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A NOTE ON THE ORTHOGONALITY OF JACKSON'S q-BESSEL FUNCTIONS

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ABSTRACT. A q-analogue of the orthogonality property of the Bessel functions on the zeros is obtained in terms of a q-integral.

1. **Introduction.** The main objective of this paper is to find a q-analogue of the formula

$$\int_0^1 xJ_\nu(\lambda_r x)J_\nu(\lambda_s x)dx = (1/2)J_{\nu+1}^2(\lambda_r)\delta_{r,s}, \tag{1.1}$$

where λ_r, λ_s are two positive zeros of the Bessel function $J_\nu(x), \nu > -1$, defined by

$$J_\nu(x) = \Gamma^{-1}(\nu+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\nu+2n}}{n!(\nu+1)_n}. \tag{1.2}$$

For integer values of ν, Jackson [6] introduced the following q-analogues:

$$J_\nu^{(1)}(x;q) = \Gamma_q^{-1}(\nu+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\nu+2n}}{(q, q^{\nu+1};q)_n}, \tag{1.3}$$

$$J_\nu^{(2)}(x;q) = \Gamma_q^{-1}(\nu+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\nu+2n}}{(q, q^{\nu+1};q)_n}q^{n(\nu+n)}, \tag{1.4}$$

where the q-shifted factorials are defined by

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n=1,2,\dots, \end{cases} \tag{1.5}$$

$$(a_1, a_2, \dots, a_k;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_k;q)_n, \tag{1.6}$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad |q|<1, \tag{1.7}$$

and the q-gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1, \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x). \tag{1.8}$$

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If $0 < q < 1$, which we shall assume to be true, then the infinite series for $J_\nu^{(2)}(x;q)$ in (1.4) is absolutely convergent for all x while the radius of convergence of the series in (1.3) is 2. However, Hahn [3] found that these q-analogues need not be restricted to integer values of ν and that, for $|x| < 2$,

$$(1.9) \qquad J_\nu^{(1)}(x;q) = J^{(2)}\nu(x;q)/(-x^2/4;q)_\infty.$$

More recently, Ismail [4, 5] found the recurrence relations for these analogues, derived the associated q-Lommel polynomials and proved that $J_\nu^{(2)}(x;q)$ has infinitely many real positive zeroes for $\nu > -1$ (this was also stated in Hahn [3]) which are simple and that the zeros of $J_\nu^{(2)}(x;q)$ and $J_{\nu+1}^{(2)}(x;q)$ interlace. Because of the finite radius of convergence of the series in (1.3), Ismail [4] remarks that $J_\nu^{(1)}(x;q)$ has only finitely many positive zeros. However, we shall use $J_\nu^{(1)}(x;q)$ to mean an analytic continuation of the series in (1.3) and thus defined more properly by (1.9) for all real x. In [9] the author used this idea to compute some infinite integrals of products of $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$. Also, we shall use the simpler notation

$$(1.10) \qquad J_\nu^{(2)}(x|q) = J_\nu^{(2)}(2x(1-q^{1/2});q)$$
$$= \Gamma_\nu^{-1}(\nu+1)\left(\frac{x}{1+q^{1/2}}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(x(1-q^{1/2}))^{2n}}{(q,q^{\nu+1};q)_n} q^{n(\nu+n)},$$

$$(1.11) \qquad J_\nu^{(1)}(x|q) = J_\nu^{(2)}(x|q)/(-(1-q^{1/2})^2x^2;q)_\infty,$$

so that

$$(1.12) \qquad \lim_{q\to 1^-} J_\nu^{(1)}(x|q) = \lim_{q\to 1^-} J_\nu^{(2)}(x|q) = J_\nu(x).$$

There is another q-analogue of the Bessel functions that was introduced recently by Exton [2] which can be written in the form

$$(1.13) \qquad J_\nu(x;q) = \Gamma_q^{-1}(\nu+1)\left(\frac{x}{1+q^{1/2}}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(x(1-q^{1/2}))^{2n}}{(q,q^{\nu+1};q)_n} q^{(1/2)n(\nu+n-1)}.$$

This differs slightly from Exton's definition by a constant factor as well as in notation. Note that the defining series in (1.10) and (1.13) are very similar and yet they define two entirely unrelated q-analogues.

Defining the q-difference operator D_q by

$$(1.14) \qquad D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x},$$

Exton was able to show that

$$(1.15) \qquad D_{q^{1/2}}[xD_{q^{1/2}}J_\nu(\lambda x;q)] + \left[\lambda^2 x - q^{-\nu/2}\left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 x^{-1}\right] J_\nu(\lambda x q^{1/2};q) = 0.$$

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Exton's approach was to prove a q-analogue of the general Sturm-Liouville theorem, namely, that if $r(x)$, $\ell(x)$ and $w(x)$ satisfy certain suitable conditions and if $y(x)$ satisfies the q-difference equation

$$D_q[r(x)D_qy(x)] + [\ell(x) + \lambda w(x)]y(qx) = 0 \tag{1.16}$$

and the boundary conditions

$$\begin{aligned} h_1y(x) + h_2D_qy(x) &= 0 \quad \text{at } x = a, \\ k_1y(x) + k_2D_qy(x) &= 0 \quad \text{at } x = b, \end{aligned} \tag{1.17}$$

where h_1, h_2, k_1, k_2 are constants (not all zero), then the eigenfunctions $y_m(x)$ and $y_n(x)$ corresponding to the eigenvalues λ_m, λ_n are q-orthogonal, in the sense that

$$\int_a^b w(x)y_m(qx)y_n(qx)d_qx = 0, \quad m \neq n, \tag{1.18}$$

where the q-integral above is defined by

$$\begin{aligned} \int_0^c f(x)d_qx &= c(1-q)\sum_{k=0}^{\infty} f(cq^k)q^k, \\ \int_a^b f(x)d_qx &= \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \end{aligned} \tag{1.19}$$

for any continuous function $f(x)$. In fact, Exton discovered (1.13) by taking

$$r(x) = x, w(x) = x \text{ and } \ell(x) = -\frac{q^{-\nu}}{x}\left(\frac{1-q^{\nu}}{1-q}\right)^2$$

and solving the corresponding equation (1.16). So, for Exton's q-analogue (1.13), the orthogonailty relation

$$\int_0^1 xJ_\nu(\lambda_r x; q)J_\nu(\lambda_s x; q)d_{q^{1/2}}x = 0, \quad r \neq s \tag{1.20}$$

is automatically satisfied, where the λ's are the positive zeros of $J_\nu(x; q)$ i.e $J_\nu(\lambda_r; q) = 0$, $r = 1, 2, \dots$.

The same, however, is not true for Jackson's q-Bessel function given in (1.10) and (1.11). In section 2 we will first show that $J_\nu^{(k)}(\lambda x|q)$ satisfy the q-difference equations

$$D_{q^{1/2}}[xD_{q^{1/2}}J_\nu^{(k)}(\lambda x|q)] - q^{-\nu/2}\left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 x^{-1}J_\nu^{(k)}(\lambda xq^{1/2}|q) \tag{1.21}$$

$$= -\lambda^2 \begin{cases} xJ_\nu^{(1)}(\lambda x|q) \\ qxJ_\nu^{(2)}(\lambda xq|q), \end{cases}$$

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for $k = 1, 2$, respectively. These equations are essentially the same as (2.1) and (2.2) of [4], but written in this form, they are similar to (1.15) and are clearly q-analogues of the standard Sturm-Liouville type differential equation for the Bessel functions:

$$[xJ_\nu'(\lambda x)]' - \nu^2 x^{-1} J_\nu(\lambda x) = -\lambda^2 x J_\nu(\lambda x). \tag{1.22}$$

Since equation (1.21) is not quite a Sturm-Liouville equation the orthogonality of $J_\nu^{(k)}(\lambda x|q)$ does not seem to follow from a general result, so we do some detailed calculations in sections 2 and 3 to show that

$$\int_0^1 xJ_\nu^{(1)}(\lambda_r x|q) J_\nu^{(2)}(\lambda_s x q^{1/2}|q) d_{q^{1/2}} x = 0, \tag{1.23}$$

where λ_r and λ_s are two distinct positive zeros of $J_\nu^{(2)}(\lambda|q), \nu > -1$. The $q \longrightarrow 1^-$ limit of both (1.20) and (1.23) is the same well-known orthogonality relation for the Bessel functions, see for example [1],

$$\int_0^1 xJ_\nu(\lambda_r x) J_\nu(\lambda_s x) dx = 0 \quad r \neq s, \tag{1.24}$$

where $J_\nu(\lambda_r) = 0$, $r = 1, 2, \ldots$. Computation of the q-integral in (1.23) when $r = s$ is a bit problematic, but we shall obtain a formula in section 3 which, unfortunately, is not as readily usable as the one for the ordinary Bessel functions.

2. The q-difference equation. We use (1.10) to find that

$$\begin{aligned} (2.1)\quad & D_{q^{1/2}}\left[x^{-\nu} J_\nu^{(2)}(\lambda x|q)\right] \\ &= [x(1+q^{1/2})^\nu \Gamma_q(\nu+1)]^{-1} \lambda^\nu \sum_{n=0}^\infty \frac{(-1)^n q^{n(\nu+n)}}{(q, q^{\nu+1}; q)_n} (\lambda x(1-q^{1/2}))^{2n} \frac{1-q^n}{1-q^{1/2}} \\ &= -\lambda x^{-\nu} q^{\frac{\nu+1}{2}} J_{\nu+1}^{(2)}(\lambda x q^{1/2}|q), \end{aligned}$$

$$\begin{aligned} (2.2)\quad & D_{q^{1/2}}\left[x^{2\nu+1} D_{q^{1/2}}\{x^{-\nu} J_\nu^{(2)}(\lambda x|q)\}\right] \\ &= [x(1-q^{1/2})^\nu \Gamma_q(\nu+1)]^{-1} (\lambda x^2)^\nu \sum_{n=0}^\infty \frac{(-1)^n q^{n(\nu+n)}}{(q, q^{\nu+1}; q)_n} (\lambda x(1-q^{1/2}))^{2n} \\ &\times \frac{(1-q^n)(1-q^{\nu+n})}{(1-q^{1/2})^2} \\ &= -\lambda^2 q x^{\nu+1} J_\nu^{(2)}(\lambda x q|q). \end{aligned}$$

Similarly,

$$D_{q^{1/2}}[x^{-\nu} J_\nu^{(1)}(\lambda x|q)] = -\lambda x^{-\nu} J_\nu^{(1)}(\lambda x|q) \tag{2.3}$$

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and

$$(2.4)\qquad D_{q^{1/2}}[x^{2\nu+1}D_{q^{1/2}}\{x^{-\nu}J_\nu^{(1)}(\lambda x|q)\}] = -\lambda^2 x^{\nu+1}J_\nu^{(1)}(\lambda x|q).$$

However,

$$D_{q^{1/2}}[x^{-\nu}J_\nu^{(k)}(\lambda x|q)]$$
$$= x^{-\nu-1}\frac{1-q^{-\nu/2}}{1-q^{1/2}}J_\nu^{(k)}(\lambda x|q) + x^{-\nu}q^{-\nu/2}D_{q^{1/2}}J_\nu^{(k)}(\lambda x|q)$$

and hence

$$(2.5)\quad D_{q^{1/2}}\left[x^{2\nu+1}D_{q^{1/2}}\left\{x^{-\nu}J_\nu^{(k)}(\lambda x|q)\right\}\right]$$
$$= q^{1/2}x^{\nu+1}D_{q^{1/2}}J_\nu^{(k)}(\lambda x|q) + \frac{q^{\nu/2}+q^{-\nu/2}-1-q^{1/2}}{1-q^{1/2}}x^\nu D_{q^{1/2}}J_\nu^{(k)}(\lambda x|q)$$
$$- q^{-\nu/2}\left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 x^{\nu-1}J_\nu^{(k)}(\lambda x|q).$$

Since

$$(2.6)\qquad D_{q^{1/2}}[xD_{q^{1/2}}J_\nu^{(k)}(\lambda x|q)]$$
$$= q^{1/2}xD^k_{q^{1/2}}J_\nu^{(k)}(\lambda x|q) + D_{q^{1/2}}J_\nu^{(k)}(\lambda x|q)$$

we obtain (1.21) by using (2.2), (2.4), (2.5) and (2.6).

For $\lambda_1 \neq \lambda_2$ let us now rewrite (1.21) in the form

$$(2.7)\qquad D_{q^{1/2}}\left[xD_{q^{1/2}}J_\nu^{(1)}(\lambda_1 x|q)\right] - \left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 (xq^{\nu/2})^{-1}J_\nu^{(1)}(\lambda_1 xq^{1/2}|q)$$
$$= -\lambda_1^2 xJ_\nu^{(1)}(\lambda_1 x|q)$$

and

$$(2.8)\qquad D_{q^{1/2}}[xD_{q^{1/2}}J_\nu^{(2)}(\lambda_2 xq^{-1/2}|q)] - \left(\frac{1-q^{\nu/2}}{1-q^{1/2}}\right)^2 (xq^{\nu/2})^{-1}J_\nu^{(2)}(\lambda_2 x|q)$$
$$= -\lambda_2^2 xJ_\nu^{(2)}(\lambda_2 xq^{1/2}|q).$$

We now multiply (2.7) by $J_\nu^{(2)}(\lambda_2 x^{1/2}|q)$, (2.8) by $J_\nu^{(1)}(\lambda_1 x|q)$ and subtract one from the other to get

$$(2.9)\qquad (\lambda_2^2-\lambda_1^2)xJ_\nu^{(1)}(\lambda_1 x|q)J_\nu^{(2)}(\lambda_2 xq^{1/2}|q)$$
$$- \frac{(q^{-\nu/4}-q^{\nu/4})^2}{1-q^{1/2}}D_{q^{1/2}}[J_\nu^{(1)}(\lambda_1 x|q)J_\nu^{(2)}(\lambda_2 x|q)]$$
$$= J_\nu^{(2)}(\lambda_2 xq^{1/2}|q)D_{q^{1/2}}[xD_{q^{1/2}}J_\nu^{(1)}(\lambda_1 x|q)]$$
$$- J_\nu^{(1)}(\lambda_1 x|q)D_{q^{1/2}}[xD_{q^{1/2}}J_\nu^{(2)}(\lambda_2 xq^{-1/2}|q)].$$

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By a somewhat lengthy but straightforward calculation it can be shown that the expression on the right side of (2.9) equals

$$(1-q^{1/2})^{-1}D_{q^{1/2}}\{2J_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2x|q)-J_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2xq^{-1/2}|q)$$
$$-J_\nu^{(1)}(\lambda_1xq^{1/2}|q)J_\nu^{(2)}(\lambda_2x|q)\}$$

and so we get

$$(2.10)\qquad (\lambda_2^2-\lambda_1^2)xJ_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2xq^{1/2}|q)=(1-q^{1/2})^{-1}D_{q^{1/2}}g_\nu(x|q),$$

where

$$(2.11)\qquad g_\nu(x|q)=(q^{\nu/2}+q^{-\nu/2})J_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2x|q)$$
$$-J_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2xq^{-1/2}|q)-J_\nu^{(1)}(\lambda_1xq^{1/2}|q)J_\nu^{(2)}(\lambda_2x|q).$$

It follows that

$$(2.12)\qquad (\lambda_2^2-\lambda_1^2)\int_0^1 xJ_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2xq^{1/2}|q)d_{q^{1/2}}x$$
$$=\sum_{r=0}^\infty q^{r/2}D_{q^{1/2}}g_\nu(q^{r/2})$$
$$=(1-q^{1/2})^{-1}\left\{g_\nu(1)-\lim_{N\to\infty}g_\nu(q^{\frac{N+1}{2}})\right\}$$

For $\nu>1$ it can be shown that

$$(2.13)\qquad \lim_{N\to\infty}g_\nu(q^{\frac{N+1}{2}})=0$$

and so we find that

$$(2.14)\qquad (\lambda_2^2-\lambda_1^2)\int_0^1 xJ_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2xq^{1/2}|q)d_{q^{1/2}}x=g_\nu(1)/(1-q^{1/2}).$$

Using the easily verified recurrence formulas, see also [4],

$$(2.15)\qquad J_\nu^{(1)}(xq^{1/2}|q)-q^{\nu/2}J_\nu^{(1)}(x|q)=x(1-q^{1/2})q^{\nu/2}J_{\nu+1}^{(1)}(x|q),$$

$$(2.16)\qquad J_\nu^{(2)}(xq^{1/2}|q)-q^{\nu/2}J_\nu^{(2)}(x|q)=x(1-q^{1/2})q^{\nu+1/2}J_{\nu+1}^{(2)}(xq^{1/2}|q),$$

we finally obtain the results

$$(2.17)\qquad (\lambda_2^2-\lambda_1^2)\int_0^1 xJ_\nu^{(1)}(\lambda_1x|q)J_\nu^{(2)}(\lambda_2x^{1/2}|q)d_{q^{1/2}}x$$
$$=-q^{\nu/2}\begin{vmatrix} J_\nu^{(2)}(\lambda_2|q) & J_\nu^{(1)}(\lambda_1|q)\\ \lambda_2J_{\nu+1}^{(2)}(\lambda_2|q) & \lambda_1J_{\nu+1}^{(1)}(\lambda_1|q)\end{vmatrix}.$$

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3. Orthogonality. It is now clear from (2.17) that if λ_r and λ_s are two distinct positive zeros of $J_\nu^{(2)}(x|q)$ then (1.23) follows. However, (1.23) is true even under the more general conditions that λ_r, λ_s are distinct roots of

$$J_\nu^{(k)}(\lambda|q) + a\lambda J_{\nu+1}^{(k)}(\lambda|q) = 0, \tag{3.1}$$

for a given real a.

To evaluate the q-integral in (1.23) when $r = s$ we divide (2.17) by $\lambda_2^2 - \lambda_1^2$, set $\lambda_1 = \lambda, \lambda_2 = \lambda\sqrt{z}$, and then take the limit $z \longrightarrow 1$. By L'Hôpital's rule we obtain

$$\begin{aligned} &\int_0^1 xJ_\nu^{(1)}(\lambda_r x|q)J_\nu^{(2)}(x\lambda_r q^{1/2}q)d_{q^{1/2}}x \\ &= -\frac{q^{\nu/2}}{2}\begin{vmatrix} J_\nu^{(2)\prime}(\lambda_r|q) & J_\nu^{(1)}(\lambda_r|q) \\ J_{\nu+1}^{(2)}(\lambda_r|q) + J_{\nu+1}^{(2)\prime}(\lambda_r|q) & J_{\nu+1}^{(1)}(\lambda_r|q) \end{vmatrix}, \end{aligned} \tag{3.2}$$

where a prime indicates differentiation with respect to λ_r. If λ_r is a positive zero of $J_\nu^{(2)}(\lambda_r|q)$, then (3.2) reduces further to

$$\int_0^1 xJ_\nu^{(1)}(x\lambda_r|q)J_\nu^{(2)}(x\lambda_r q^{1/2}|q)d_{q^{1/2}}x = -\frac{q^{\nu/2}}{2}J_{\nu+1}^{(1)}(\lambda_r|q)J_\nu^{(2)\prime}(\lambda_r|q), \tag{3.3}$$

which, of course, goes to the limit

$$\int_0^1 xJ_\nu^2(\lambda_r x)dx = \frac{1}{2}J_{\nu+1}^{(2)}(\lambda_r) \tag{3.4}$$

as $q \longrightarrow 1^-$.

It must abe noted, however, that the presence of an ordinary derivative in a basic hypergeometric series is ominous and should be avoided whenever possible. Ideally, one would like to have the derivatives in (3.2) and (3.3) replaced by the q-derivatives defined by (1.14). Unfortunately, we were unable to find such expressions. The situation is not much better for Exton's q-analogue since the value of the integral in (1.20) for $r = s$ also contains a derivative, see section 5.4.5 in Exton [2].

4. Concluding remarks. It might appear from this work that Exton's q-analogue (1.13) is a bit nicer than Jackson's analogues (1.10) and (1.11), at least as far as the Sturm-Liouville theory is concerned. But it has already been found through the works of Hahn [3] and Ismail [4, 5] and more recently of the author [7, 8, 9] that Jackson's analogues have very nice properties that are analogous to those of the ordinary Bessel functions. It remains to be seen if Exton's $J_\nu(x;q)$ has the same nice properties. My guess is that in problems like addition theorems and product formulas and Poisson-type integral representations $J_\nu(x;q)$ will not behave as well as $J_\nu^{(2)}(x|q)$. One reason for this suspicion is that $J_\nu^{(2)}(x|q)$ can be obtained as the limit of a well-poised ${}_2\phi_1$ series, namely,

$$J_\nu^{(2)}(x|q) = \Gamma_q^{-1}(\nu+1)\left(\frac{x}{1+q^{1/2}}\right)^\nu \times \lim_{a\to\infty} {}_2\phi_1\left[\begin{matrix} a, & aq^{-\nu} \\ & q^{\nu+1}\end{matrix} ; q, -(1-q^{1/2})^2 x^2 \frac{q^{2\nu+1}}{a^2}\right] \tag{4. 1}$$

and so, via many different transformations for well-poised series, $J_\nu^{(2)}(x|q)$ can be expressed in many different basic hypergeometric forms. Such possibilities do not seem to exist for Exton's q-analogue. But more work needs to be done to come to a definite conclusion.

References

1. A. Erdélyi, ed., *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.
2. H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Ltd., Chichester, U.K., 1983.
3. W. Hahn, *Beiträge zur theorie der Heineschen Reichen*, Math. Nachr. **2** (1949), 340–379.
4. M. E. H. Ismail, *The zeroes of basic Bessel functions, the function $J_{\nu+ax}$ and associated orthogonal polynomials*, J. Math. Anal. Appl., **86** (1982), 1–19.
5. ———, *The basic Bessel functions and polynomials*, SIAM J. Math. Anal. **12** (1981), 454–468.
6. F. H. Jackson, *On generalized functions of Legendre and Bessel*, Trans. Roy. Soc. Edin. **41** (1903), 1–28.
7. Mizan Rahman, *An integral representation and some transformation properties of q-Bessel functions*, J. Math. Anal. Appl. **125** (1987), 58–71.
8. ———, *An addition theorem and some product formulas for q-Bessel functions*, Can. J. Math., **40** (1988), 1203–1221.
9. ———, *Some infinite integrals of q-Bessel functions*, to appear.

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