A THEOREM ON PARTITIONS

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Certain integers have the property that they can be partitioned into distinct positive integers whose reciprocals sum to 1, e.g.,

$$11 = 2+3+6, \qquad 1 = 2^{-1}+3^{-1}+6^{-1}$$

and

$$24 = 2+4+6+12, \qquad 1 = 2^{-1}+4^{-1}+6^{-1}+12^{-1}.$$

In this paper we prove that all integers exceeding 77 possess this property. This result can then be used to establish the more general theorem that for any positive rational numbers α and β, there exists an integer $r(\alpha, \beta)$ such that any integer exceeding $r(\alpha, \beta)$ can be partitioned into distinct positive integers exceeding β whose reciprocals sum to α.

THEOREM 1. *If n is an integer exceeding* 77 *then there exist positive integers k, $a_1, a_2, \cdots, a_k$* such that:

1. $1 < a_1 < a_2 < \cdots < a_k$.
2. $n = a_1+a_2+\cdots+a_k$.
3. $1 = a_1^{-1}+a_2^{-1}+\cdots+a_k^{-1}$.

PROOF. Consider the following table. The entry

$$n\colon \quad a_1, a_2, \cdots, a_k$$

indicates that

$$a_1+a_2+\cdots+a_k = n \quad \text{and} \quad a_1^{-1}+a_2^{-1}+\cdots+a_k^{-1} = 1.$$

78: 2, 6, 8, 10, 12, 40	90: 3, 4, 6, 9, 18, 20, 30
79: 2, 3, 10, 24, 40	91: 3, 4, 6 11, 12, 22, 33
80: 2, 4, 10, 15, 21, 28	92: 3, 4, 8, 9, 10, 18, 40
81: 2, 4, 10, 15, 20, 30	93: 3, 5, 6, 9, 10, 15, 45
82: 2, 4, 9, 18, 21, 28	94: 4, 5, 6, 9, 10, 12, 18, 30
83: 2, 4, 9, 18, 20, 30	95: 3, 4, 6, 8, 20, 24, 30
84: 2, 6, 7, 9, 18, 42	96: 2, 6, 9, 12, 18, 21, 28
85: 2, 4, 10, 15, 18, 36	97: 2, 6, 9, 12, 18, 20, 30
86: 2, 5, 9, 10, 15, 45	98: 3, 4, 5, 12, 18, 20, 36
87: 2, 4, 6, 15, 60	99: 3, 4, 6, 8, 18, 24, 36
88: 3, 4, 6, 10, 15, 20, 30	100: 3, 4, 7, 8, 12, 24, 42
89: 2, 3, 9, 30, 45	101: 2, 4, 5, 30, 60

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102: 3, 4, 6, 10, 15, 16, 48
103: 2, 5, 9, 18, 20, 21, 28
104: 3, 4, 6, 9, 16, 18, 48
105: 2, 5, 11, 12, 20, 22, 33
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107: 2, 6, 7, 14, 20, 28, 30
108: 3, 4, 5, 6, 30, 60
109: 3, 4, 6, 8, 16, 24, 48
110: 2, 3, 9, 24, 72
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112: 3, 4, 6, 7, 20, 30, 42
113: 2, 3, 8, 40, 60
114: 3, 4, 7, 8, 10, 40, 42
115: 3, 4, 6, 9, 12, 27, 54
116: 3, 4, 6, 7, 18, 36, 42
117: 2, 6, 9, 10, 15, 30, 45
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119: 2, 4, 7, 16, 42, 48
120: 3, 4, 9, 10, 12, 18, 24, 40
121: 2, 3, 8, 36, 72
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123: 2, 6, 7, 12, 18, 36, 42
124: 2, 6, 8, 9, 24, 30, 45
125: 3, 5, 6, 9, 16, 18, 20, 48
126: 3, 4, 6, 7, 16, 42, 48
127: 2, 6, 8, 12, 15, 24, 60
128: 3, 6, 8, 10, 12, 15, 20, 24, 30
129: 3, 4, 6, 12, 18, 20, 30, 36
130: 2, 3, 15, 20, 30, 60
131: 3, 4, 6, 8, 12, 42, 56
132: 3, 6, 8, 10, 12, 15, 18, 24, 36
133: 3, 4, 6, 8, 12, 40, 60
134: 2, 3, 8, 33, 88
135: 2, 3, 9, 22, 99
136: 2, 4, 5, 25, 100
137: 3, 5, 6, 7, 18, 20, 36, 42
138: 3, 4, 6, 9, 14, 18, 84
139: 2, 3, 12, 24, 42, 56
140: 2, 6, 11, 12, 18, 22, 33, 36
141: 2, 3, 8, 32, 96
142: 3, 6, 8, 10, 12, 15, 16, 24, 48
143: 3, 4, 6, 8, 14, 24, 84
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159: 3, 4, 10, 11, 12, 22, 24, 33, 40
160: 3, 4, 6, 7, 14, 42, 84
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162: 2, 6, 7, 9, 24, 42, 72
163: 2, 3, 8, 30, 120
164: 2, 4, 8, 20, 30, 40, 60
165: 2, 8, 9, 12, 16, 18, 20, 80
166: 3, 6, 7, 8, 12, 16, 24, 42, 48
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175: 2, 5, 8, 12, 20, 32, 96
177: 4, 5, 6, 8, 12, 14, 20, 24, 84
179: 3, 4, 8, 9, 20, 24, 27, 30, 54
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183: 3, 6, 8, 11, 12, 16, 22, 24, 48
185: 3, 6, 8, 10, 14, 15, 18, 35, 36, 40
187: 3, 6, 8, 10, 11, 18, 22, 33, 36, 40
189: 3, 4, 11, 12, 18, 20, 22, 30, 33, 36
191: 3, 4, 8, 11, 12, 22, 33, 42, 56
193: 3, 4, 8, 11, 12, 22, 33, 40, 60
195: 3, 4, 8, 14, 15, 16, 20, 35, 80
197: 3, 4, 8, 11, 16, 20, 22, 33, 80
199: 3, 4, 11, 12, 16, 20, 22, 30, 33, 48
201: 3, 4, 8, 11, 12, 22, 33, 36, 72
203: 3, 4, 11, 12, 16, 18, 22, 33, 36, 48
205: 3, 4, 6, 14, 15, 28, 30, 35, 70
207: 3, 4, 6, 11, 22, 28, 30, 33, 70
209: 3, 4, 6, 14, 15, 24, 35, 36, 72
211: 3, 4, 6, 11, 22, 24, 33, 36, 72
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223: 3, 4, 12, 14, 15, 16, 20, 24, 35, 80
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237: 3, 4, 11, 12, 14, 18, 22, 33, 36, 72
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259: 3, 6, 8, 11, 12, 16, 22, 33, 40, 48, 60
261: 3, 4, 11, 12, 18, 22, 24, 33, 36, 42, 56
263: 3, 4, 11, 12, 18, 22, 24, 33, 36, 40, 60
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267: 3, 6, 8, 11, 12, 16, 22, 33, 36, 48, 72
269: 3, 6, 8, 11, 14, 16, 22, 24, 33, 48, 84
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303: 3, 4, 11, 12, 18, 20, 22, 30, 33, 60, 90
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311: 3, 4, 6, 11, 16, 22, 33, 72, 144
313: 3, 4, 11, 12, 18, 22, 24, 30, 33, 36, 120
315: 3, 4, 11, 12, 14, 22, 24, 33, 36, 72, 84
317: 3, 4, 11, 12, 16, 18, 22, 33, 48, 60, 90
319: 3, 6, 10, 11, 12, 18, 22, 32, 33, 36, 40, 96
321: 3, 6, 8, 11, 12, 14, 22, 32, 33, 84, 96
323: 3, 4, 11, 12, 16, 22, 24, 30, 33, 48, 120
325: 3, 6, 11, 12, 16, 20, 22, 24, 30, 33, 40, 48, 60
327: 3, 6, 8, 11, 12, 22, 24, 33, 36, 40, 60, 72
329: 3, 4, 6, 11, 22, 30, 33, 40, 60, 120
331: 3, 6, 8, 11, 12, 16, 22, 24, 33, 84, 112
333: 3, 6, 11, 12, 16, 20, 22, 24, 30, 33, 36, 48, 72

Now notice that each of the transformations

(a) $$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_k} = \frac{1}{2} + \frac{1}{2d_1} + \cdots + \frac{1}{2d_k}$$

and

(b) $$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_k} = \frac{1}{3} + \frac{1}{7} + \frac{1}{78} + \frac{1}{91} + \frac{1}{2d_1} + \cdots + \frac{1}{2d_k}$$

keeps the denominators distinct as long as no d_i is 1 or 39. The new sums of the denominators are $2U+2$ and $2U+179$ respectively, where

$$U = d_1+d_2+ \cdots +d_k.$$

Since no 1 or 39 is used in the table, then by using (a) we can first extend the table to the even integers from $2.88+2 = 168$ through $2.166+2 = 334$ where none of the new denominators is 1 or 39. Then by using (a) and (b) we can next extend the table from $2.78+179 = 335$ to $2.334+2 = 670$. But none of the new denominators is 1 or 39 so we can again extend the table, and so on. This proves the theorem.

THEOREM 2. *For any integer m there exists an $r = r(m)$ such that if the integer n exceeds r then there exist positive integers $k, a_1, a_2, \cdots, a_k$ such that:*

1. $m < a_1 < a_2 < \cdots < a_k$.
2. $n = a_1+a_2+ \cdots +a_k$.
3. $1 = a_1^{-1}+a_2^{-1}+ \cdots +a_k^{-1}$.

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PROOF. The proof rests on the following lemma which is a special case of a theorem of the author [1]:

LEMMA. *Let p/q be a positive rational and let t be an integer relatively prime to q. Then for all s there exist positive integers $k, c_1, \cdots, c_k$ such that*

$$s < c_1 < c_2 < \cdots < c_k$$

and

$$\frac{p}{q} = \sum_{i=1}^{k} \frac{1}{tc_i - 1}.$$

Let m be an arbitrary integer. If $m \leqq 1$ then the theorem is true by Theorem 1. Thus, we may assume that $m \geqq 2$. By the Dirichlet theorem on primes in an arithmetic progression (cf. [2]) there is an h such that $mh-1$ is a prime greater than 13. Consider the quantity

$$\frac{m-1}{m} - \frac{1}{m(mh-1)} = \frac{(m-1)h-1}{mh-1}.$$

By the lemma (and the fact that there are infinitely many primes) there exist primes $q_1, q_2, \cdots, q_m$ and positive integers $k, c_1, c_2, \cdots, c_k$ such that:

1. $mh-1 < q_1$.
2. $mq_i(mq_i-1) < q_{i+1}$ for $1 \leqq i \leqq m$.
3. $\dfrac{(m-1)h-1}{mh-1} > \dfrac{1}{mq_1-1} + \cdots + \dfrac{1}{mq_m-1}$.
4. $mq_m(mq_m-1) < mc_1-1$.
5. $c_i < c_{i+1}$ for $1 \leqq i \leqq k-1$.
6. $\dfrac{(m-1)h-1}{mh-1} - \left(\dfrac{1}{mq_1-1} + \cdots + \dfrac{1}{mq_m-1}\right) = \dfrac{1}{mc_1-1} + \cdots + \dfrac{1}{mc_k-1}$.

The preceding lemma implies that 4., 5., and 6. can be satisfied. Thus we have

$$1 = \frac{1}{m} + \frac{1}{m(mh-1)} + \frac{1}{mq_1-1} + \cdots + \frac{1}{mq_m-1} + \frac{1}{mc_1-1} + \cdots + \frac{1}{mc_k-1}.$$

Notice that if $(mq_i-1)^{-1}$ is replaced by

$$\frac{1}{mq_i} + \frac{1}{mq_i(mq_i-1)}$$

then all the denominators are still distinct and their sum modulo m has been

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increased by 1. Consider the m representations of 1 given by:

$$1=\frac{1}{m}+\frac{1}{m(mh-1)}+\left(\frac{1}{mq_1}+\frac{1}{mq_1(mq_1-1)}\right)+\cdots+\left(\frac{1}{mq_j}+\frac{1}{mq_j(mq_j-1)}\right)$$
$$+\frac{1}{mq_{j+1}-1}+\cdots+\frac{1}{mq_m-1}+\frac{1}{mc_1-1}+\cdots+\frac{1}{mc_k-1}$$

for $1\leqq j\leqq m$. Let U_j denote the sum of the denominators of the jth representation of 1. Then for $1\leqq j\leqq m$ the U_j run through a complete residue system modulo m. By Theorem 1 any n exceeding 77 is the sum of the denominators of some representation of 1, where the denominators are distinct and contain only 2, 3, 5, 7, 11, 13 as prime factors. If

$$1=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_w}$$

for some w where the d_i are distinct and $U=d_1+d_2+\cdots+d_w$ then

$$1=\frac{1}{md_1}+\cdots+\frac{1}{md_w}+\frac{1}{m(mh-1)}+\left(\frac{1}{mq_1}+\frac{1}{mq_1(mq_1-1)}\right)+\cdots$$
$$+\left(\frac{1}{mq_j}+\frac{1}{mq_j(mq_j-1)}\right)+\frac{1}{mq_{j+1}-1}+\cdots+\frac{1}{mq_m-1}+\frac{1}{mc_1-1}+\cdots+\frac{1}{mc_k-1}$$

where the sum of the denominators in this new representation of 1 is

$$mU+U_j-m.$$

Thus, by using the representations of all numbers greater than 77 given by Theorem 1 and applying to each of these the m transformations arising from the m representations of 1 given in the previous paragraph, it follows at once that every integer exceeding $78m+U_m-m$ occurs as the sum of the denominators of at least one of the new representations of 1. But in each one of these representations all the denominators used are greater than m. It remains only to check that any one of these representations contains distinct denominators. The only way this could fail to happen is for some md_i to be equal to one of the integers mq_j, $mq_j(mq_j-1)$ or $m(mh-1)$. However, since $mh-1$ is a prime greater than 13 as are the q_j and 13 is the largest prime ever used in any d_i, then we can never have equality. Thus, the denominators in each new representation are distinct. This proves the theorem.

THEOREM 3. *For any positive rationals α and β there exists an $r=r(\alpha,\beta)$ such that if the integer n exceeds r then there exist positive integers $k, a_1, a_2, \cdots, a_k$ such that:*

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1. $\beta < a_1 < a_2 < \cdots < a_k$.
2. $n = a_1 + a_2 + \cdots + a_k$.
3. $\alpha = a_1^{-1} + a_2^{-1} + \cdots + a_k^{-1}$.

PROOF. By the lemma mentioned in Theorem 2 it follows that there exist positive integers $k, c_1, c_2, \cdots, c_k$ such that

$$\beta < c_1 < c_2 < \cdots < c_k$$

and

$$\alpha = \sum_{i=1}^{k} \frac{1}{c_i}.$$

Therefore, if we let $c = 2c_k$ then

$$\begin{aligned}\alpha &= \frac{1}{c_1} + \cdots + \frac{1}{c_k} = \frac{1}{c_1} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c} + \frac{1}{c} \\ &= \frac{1}{c_1} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c} + \frac{1}{c}\left(\frac{1}{d_1} + \cdots + \frac{1}{d_w}\right)\end{aligned}$$

where

$$1 = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_w}$$

for some w and $d_1 < d_2 < \cdots < d_w$. Therefore we have

$$\alpha = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c} + \frac{1}{cd_1} + \cdots + \frac{1}{cd_w} \tag{1}$$

and as $U = d_1 + d_2 + \cdots + d_w$ runs through all sufficiently large integers (by Theorem 1), then representation (1) has the sum of its denominators run through all sufficiently large numbers congruent to $c_1 + c_2 + \cdots + c_{k-1}$ (modulo c). But we also have

$$\alpha = \frac{1}{c_1} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c+1} + \frac{1}{c(c+1)} + \frac{1}{cd_1} + \cdots + \frac{1}{cd_w}. \tag{2}$$

By restricting the d_i so that $d_i > c+1$, then by Theorem 2 we can still have $U = d_1 + d_2 + \cdots + d_w$ run through all sufficiently large integers (keeping the d_i distinct) and hence the denominators in (2) are distinct and their sum runs through all sufficiently large integers congruent to $c_1 + \cdots + c_{k-1} + 1$ (modulo c). Similarly,

$$\begin{aligned}\alpha = {} & \frac{1}{c_1} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c+1} + \frac{1}{c(c+1)+1} \\ & + \frac{1}{c(c+1)(c(c+1)+1)} + \frac{1}{cd_1} + \cdots + \frac{1}{cd_w}\end{aligned} \tag{3}$$

and restricting the d_i so that $d_i > (c+1)(c(c+1)+1)$ then as $U = d_1+d_2 + \cdots + d_w$ runs through all sufficiently large integers (by Theorem 2), the denominators in (3) are distinct and their sum assumes all sufficiently large integers congruent to $c_1 + \cdots + c_{k-1} + 2$ (modulo c). Continuing in this manner for c steps, we see that the sums of the denominators in (1), (2), (3), $\cdots$, (c) run over all sufficiently large integers and since in each representation the denominators are distinct, then the theorem is proved.

REMARKS. It seems to be a difficult question to determine exactly the least integer value that $r(\alpha, \beta)$ may assume. Theorem 1 shows that we may take $r(1, 1)$ to be any integer $\geqq 77$. On the other hand, in some recent unpublished work of D. H. Lehmer, it has been shown that we must have $r(1, 1) \geqq 77$, i.e., 77 cannot be partitioned into distinct positive integers whose reciprocals sum to 1.

It would not be unreasonable to conjecture that 2. in Theorem 3 could be replaced by:

2′. $$n = f(a_1) + f(a_2) + \cdots + f(a_k)$$

where f is any polynomial mapping integers into integers which has a positive leading coefficient and such that for any prime p there is an m such that p does not divide $f(m)$. At present, however, very little is known about this problem.

References

[1] R. Graham, On Finite Sums of Unit Fractions, (to appear in *Proc. London Math. Soc.*).
[2] W. Levèque, *Topics in Number Theory* (Addison-Wesley, Reading, 1956), p. 76.

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