as } n \to +\infty,$$

for every solution x_n *of* (1.1) *and* $\lim_{n\to+\infty} c_{n-1}(x_n - x_{n-1})$ *is finite.*

PROOF. From (2.5), conditions (b) and (c) and some simple calculations we get

$$\begin{aligned}
|x_n| &\leq |x_1| + |y_0| \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} g_j + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} |d_j||x_j|^\alpha \\
&\leq |x_1| + C \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \sum_{j=1}^{i-1} |d_j||x_j|^\alpha \\
&= |x_1| + C \sum_{i=2}^{n} \frac{1}{c_{i-1}} + \sum_{j=1}^{n-1} |d_j||x_j|^\alpha \sum_{i=j+1}^{n} \frac{1}{c_{i-1}} \\
&\leq |x_1| + \sum_{i=2}^{n} \frac{1}{c_{i-1}} \left(C + \sum_{j=1}^{n-1} |d_j||x_j|^\alpha \right),
\end{aligned} \tag{3.2}$$

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where $C = |y_0| + \sum_{n=1}^{\infty} g_n$.

From (3.2) we have

$$\frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \le C_1|x_1| + C + \sum_{j=1}^{n-1}\left(\sum_{i=2}^{j}\frac{1}{c_{i-1}}\right)^{\alpha}|d_j|\frac{|x_j|^{\alpha}}{\left(\sum_{i=2}^{j}1/c_{i-1}\right)^{\alpha}}.$$

For $\alpha \in [0, 1)$, by Lemma 2.3 we get

$$\frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \le \left((C_1|x_1| + C)^{1-\alpha} + (1-\alpha)\sum_{i=1}^{n-1}\left(\sum_{j=2}^{i}\frac{1}{c_{j-1}}\right)^{\alpha}|d_i|\right)^{1/(1-\alpha)} < \infty,$$

in view of condition (c), from which the first part of the theorem follows.

From the above we know that there exists $M > 0$, such that $|x_n| \le M\sum_{i=2}^{n} 1/c_{i-1}$, for every $n \in \mathbb{N}$. Summing (2.3) from $n+1$ to $n+p$, we obtain

$$y_{n+p} - y_n = \sum_{i=n+1}^{n+p}(f(i, x_i, \Delta x_{i-1}) + g(i, x_i, \Delta x_{i-1})).$$

Hence

$$|y_{n+p} - y_n| \le \sum_{i=n+1}^{n+p} g_i + \sum_{i=n+1}^{n+p}|d_i||x_i|^{\alpha} \le \sum_{i=n+1}^{n+p} g_i + M^{\alpha}\sum_{i=n+1}^{n+p}\left(\sum_{j=2}^{i}\frac{1}{c_{j-1}}\right)^{\alpha}|d_i|.$$

By the conditions of the theorem and Cauchy's criteria we obtain the result.

EXAMPLE 2. Consider the equation

$$\Delta(\sqrt{n+1}\Delta x_{n-1}) = \frac{\sqrt{|x_n|}}{n^{3/2}(1 + (\Delta x_{n-1})^2)} + \frac{1}{n^2 + (\Delta x_{n-1})^2}, \quad n \in \mathbb{N}.$$

Here $c_n = \sqrt{n+2}$, $d_n = 1/n^{3/2}$ and $\alpha = 1/2$. As in Example 1 it can be shown that the conditions in Theorem 3.5 are satisfied. Therefore

$$x_n = \mathcal{O}\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i+1}}\right) = \mathcal{O}(\sqrt{n})$$

and the limit $\lim_{n\to\infty}\sqrt{n+1}(x_n - x_{n-1})$ is finite.

Note that this equation does not satisfy condition (c) in Corollary 3.3, since

$$\sum_{j=1}^{+\infty}|d_j|\sum_{i=2}^{j}\frac{1}{c_{i-1}} \asymp \sum_{j=1}^{+\infty}\frac{1}{j} = \infty.$$

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The following result also includes the case $\alpha = 1$, moreover, we impose a condition on the rate of the growth of the function f which also includes the difference Δx_{n-1}.

THEOREM 3.6. *Consider* (1.1) *where*

(a) $c_n \geq \delta > 0$, $n \geq 0$;

(b) $|g(n,u,v)| \leq g_n$, *where* g_n *is a real sequence such that* $\sum_{i=1}^{\infty} g_n < \infty$;

(c) f *is a real function such that* $|f(n,u,v)| \leq d_n|u|^\alpha + f_n|v|^\beta$, $u, v \in \mathbb{R}$, $n \in \mathbb{N}$, *for some* $\alpha, \beta \in [0,1]$ *and real sequences* d_n *and* f_n; *and*

(d) *the series* $\sum_{n=1}^{\infty} |d_i| \left(\sum_{j=2}^{i} 1/c_{j-1}\right)^\alpha$ *and* $\sum_{n=1}^{\infty} f_n$ *are convergent.*

Then $x_n = \mathcal{O}\left(\sum_{i=2}^{n} 1/c_{i-1}\right)$ *as* $n \to +\infty$, *for every solution* x_n *of* (1.1) *and the limit* $\lim_{n\to+\infty} c_{n-1}(x_n - x_{n-1})$ *is finite.*

PROOF. Since the conditions of Lemma 2.4 are satisfied, summing inequalities (2.1) and (2.2), and using condition (a), we obtain

$$|\Delta x_{n-1}| + \frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \leq \frac{|y_0|}{\delta} + c_1|x_1| + |y_0| + \left(\frac{1}{\delta} + 1\right)\left(\sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} |f(i, x_i, \Delta x_{i-1})|\right).$$

hence by conditions (b) and (c) we get

$$\begin{aligned} |\Delta x_{n-1}| + \frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} &\leq C + C\left(\sum_{i=1}^{n-1} g_i + \sum_{i=1}^{n-1} (|d_j||x_j|^\alpha + |f_j||\Delta x_{j-1}|^\beta)\right) \\ &\leq C_1 + C\sum_{i=1}^{n-1} (|d_j||x_j|^\alpha + |f_j||\Delta x_{j-1}|^\beta), \end{aligned}$$

where $C_1 = C(1 + \sum_{n=1}^{\infty} g_n)$, or equivalently

$$\begin{aligned} &|\Delta x_{n-1}| + \frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \\ &\leq C_1 + C\sum_{j=1}^{n-1}\left(\left(\sum_{i=2}^{j} \frac{1}{c_{i-1}}\right)^\alpha |d_j| \frac{|x_j|^\alpha}{\left(\sum_{i=2}^{j} 1/c_{i-1}\right)^\alpha} + |f_j||\Delta x_{j-1}|^\beta\right). \end{aligned}$$

Using the inequality $x^t \leq 1 + x + y$, for $x, y \geq 0$ and $t \in (0, 1]$, we get

$$\begin{aligned} &|\Delta x_{n-1}| + \frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \\ &\leq C_1 + C\sum_{j=1}^{n-1}\left(\left(\sum_{i=2}^{j} \frac{1}{c_{i-1}}\right)^\alpha |d_j| + |f_j|\right)\left(1 + \frac{|x_j|}{\sum_{i=2}^{j} 1/c_{i-1}} + |\Delta x_{j-1}|\right). \end{aligned}$$

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By Corollary 2.2 we have

$$1 + |\Delta x_{n-1}| + \frac{|x_n|}{\sum_{i=2}^{n} 1/c_{i-1}} \leq (1 + C_1)\exp\left(C\left(\sum_{j=1}^{n-1}\left(\sum_{i=2}^{j}\frac{1}{c_{i-1}}\right)^{\alpha}|d_j| + |f_j|\right)\right) < \infty,$$

in view of condition (d), from which the first part of the theorem follows.

The rest of the proof is similar to the corresponding proof of Theorem 3.5 and will be omitted.

EXAMPLE 3. Consider the equation

$$\Delta(\sqrt[3]{n+1}\Delta x_{n-1}) = \frac{\sqrt{|x_n|} + n^{1/4}\sqrt[4]{\Delta x_{n-1}}}{n^{3/2}(1 + (\Delta x_{n-1})^2)} + \frac{n}{2^n + |\Delta x_{n-1}|}, \quad n \in \mathbb{N}.$$

Here $c_n = \sqrt[3]{n+2}$, $d_n = 1/n^{3/2}$, $f_n = 1/n^{5/4}$, $\alpha = 1/2$ and $\beta = 1/4$. It is easy to see that the conditions in Theorem 3.6 are satisfied. Thus $x_n = \mathcal{O}\left(\sum_{i=2}^{n} 1/\sqrt[3]{i+1}\right) = \mathcal{O}(n^{2/3})$ and $\lim_{n\to\infty}\sqrt[3]{n+1}(x_n - x_{n-1})$ is finite.

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