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MINIMAX INEQUALITIES AND GENERALISATIONS OF THE GALE-NIKAIDO-DEBREU LEMMA

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Some minimax inequalities are first proved both in the compact case and in the non-compact case using the concept of escaping sequences introduced by Border. Applications are given to deduce a generalisation of the Gale-Nikaido-Debreu Lemma due to Mehta and Tarafdar and to obtain a new generalisation of the Gale-Nikaido-Debreu Lemma from which the corresponding generalisation due to Grandmont is derived.

1. INTRODUCTION

The Gale-Nikaido-Debreu Lemma (in short, the GND Lemma), see [**4**, **6**, **12**] (see also [**2**, Theorem 18.1, p.81]), is fundamental to proving the existence of a market equilibrium of an economy, for example, see Border [**2**] and Debreu [**5**]. Recently, there have been many generalisation of this Lemma, see [**7**, **8**, **10**, **11**, **14**]. The objective of this paper is two-fold:

(1) we first give some minimax inequalities both in the compact case and in the non-compact case using the concept of escaping sequences introduced by Border [**2**];

(2) as applications of the minimax inequalities,

- (a) we deduce a generalisation of the Gale-Nikaido-Debreu Lemma due to Mehta and Tarafdar [**10**] which in turn generalises that of Yannelis [**14**] and
- (b) we obtain a new generalisation of the Gale-Nikaido-Debreu Lemma, from which the corresponding generalisation due to Grandmont [**8**] is derived.

2. PRELIMINARIES

If A is a set, 2^A denotes the family of all subsets of A. If A is a subset of a vector space, coA denotes the convex hull of A. We shall denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all natural numbers respectively. If A is a non-empty subset of $\mathbb{R}^m$ and $p \in \mathbb{R}^m$, $\operatorname{dist}(p, A)$ denotes the distance from p to A. If $p = (p_1, \cdots, p_m) \in \mathbb{R}^m$, then $p \geqslant 0$ (respectively, $p > 0$, $p \leqslant 0$) if $p_i \geqslant 0$

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(respectively, $p_i > 0$, $p_i \leqslant 0$) for all $i = 1, \cdots, m$. If X and Y are topological spaces and $T : X \to 2^Y$, then T is upper semicontinuous if for each open subset U of Y, the set $\{x \in X : T(x) \subset U\}$ is open in X. If X is a non-empty convex subset of a vector space, then $f : X \to \mathbb{R}$ is quasi-concave (respectively, quasi-convex) if for each $\lambda \in \mathbb{R}$, the set $\{x \in X : f(x) > \lambda\}$ (respectively, the set $\{x \in X : f(x) < \lambda\}$) is convex.

3. Generalisations of the GND Lemma: The compact case

We first prove the following minimax inequality:

Theorem 1. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E and let Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose that the real-valued function $f : X \times Y \to \mathbb{R}$ and the set-valued map $T : X \to 2^Y$ satisfy the following conditions:*

(i) *for each fixed $y \in Y$, $x \to f(x, y)$ is upper semicontinuous and quasi-concave (respectively, $x \to f(x, y)$ is concave);*

(ii) *for each fixed $x \in X$, $y \to f(x, y)$ is lower semicontinuous and quasi-convex (respectively, $y \to f(x, y)$ is lower semicontinuous and convex);*

(iii) *for each $x \in X$, $T(x)$ is non-empty compact, convex and $\min_{y \in T(x)} f(x, y) \leqslant 0$;*

(iv) *for each $x \in X$ with $\{u \in X : \min_{y \in T(x)} f(u, y) > 0\} \neq \emptyset$, there is $\overline{x} \in X$ such that $x \in \operatorname{int}\{v \in X : \min_{y \in T(v)} f(\overline{x}, y) > 0\}$.*

Then there exists $(x^*, y^*) \in X \times Y$ with $y^* \in T(x^*)$ such that $f(x, y^*) \leqslant 0$ for all $x \in X$.

Proof: Define the set-valued map $F : X \to 2^Y$ by $F(x) = \{u \in X : \min_{y \in T(x)} f(u, y) > 0\}$ for each $x \in X$. Fix an $x \in X$. For any $u_1 \in F(x)$ and $u_2 \in F(x)$, by (ii), (iii), there exist $y_1 \in T(x)$ and $y_2 \in T(x)$ such that $f(u_1, y_1) = \min_{y \in T(x)} f(u_1, y) > 0$ and $f(u_2, y_2) = \min_{y \in T(x)} f(u_2, y) > 0$. For any $\alpha \in [0, 1]$, there exists $y_\alpha \in T(x)$ such that $f(\alpha u_1 + (1 - \alpha) u_2, y_\alpha) = \min_{y \in T(x)} f(\alpha u_1 + (1 - \alpha) u_2, y)$. By (i),

$$\begin{aligned} f(\alpha u_1 + (1 - \alpha) u_2, y_\alpha) &\geqslant \min\{f(u_1, y_\alpha), f(u_2, y_\alpha)\} \\ &\geqslant \min\{f(u_1, y_1), f(u_2, y_2)\} > 0 \end{aligned}$$

so that $\alpha u_1 + (1 - \alpha) u_2 \in F(x)$. Thus $F(x)$ is convex for each $x \in X$.

By (iii), $x \notin F(x)$ for all $x \in X$.

For each $x' \in X$, $F^{-1}(x') = \{v \in X : \min_{y \in T(v)} f(x', y) > 0\}$. By (iv), if $F(x) \neq \emptyset$, then there is $\overline{x} \in X$ such that $x \in \operatorname{int} F^{-1}(\overline{x})$.

Theorem 4 of [**10**] implies that there exists $x^* \in X$ such that $F(x^*) = \emptyset$, that is, $\min_{y \in T(x^*)} f(u,y) \leqslant 0$ for all $u \in X$.

Now, since conditions (i), (ii) and (iii) hold, by [**13**, Theorem 3.4] (respectively by [**7**, Corollary 9.4 (b)]), we have

$$\min_{y \in T(x^*)} \max_{x \in X} f(x,y) = \max_{x \in X} \min_{y \in T(x^*)} f(x,y) \leqslant 0$$

(respectively, $\min_{y \in T(x^*)} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in T(x^*)} f(x,y) \leqslant 0$).

Hence there exists $y^* \in F(x^*)$ such that $\max_{x \in X} f(x,y^*) \leqslant 0$ (respectively, $\sup_{x \in X} f(x,y^*) \leqslant 0$), that is, $f(x,y^*) \leqslant 0$ for all $x \in X$. ∎

As an application of Theorem 1, we have the following generalisation of the Gale-Nikaido-Debreu Lemma due to Mehta and Tarafdar [**10**, Theorem 8]:

COROLLARY 1. *Let E be a real Hausdorff locally convex topological vector space, E^* be the topological dual of E equipped with the weak* topology, C be a closed convex cone of E having an interior point e, $C^* = \{p \in E^* : \langle p,y\rangle \leqslant 0$ for all $y \in C\} \neq \{0\}$ be the dual cone of C, and $\Delta = \{p \in C^* : \langle p,e\rangle = -1\}$. Suppose that the set-valued map $T : \Delta \to 2^E$ satisfies the following conditions:*

(i) *for each $p \in \Delta$, $T(p)$ is non-empty compact convex and $\min_{y \in T(p)} \langle p,y\rangle \leqslant 0$;*

(ii) *for each $p \in \Delta$ with $\{q \in \Delta : \min_{y \in T(p)} \langle q,y\rangle > 0\} \neq \emptyset$, there is $\overline{p} \in \Delta$ such that $p \in \operatorname{int}\{\overline{q} \in \Delta : \min_{y \in T(\overline{q})} \langle \overline{p},y\rangle > 0\}$.*

Then there exists $p^* \in \Delta$ such that $T(p^*) \cap C \neq \emptyset$.

PROOF: Set $X = \Delta$, then X is convex. Since Δ is equicontinuous and w^*-closed, the Alaoglu theorem [**9**, Theorem 3.8, p.123] implies that it is w^*-compact.

Set $Y = E$ and $f(p,y) = \langle p,y\rangle$, then the conditions of Theorem 1 hold so that there exist $p^* \in \Delta$ and $y^* \in T(p^*)$ such that $\langle p,y^*\rangle \leqslant 0$ for all $p \in \Delta$.

We shall prove that $y^* \in C$ and hence $y^* \in T(p^*) \cap C$.

If $y^* \notin C$, since E is locally convex and C is closed convex, by [**3**, p.111, Corollary 3.10], there exists $r \in E^*$ with $r \neq 0$ such that $\sup_{y \in C} \langle r,y\rangle < \langle r,y^*\rangle$. Since $0 \in C$ and C is a cone, we must have $\sup_{y \in C} \langle r,y\rangle = 0$. It follows that $r \in C^*$ and $\langle r,y^*\rangle > 0$. Since $e \in \operatorname{int} C$, we have $r(e) < 0$. Let $\overline{r} = -(r/r(e))$, then $\overline{r} \in \Delta$. But $\langle \overline{r},y^*\rangle > 0$ which contradicts the fact that $\langle p,y^*\rangle \leqslant 0$ for all $p \in \Delta$. Hence we must have $y^* \in C$. ∎

By [**10**, Remark 1] Corollary 1 is more general than Theorem 3.1 of [**14**]. Hence Theorem 1 also generalises Theorem 3.1 of [**14**].

In what follows we deduce another minimax inequality from Theorem 1.

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THEOREM 2. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E, and let Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose that the real-valued function $f : X \times Y \to \mathbb{R}$ and the set-valued map $T : X \to 2^Y$ satisfy the following conditions:*

(i) *for each fixed $y \in Y$, $x \to f(x,y)$ is upper semicontinuous and quasi-concave (respectively, $x \to f(x,y)$ is concave);*

(ii) *for each fixed $x \in X$, $y \to f(x,y)$ is lower semicontinuous and quasi-convex (respectively, $y \to f(x,y)$ is lower semicontinuous and convex);*

(iii) *for each $x \in X$, $T(x)$ is non-empty compact convex and $\min_{y \in T(x)} f(x,y) \leqslant 0$;*

(iv) *for each $x \in X$, $\{u \in X : \min_{y \in T(u)} f(x,y) \leqslant 0\}$ is closed.*

Then there exists $(x^*, y^*) \in X \times Y$ with $y^* \in T(x^*)$ such that $f(x,y^*) \leqslant 0$ for all $x \in X$.

PROOF: We only need to prove that the condition (iv) of Theorem 1 holds: Indeed, let $x \in X$ be such that $\{u \in X : \min_{y \in T(x)} f(u,y) > 0\} \neq \emptyset$ and take $\overline{x} \in \{u \in X : \min_{y \in T(x)} f(u,y) > 0\}$, that is, $\min_{y \in T(x)} f(\overline{x},y) > 0$. Thus $x \notin \{u \in X : \min_{y \in T(u)} f(\overline{x},y) \leqslant 0\}$. Since $\{u \in X : \min_{y \in T(u)} f(\overline{x},y) \leqslant 0\}$ is closed, it follows that $x \in \operatorname{int}\{u \in X : \min_{y \in T(u)} f(\overline{x},y) > 0\}$. □

As an application of Theorem 2, we derive the following minimax inequality due to Granas and Liu [7, Theorem (13.1)]:

COROLLARY 2. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E, and let Y be a non-empty convex subset of a Hausdorff topological vector space F. Let $T : X \to 2^Y$ be upper semicontinuous with non-empty compact convex values and $g : X \times Y \to \mathbb{R}$ satisfy one of the following conditions:*

(I) $\begin{cases} \text{For each fixed } y \in Y, \quad x \to g(x,y) \text{ is lower semicontinuous} \\ \text{and quasi-convex;} \\ \text{for each fixed } x \in X, \quad y \to g(x,y) \text{ is upper semicontinuous} \\ \text{and quasi-concave.} \end{cases}$

(II) $\begin{cases} \text{For each fixed } y \in Y, \quad x \to g(x,y) \text{ is convex;} \\ \text{for each fixed } x \in X, \quad y \to g(x,y) \text{ is upper semicontinuous} \\ \text{and concave.} \end{cases}$

Then for each $\lambda \in \mathbb{R}$, one of the following properties holds:

(A) *there exists $\overline{x} \in X$ such that $\max_{y \in T(\overline{x})} g(\overline{x},y) < \lambda$;*

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(B) *there exists* $(x^*, y^*) \in X \times Y$ *with* $y^* \in T(x^*)$ *such that* $\min_{x \in X} g(x, y^*) \geqslant \lambda$.

PROOF: Assume that (I) (respectively, (II)) holds. If (A) is not true, then for each $x \in X$, $\max_{y \in T(x)} [g(x,y) - \lambda] \geqslant 0$. Define $f : X \times Y \to \mathbb{R}$ by $f(x,y) = \lambda - g(x,y)$ for each $(x,y) \in X \times Y$. Then the conditions (i), (ii) and (iii) of Theorem 2 hold. Now fix an $x \in X$. Define $W : X \times Y \to \mathbb{R}$ by $W(u,y) = g(x,y) - \lambda$ for each $(u,y) \in X \times Y$. Then W is upper semicontinuous. Since T is upper semicontinuous such that for each $u \in X$, $T(u)$ is compact, by [**1**, p.52, Theorem 5], the map $V : X \to \mathbb{R}$ defined by $V(u) = \max_{y \in T(u)} [g(x,y) - \lambda]$ for each $u \in X$ is upper semicontinuous. It follows that the set $\{u \in X : \min_{y \in T(u)} f(x,y) \leqslant 0\} = \{u \in X : V(u) \geqslant 0\}$ is closed. Thus the condition (iv) of Theorem 2 also holds. Hence there exists $(x^*, y^*) \in X \times Y$ with $y^* \in T(x^*)$ such that $f(x, y^*) \leqslant 0$ for all $x \in X$, that is, $\min_{x \in X} g(x,y) \geqslant \lambda$. ∎

The following simple example shows that Theorem 2 is a true generalisation of [**7**, Theorem (13.1)].

EXAMPLE 1. Let $E = F = \mathbb{R}$, $X = [0, \pi/2]$ and $Y = \mathbb{R}$. Define $g : X \times Y \to \mathbb{R}$ by $g(x,y) = y - \sin x$ for all $(x,y) \in X \times Y$. Then for each fixed $y \in Y$, $x \to g(x,y)$ is continuous and convex and for each fixed $x \in X$, $y \to g(x,y)$ is continuous and concave. Define $T : X \to 2^Y$ by

$$T(x) = \begin{cases} [\frac{1}{2}\sin x, \sin x], & \text{if } x \neq 0, \\ \{1\}, & \text{if } x = 0 \end{cases}$$

for each $x \in X$. Then T has non-empty compact convex values but T is not upper semicontinuous at $x = 1$ so that [**7**, Theorem (13.1)] is not applicable. However, if we let $f = -g$, then we have $\min_{y \in T(x)} f(x,y) \leqslant 0$ for each $x \in X$. We shall show that for each $x \in X$, the set $\{u \in X : \min_{y \in T(u)} f(x,y) > 0\}$ is open in X. Indeed, let $x \in X$ be arbitrarily fixed. If $u \in X$ is such that $\min_{y \in T(u)} f(x,y) > 0$, we must have $0 < u < x$. Let $\delta > 0$ be such that $(u - \delta, u + \delta) \subset (0, x)$. It follows that for each $v \in (u - \delta, u + \delta)$, $\min_{y \in T(v)} f(x,y) = \sin x - \sin v > 0$. This shows that the set $\{u \in X : \min_{y \in T(u)} f(x,y) > 0\}$ is open in X so that the set $\{u \in X : \min_{y \in T(u)} f(x,y) \leqslant 0\}$ is closed in X. Therefore Theorem 2 is applicable.

4. GENERALISATIONS OF THE GND LEMMA: THE NON-COMPACT CASE

We need the concept of an escaping sequence introduced in [**2**, p.34]: Let X be a topological space such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of

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non-empty compact sets. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be escaping from X (relative to $\{C_n\}_{n=1}^{\infty}$) if for each $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $y_k \notin C_n$ for all $k \geqslant M$.

We shall prove the following minimax inequality on a non-compact set.

THEOREM 3. *Let X be a non-empty subset of a Hausdorff topological vector space E such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X, and let Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose that the real-valued function $f : X \times Y \to \mathbb{R}$ and the set-valued map $T : X \to 2^Y$ satisfy the following conditions:*

(i) *for each fixed $y \in Y$, $x \to f(x,y)$ is upper semicontinuous and quasi-concave (respectively, $x \to f(x,y)$ is concave);*
(ii) *for each fixed $x \in X$, $y \to f(x,y)$ is lower semicontinuous and quasi-convex (respectively, $y \to f(x,y)$ is lower semicontinuous and convex);*
(iii) *for each $x \in X$, $T(x)$ is non-empty compact convex and $\min_{y \in T(x)} f(x,y) \leqslant 0$;*
(iv) *for each $n \in \mathbb{N}$ and each $x \in C_n$ with $\{u \in C_n : \min_{y \in T(x)} f(u,y) > 0\} \neq \emptyset$, there is $\bar{x} \in C_n$ such that $x \in \operatorname{int}\{v \in C_n : \min_{y \in T(v)} f(\bar{x},y) > 0\}$;*
(v) *for each sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in C_n$ for each $n = 1, 2, \cdots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ and each sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n \in T(x_n)$ for each $n = 1, 2, \cdots$, there exist $n_0 \in \mathbb{N}$ and $x'_{n_0} \in C_{n_0}$ with $f(x'_{n_0}, y_{n_0}) > 0$.*

Then there exists $(x^*, y^*) \in X \times Y$ with $y^* \in T(x^*)$ such that $f(x, y^*) \leqslant 0$ for all $x \in X$.

PROOF: For each $n \in \mathbb{N}$ by Theorem 1, there exists $(x_n, y_n) \in C_n \times Y$ with $y_n \in T(x_n)$ such that $f(x, y_n) \leqslant 0$ for all $x \in C_n$.

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ were escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. By (v), there exist $n_0 \in N$ and $x'_{n_0} \in C_{n_0}$ with $f(x'_{n_0}, y_{n_0}) > 0$ which is a contradiction. Therefore the sequence $\{x_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, so that some subsequence of $\{x_n\}_{n=1}^{\infty}$ must lie entirely in some C_{n_1}. Since C_{n_1} is compact, there exist a subnet $\{z_\alpha\}_{\alpha \in \Gamma}$ of $\{x_n\}_{n=1}^{\infty}$ in C_{n_1} and $x^* \in C_{n_1}$ such that $z_\alpha \to x^*$. Let $z_\alpha = x_{n(\alpha)}$ where $n(\alpha) \to \infty$.

If $\{u \in X : \min_{y \in T(x^*)} f(u,y) > 0\} \neq \emptyset$, there exists $n_2 \geqslant n_1$ such that $\{u \in C_{n_2} : \min_{y \in T(x^*)} f(u,y) > 0\} \neq \emptyset$. By (iv), there is $\bar{x} \in C_{n_2}$ such that $x^* \in int\{v \in C_{n_2} : \min_{y \in T(v)} f(\bar{x}, y) > 0\}$. Since $z_\alpha \to x^*$, there is α_0 such that $n(\alpha_0) \geqslant n_2$ and $\min_{y \in T(z_{\alpha_0})} f(\bar{x}, y) > 0$, hence $f(\bar{x}, y_{n(\alpha_0)}) > 0$. This contradicts the fact that

$\overline{x} \in C_{n(\alpha_0)}$ and $f(\overline{x}, y_{n(\alpha_0)}) \leqslant 0$. Therefore $\{u \in X : \min_{y \in T(x^*)} f(u,y) > 0\} = \emptyset$, that is, $\min_{y \in T(x^*)} f(u,y) \leqslant 0$ for all $u \in X$.

By [**13**, Corollary 3.5] (respectively, by [**7**, Corollary 9.4 (b)]), we have

$$\min_{y \in T(x^*)} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in T(x^*)} f(x,y) \leqslant 0.$$

Hence there exists $y^* \in T(x^*)$ such that $\sup\limits_{x \in X} f(x, y^*) \leqslant 0$, that is, $f(x,y^*) \leqslant 0$ for all $x \in X$. ∎

As an application of Theorem 3, we obtain the following generalisation of the Gale-Nikaido-Debreu Lemma:

Theorem 4. *Let* $\Delta = \{p \in \mathbb{R}^m : p \geqslant 0, \sum\limits_{i=1}^{m} p_i = 1\}$, $S = \{p \in \mathbb{R}^m : p > 0, \sum\limits_{i=1}^{m} p_i = 1\}$ *and* $C_n = co\{p \in S : \operatorname{dist}(p, \Delta \backslash S) \geqslant 1/n\}$ *for* $n = 1, 2, \cdots$. *Suppose that the set-valued map* $T : S \to 2^{R^m}$ *satisfies the following conditions:*

(i) *T is upper semicontinuous such that for each $p \in S$, $T(p)$ is non-empty compact convex;*

(ii) *for each $p \in S$ and each $y \in T(p)$, $\langle p, y \rangle = 0$;*

(iii) *for each sequence $\{p_n\}_{n=1}^{\infty}$, where $p_n \in C_n$ for each $n = 1, 2, \cdots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ and for each sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n \in T(p_n)$ for each $n = 1, 2, \cdots$, there exist $n_0 \in \mathbb{N}$ and $p'_{n_0} \in C_{n_0}$ with $\langle p'_{n_0}, y_{n_0} \rangle > 0$.*

Then there exists $p^* \in S$ such that $0 \in T(p^*)$.

Proof: The conclusion clearly holds for $m = 1$. Now suppose that $m > 1$. We may assume that each C_n is non-empty. Note that each C_n is compact and convex and $S = \bigcup\limits_{n=1}^{\infty} C_n$. Set $X = S$ and $Y = \mathbb{R}^m$. Let $f : X \times Y \to \mathbb{R}$ be defined by $f(p,y) = \langle p, y \rangle$ for each $(p,y) \in X \times Y$. Then similar to the proof of Corollary 2, we can prove that for each $n \in \mathbb{N}$ and each $p \in C_n$, the set $\{u \in C_n : \min\limits_{y \in T(u)} f(p,y) \leqslant 0\}$ is closed. Also, similar to the proof of Theorem 2, the condition (iv) of Theorem 3 holds. By Theorem 3, there exists $(p^*, y^*) \in S \times Y$ with $y^* \in T(p^*)$ such that $\langle p, y^* \rangle \leqslant 0$ for all $p \in S$.

If $y^* \leqslant 0$ does not hold, there is $i \in \{1, \cdots, m\}$ such that $y_i^* > 0$. We choose a with $0 < a < 1$ such that

$$\frac{1-a}{m-1} \sum_{\substack{j \neq i \\ 1 \leqslant j \leqslant m}} y_j^* + a y_i^* > 0.$$

Let $p_i = a$, $p_j = (1-a)/(m-1)$ $(j \neq i)$. Then $p \in S$ and $\langle p, y^* \rangle > 0$, which is impossible. Thus we must have $y^* \leqslant 0$. On the other hand, since $p^* \in S$ and $\langle p^*, y^* \rangle = 0$, we obtain $y^* = 0$ and hence $0 \in T(p^*)$. ∎

Finally, we shall deduce the following generalisation of the Gale-Nikaido-Debreu Lemma due to Grandmont [**8**, Lemma 1]:

COROLLARY 3. *Let* $\Delta = \{p \in \mathbb{R}^m : p \geqslant 0,\ \sum_{i=1}^{m} p_i = 1\}$ *and* $S = \{p \in \mathbb{R}^m : p > 0,\ \sum_{i=1}^{m} p_i = 1\}$. *Suppose that the set-valued map* $T : S \to 2^{\mathbb{R}^m}$ *satisfies the following conditions:*

(i) *T is upper semicontinuous such that for each $p \in S$, $T(p)$ is non-empty compact convex;*
(ii) *for each $p \in S$ and each $y \in T(p)$, $\langle p, y \rangle = 0$;*
(iii) *for each sequence $\{p_n\}_{n=1}^{\infty}$ in S with $p_n \to p \in \Delta \backslash S$ and each sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n \in T(p_n)$ for each $n = 1, 2, \cdots$, there is $\overline{p} \in S$ such that $\langle \overline{p}, y_n \rangle > 0$ for infinitely many n.*

Then there exists $p^* \in S$ such that $0 \in T(p^*)$.

PROOF: We shall show that the condition (iii) of Theorem 4 holds. Indeed, let $\{p_n\}_{n=1}^{\infty}$ be a sequence, where $p_n \in C_n = co\{p \in S : \operatorname{dist}(p, \Delta \backslash S) \geqslant 1/n\}$ for $n = 1, 2, \cdots$, which is escaping from S relative to $\{C_n\}_{n=1}^{\infty}$ and let $\{y_n\}_{n=1}^{\infty}$ be another sequence, where $y_n \in T(p_n)$ for $n = 1, 2, \cdots$. Since $\{p_n\}_{n=1}^{\infty}$ is a sequence in Δ and Δ is compact, without loss of generality, we may suppose that $p_n \to p^* \in \Delta \backslash S$. By (iii), there is $\overline{p} \in S$ such that $\langle \overline{p}, y_n \rangle > 0$ for infinitely many n. Since $S = \bigcup_{n=1}^{\infty} C_n$, there is $n_1 \in \mathbb{N}$ such that $\overline{p} \in C_n$ for all $n \geqslant n_1$. Choose any $n_0 \geqslant n_1$ such that $\langle \overline{p}, y_{n_0} \rangle > 0$. The condition (iii) of Theorem 4 holds so that the conclusion follows. ∎

By [**2**, p.86, Remark 18.15], the hypotheses of [**8**, Lemma 1] are weaker than the hypotheses of [**11**, Lemma 2]. Hence Theorem 4 also generalises [**11**, Lemma 2].

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