On invariant subalgebras of group *C** and von Neumann algebras

MEHRDAD KALANTAR ⓘ and NIKOLAOS PANAGOPOULOS

Department of Mathematics, University of Houston, Houston, TX 77004, USA
(e-mail: mkalantar@uh.edu, npanagopoulos@uh.edu)

Abstract. Given an irreducible lattice Γ in the product of higher rank simple Lie groups, we prove a co-finiteness result for the Γ-invariant von Neumann subalgebras of the group von Neumann algebra $\mathcal{L}(\Gamma)$, and for the Γ-invariant unital C^*-subalgebras of the reduced group C^*-algebra $C^*_{\mathrm{red}}(\Gamma)$. We use these results to show that: (i) every Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$ is generated by a normal subgroup; and (ii) given a weakly mixing unitary representation π of Γ, every Γ-equivariant conditional expectation on $C^*_\pi(\Gamma)$ is the canonical conditional expectation onto the C^*-subalgebra generated by a normal subgroup.

1. *Introduction*

Given a discrete group Γ and a subgroup $\Lambda \leq \Gamma$, we have a canonical inclusion of the reduced group C^*-algebras $C^*_{\mathrm{red}}(\Lambda) \subseteq C^*_{\mathrm{red}}(\Gamma)$, and the characteristic function $\mathbf{1}_\Lambda$ of Λ extends to a conditional expectation $\mathcal{E}_\Lambda : C^*_{\mathrm{red}}(\Gamma) \to C^*_{\mathrm{red}}(\Lambda)$. Considering the canonical action of Γ on $C^*_{\mathrm{red}}(\Gamma)$ by unitary conjugations, one can observe that Λ is normal in Γ if and only if $C^*_{\mathrm{red}}(\Lambda)$ is Γ-invariant inside $C^*_{\mathrm{red}}(\Gamma)$, or equivalently if the conditional expectation $\mathcal{E}_\Lambda$ is Γ-equivariant. Similar observations hold for the inclusion $\mathcal{L}(\Lambda) \subset \mathcal{L}(\Gamma)$ of the corresponding group von Neumann algebras.

In this sense, Γ-invariant unital C^*-subalgebras of $C^*_{\mathrm{red}}(\Gamma)$ and Γ-invariant von Neumann subalgebras of $\mathcal{L}(\Gamma)$ can be viewed as 'non-commutative normal subgroups' of Γ. Similarly, Γ-equivariant conditional expectations on $C^*_{\mathrm{red}}(\Gamma)$ or $\mathcal{L}(\Gamma)$ can be considered as alternative non-commutative generalizations of normal subgroups of Γ.

This note is concerned with these objects. We are mostly interested in the case of lattices Γ in higher rank semisimple groups, where some of the most prominent results (e.g., Margulis' normal subgroup theorem [**Mar91**], Stuck–Zimmer's rigidity theorem [**SZ94**], character rigidity [**BBHP20**, **Bek07**, **BH21**, **Pet15**]) are concerned with rigidity properties of their normal subgroups and various generalized objects. In this context, the above considerations naturally lead us to the questions of to what extent, and in what sense, those rigidity results can be extended to the non-commutative setting.

The main result of this paper is that irreducible lattices in higher rank semisimple Lie groups Γ have no genuine 'non-commutative normal subgroups'.

THEOREM 1.1. *Assume G is a connected semisimple Lie group with trivial center, no non-trivial compact factors, and such that all its simple factors have real rank at least two. Let $\Gamma < G$ be an irreducible lattice. Then, any Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$ is of the form $\mathcal{L}(\Lambda)$ for some normal subgroup $\Lambda \leq \Gamma$.*

The work of Alekseev and Brugger [**AB21**] and Brugger's thesis [**Bru18**] seem to be the first systematic study of Γ-invariant von Neumann subalgebras of $\mathcal{L}(\Gamma)$, where the authors proved that for Γ as in Theorem 1.1 (with the additional assumption of G being simple), the inclusion $\mathsf{M} \subset \mathcal{L}(\Gamma)$ of any non-trivial Γ-invariant subfactor M of $\mathcal{L}(\Gamma)$ has finite (Jones) index (see also [**CD20**, §3] for some related results).

Recall that if Λ is a subgroup of Γ, then Λ has finite index in Γ if and only if $\mathcal{L}(\Lambda)$ has finite index in $\mathcal{L}(\Gamma)$. Hence, their result can be considered as a non-commutative version of Margulis' normal subgroup theorem (NST) [**Mar91**], which states that every group Γ as in Theorem 1.1 is *just-infinite*, meaning that every non-trivial normal subgroup of Γ has finite index in Γ.

In the C^*-algebra setting, we prove a complete description of equivariant conditional expectations as follows. We denote by λ_Γ the left regular representation of Γ.

THEOREM 1.2. *Let Γ be as in the statement of Theorem 1.1, and let π be a weakly mixing unitary representation of Γ. If $\mathcal{E}$ is a Γ-equivariant conditional expectation on $C^*_\pi(\Gamma)$, then there is a normal subgroup $\Lambda \leq \Gamma$ such that $\mathcal{E}|_{\pi(\Gamma)} = \mathbf{1}_{\pi(\Lambda)}$.*

Here, by a conditional expectation on a unital C^*-algebra B, we mean a unital completely positive (ucp) idempotent map $\mathcal{E} : \mathsf{B} \to \mathsf{B}$ whose range is a C^*-subalgebra of B, and the action of Γ on $C^*_\pi(\Gamma)$ is by conjugations with unitaries $\pi(g)$, $g \in \Gamma$.

Recall that a unitary representation is called weakly mixing if it does not have any finite-dimensional subrepresentations. Let us point out that the weakly mixing condition is rather non-restrictive. In particular, a unitary representation of a property (T) group is weakly mixing if and only if π is non-amenable in the sense of Bekka (see [**Bek90**, **BV93**]). Examples of non-amenable representations π of groups Γ as in the statement of Theorem 1.2 include, for instance, the quasi-regular representation of Γ on $\ell^2(\Gamma/\Lambda)$ for any subgroups $\Lambda \leq \Gamma$ of infinite index, and the Koopman representation of Γ on $L^2(X, \nu)$ for any Γ-quasi-invariant measure on a compact minimal Γ-space admitting no Γ-invariant probability measure (e.g., any topological Γ-boundary X).

Since every von Neumann subalgebra of a finite von Neumann algebra is the image of a unique trace-preserving conditional expectation, Theorem 1.1 can be equivalently stated as a rigidity result for conditional expectations on $\mathcal{L}(\Gamma)$.

THEOREM 1.3. *Let Γ be as in the statement of Theorem 1.1. If $\mathcal{E}$ is an ultraweakly continuous Γ-equivariant conditional expectation on $\mathcal{L}(\Gamma)$, then there is a normal subgroup $\Lambda \leq \Gamma$ such that $\mathcal{E}|_{\lambda_\Gamma(\Gamma)} = \mathbf{1}_{\lambda_\Gamma(\Lambda)}$.*

In contrast to the von Neumann algebra case, in general, even for groups Γ with the infinite conjugacy class (icc) property, it is not the case that any Γ-invariant unital

C^*-subalgebra A of $C^*_{\mathrm{red}}(\Gamma)$ is the image of some conditional expectation (see Example 3.2). In particular, Theorem 1.2 does not offer a complete description of all Γ-invariant unital C^*-subalgebras A of $C^*_{\mathrm{red}}(\Gamma)$, although in the case of these lattices, we do not know of any concrete example of such A that is not the image of an equivariant conditional expectation of $C^*_{\mathrm{red}}(\Gamma)$.

We do however prove, in Theorem 1.5 below, a co-finiteness result for general Γ-invariant unital C^*-subalgebras of $C^*_{\mathrm{red}}(\Gamma)$, where Γ is a lattice as in Theorem 1.1.

The co-finiteness of C^*-subalgebras in our result is in the following sense.

Definition 1.4. Let π be a unitary representation of the group Γ.

A Γ-invariant von Neumann subalgebra N of $\pi(\Gamma)''$ is said to be *co-finite in* $\pi(\Gamma)''$ if the commutant N$'$ of N in $B(H_\pi)$ admits an ultraweakly continuous Γ-invariant state (commutants are taken in $B(H_\pi)$).

A Γ-invariant unital C^*-subalgebra A of $C^*_\pi(\Gamma)$ is said to be *co-finite in* $C^*_\pi(\Gamma)$ if the double commutant A$''$ is co-finite in $\pi(\Gamma)''$.

THEOREM 1.5. *Let Γ be as in the statement of Theorem 1.1. Then, for every non-trivial unital Γ-invariant C^*-subalgebra A of $C^*_{\mathrm{red}}(\Gamma)$ and every unitary representation π of Γ which is weakly contained in λ_Γ, $\pi(A)$ is co-finite in $C^*_\pi(\Gamma)$.*

We also prove a von Neumann algebraic version of Theorem 1.5, recorded as Theorem 2.17, which will be a key ingredient in the proof of Theorem 1.1.

The conclusion of Theorem 1.5 is indeed a non-commutative just-infiniteness for Γ (see Definition 2.13). Obviously, any group Γ that satisfies the conclusion of Theorem 1.5 is just-infinite, but not conversely: we will give examples of simple non-amenable groups Γ for which the conclusion of Theorem 1.5 does not hold (see Example 2.15). In this sense, the above theorem is a sharper result than Margulis' NST for the lattices Γ as in the statement.

We close this introduction by pointing out that a different concept of just-infiniteness for C^*-algebras has been studied by Grigorchuk, Musat and Rørdam [**GMR18**].

1.1. *New developments.* Several new developments have occurred since the this version of the paper appeared in February 2022. In [**AJ22**], the authors prove a similar result as our Theorem 1.1 for a large class of 'negatively curved' groups, including all torsion free non-amenable hyperbolic groups. Their result was then generalized to all acylindrically hyperbolic groups in [**CDS22**].

Most recently, a similar result as our Theorem 1.5 was proved in [**AH22**]. Indeed, they prove a co-amenability result for invariant C^*-algebras of reduced C^*-algebras of a class of groups satisfying a 'singularity' property (see [**AH22**, Examples]).

2. *Γ-invariant subalgebras*
In this section, we prove some general properties of co-finite Γ-invariant subalgebras, mostly in the case of $C^*_{\mathrm{red}}(\Gamma)$ and $\mathcal{L}(\Gamma)$. Throughout the section, Γ is always a discrete group.

We observe that if Λ is a finite-index normal subgroup of Γ, then for any unitary representation π of Γ, $C^*_\pi(\Lambda)$ is co-finite in $C^*_\pi(\Gamma)$ and $\pi(\Lambda)''$ is co-finite in $\pi(\Gamma)''$.

Indeed, in this case, the Γ-action on $M := C_\pi^*(\Lambda)' = \pi(\Lambda)'$ factors through the finite group Γ/Λ, and hence M admits a Γ-invariant ultraweakly continuous state.

One expects that for any natural notion of co-finiteness, certain properties should lift from co-finite subalgebras to the ambient space. In the following theorem, we prove that amenability and Haagerup property are among such properties.

THEOREM 2.1. *Let π be a factorial unitary representation of Γ.*

(1) *If $\pi(\Gamma)''$ has a non-trivial co-finite Γ-invariant von Neumann subalgebra with the Haagerup property, then $\pi(\Gamma)''$ also has the Haagerup property.*

(2) *If $\pi(\Gamma)''$ has an amenable non-trivial co-finite Γ-invariant von Neumann subalgebra, then $\pi(\Gamma)''$ is amenable.*

Proof. (1) Let $N \subset \pi(\Gamma)''$ be a non-trivial co-finite Γ-invariant von Neumann subalgebra that has the Haagerup property. Let $M = N' \subset B(H_\pi)$. Then, M has the Haagerup property [**OT15**, Theorem 3.12]. Let τ be the Γ-invariant ultraweakly continuous state on M guaranteed by the co-finiteness assumption, and let $\mathfrak{p} \in M$ be the support projection of τ. Then, $\mathfrak{p}$ is Γ-invariant and therefore $\mathfrak{p} \in \pi(\Gamma)'$. Furthermore, τ restricts to a faithful Γ-invariant ultraweakly continuous state on the Γ-von Neumann algebra $\mathfrak{p}M\mathfrak{p}$, which also has the Haagerup property [**OT15**, Corollary 3.13]. Since there exists a conditional expectation $\mathfrak{p}M\mathfrak{p} \to (\mathfrak{p}M\mathfrak{p})^\Gamma = \mathfrak{p}M^\Gamma\mathfrak{p} = \mathfrak{p}\pi(\Gamma)'\mathfrak{p}$ (see, e.g., [**BBHP20**, Proposition 2.7(1)]), it follows that $\mathfrak{p}\pi(\Gamma)'\mathfrak{p}$ has the Haagerup property [**OT15**, Theorem 5.9]. Since $\mathfrak{p}\pi(\Gamma)'\mathfrak{p}$ is a factor, this implies $\pi(\Gamma)'$ has the Haagerup property [**OT15**, Corollary 3.13]. Invoking [**OT15**, Theorem 3.12] once again, the result follows. (2) The proof is similar and the corresponding permanence results regarding amenability are well known (see, e.g., [**ADP**]). $\square$

COROLLARY 2.2. *Let Γ be icc. Then the following hold.*

(1) *Γ is amenable if and only if $\mathcal{L}(\Gamma)$ has a non-trivial co-finite Γ-invariant amenable von Neumann subalgebra.*

(2) *Γ has the Haagerup property if and only if $\mathcal{L}(\Gamma)$ has a non-trivial co-finite Γ-invariant von Neumann subalgebra that has the Haagerup property.*

Proof. The arguments for both parts are similar and therefore we only give the proof of item (2). If Γ has the Haagerup property, then $\mathcal{L}(\Gamma)$ has the Haagerup property and so the forward direction follows. Conversely, since Γ is icc, the regular representation of Γ is factorial. If $\mathcal{L}(\Gamma)$ has a non-trivial co-finite Γ-invariant von Neumann subalgebra that has the Haagerup property, Theorem 2.1 implies that $\mathcal{L}(\Gamma)$ has the Haagerup property. Hence, Γ has the Haagerup property [**Cho83**, Theorem 3]. $\square$

2.1. *Non-commutative just-infiniteness.* Any notion of co-finiteness in the operator algebraic setting naturally leads to non-commutative notions of just-infiniteness.

Definition 2.3. We say Γ is *π-just-infinite* if π is infinite-dimensional and every non-trivial Γ-invariant von Neumann subalgebra $N \subset \pi(\Gamma)''$ is co-finite in $\pi(\Gamma)''$.

Recall that λ_Γ denotes the left regular representation of Γ.

LEMMA 2.4. *If Γ is λ_Γ-just-infinite, then Γ is icc.*

Proof. Assume Γ is not icc. Then $\mathcal{L}(\Gamma)$ contains a non-trivial central projection $\mathfrak{p}$. Then, $N = \mathbb{C}\mathfrak{p} \oplus \mathbb{C}\mathfrak{p}^\perp$ is a non-trivial Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$, and $N' = \mathfrak{p}B(\ell^2(\Gamma))\mathfrak{p} \oplus \mathfrak{p}^\perp B(\ell^2(\Gamma))\mathfrak{p}^\perp$. Since Γ is λ_Γ-just-infinite, N' admits a Γ-invariant ultraweakly continuous state. The map $\mathfrak{a} \mapsto \mathfrak{p}\mathfrak{a}\mathfrak{p} + \mathfrak{p}^\perp \mathfrak{a}\mathfrak{p}^\perp$, from $B(\ell^2(\Gamma))$ to N', is ultraweakly continuous, Γ-equivariant, and ucp. Hence, we obtain a Γ-invariant ultraweakly continuous state on $B(\ell^2(\Gamma))$, which implies that Γ is finite. This is a contradiction as λ_Γ-just-infinite groups are, by definition, assumed to be infinite. $\square$

Example 2.5. It follows from Lemma 2.4 that there are just-infinite groups that are not λ_Γ-just-infinite. Indeed, the integer group $\Gamma = \mathbb{Z}$ is just-infinite (in fact, it even satisfies the conclusion of Stuck–Zimmer's rigidity theorem), but not icc, and hence it is not λ_Γ-just-infinite by Lemma 2.4.

PROPOSITION 2.6. *Assume Γ is λ_Γ-just-infinite and does not have the Haagerup property. Then $\mathcal{L}(\Gamma)$ does not admit any non-trivial Γ-invariant von Neumann subalgebras with the Haagerup property.*

Proof. For the sake of contradiction, assume that N is a non-trivial Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$ with the Haagerup property. Since, Γ is λ_Γ-just-infinite, N is co-finite in $\mathcal{L}(\Gamma)$, and hence Lemma 2.4 and Corollary 2.2 entail that $\mathcal{L}(\Gamma)$ has the Haagerup property. Therefore, by [**Cho83**, Theorem 3], the group Γ has the Haagerup property, which contradicts our assumption. $\square$

PROPOSITION 2.7. *Assume that Γ is λ_Γ-just-infinite and non-amenable. Then $\mathcal{L}(\Gamma)$ does not admit any non-trivial Γ-invariant amenable von Neumann subalgebras.*

Proof. The argument is similar to the proof of Proposition 2.6, using the fact that $\mathcal{L}(\Gamma)$ is amenable if and only if Γ is amenable. $\square$

In particular, we conclude the following corollary.

COROLLARY 2.8. *Assume that Γ is λ_Γ-just-infinite and non-amenable. Then $\mathcal{L}(\Gamma)$ does not admit any Cartan subalgebra with $\lambda_\Gamma(\Gamma) \subseteq \mathcal{N}_{\mathcal{L}(\Gamma)}(N)$.*

Let us point out that the current understanding of Cartan subalgebras of $\mathcal{L}(\Gamma)$ is far from complete in the higher-rank realm. Important partial results in this direction were obtained recently by Boutonnet, Ioana and Peterson in [**BIP21**], where they proved the lack of certain classes of Cartan subalgebras.

We conclude the section with another useful consequence of Proposition 2.7.

COROLLARY 2.9. *Assume Γ is λ_Γ-just-infinite and non-amenable. Then every Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$ is a factor.*

Proof. Let M be a Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$. Then the center $\mathcal{Z}(M)$ of M is a Γ-invariant abelian von Neumann subalgebra of $\mathcal{L}(\Gamma)$. Hence, $\mathcal{Z}(M) = \mathbb{C}1$ by Proposition 2.7. $\square$

2.2. *Weak just-infiniteness.* We have parallel results in the C^*-algebra setup and, as we see, a formally weaker just-infiniteness condition would suffice.

Definition 2.10. Let π be a unitary representation of Γ. We say Γ is *weakly π-just-infinite* if π is infinite-dimensional and every non-trivial Γ-invariant unital C^*-subalgebra of $C_\pi^*(\Gamma)$ is co-finite in $C_\pi^*(\Gamma)$.

The next two results are C^*-algebraic analogs of Propositions 2.6 and 2.7. By the canonical trace, we mean the (unique) state on $C_{\mathrm{red}}^*(\Gamma)$ that extends the Dirac function supported on the neutral element $e \in \Gamma$.

PROPOSITION 2.11. *Assume Γ is weakly λ_Γ-just-infinite and does not have the Haagerup property. Then, $C_{\mathrm{red}}^*(\Gamma)$ does not admit any non-trivial Γ-invariant exact C^*-subalgebra on which the restriction of the canonical trace is an amenable trace.*

Proof. For the sake of contradiction, assume that A is a non-trivial exact Γ-invariant unital C^*-subalgebra of $C_{\mathrm{red}}^*(\Gamma)$ such that the restriction of the canonical trace τ to A is an amenable trace on A. Then by [**Suz13**, Corollary 3.7], the pair (A, τ) has the Haagerup property in the sense of [**Don11**] and so from [**Suz13**, Lemma 4.5], it follows that A'' has the Haagerup property. Since A is co-finite in $C_{\mathrm{red}}^*(\Gamma)$ by λ_Γ-just-infiniteness, A'' is co-finite in $\mathcal{L}(\Gamma)$ and so, as in the proof of Proposition 2.6, we get a contradiction. $\square$

PROPOSITION 2.12. *Assume that Γ is weakly λ_Γ-just-infinite and non-amenable. Then, $C_{\mathrm{red}}^*(\Gamma)$ does not admit any non-trivial Γ-invariant nuclear C^*-subalgebras.*

Proof. For the sake of contradiction, assume that $C_{\mathrm{red}}^*(\Gamma)$ admits a non-trivial nuclear Γ-invariant unital C^*-subalgebra A. Then, A is co-finite in $C_{\mathrm{red}}^*(\Gamma)$ by C^*-just-infiniteness and so A'' is a non-trivial amenable co-finite von Neumann subalgebra of $\mathcal{L}(\Gamma)$. Corollary 2.2 implies that $\mathcal{L}(\Gamma)$ is amenable, and hence Γ is amenable. This is a contradiction and the proof is complete. $\square$

2.3. *C^*-just-infinite groups.* In this section, we prove Theorem 1.5.

Definition 2.13. A discrete group Γ is said to be *C^*-just-infinite*, if it is infinite and for every non-trivial invariant unital C^*-subalgebra A of $C_{\mathrm{red}}^*(\Gamma)$ and every unitary representation π of Γ which is weakly contained in λ_Γ, $\pi(\mathsf{A})$ is co-finite in $C_\pi^*(\Gamma)$.

We remark that for a given unitary representation π of Γ and an invariant C^*-subalgebra $\mathsf{A} \subset C_\pi^*(\Gamma)$, co-finiteness of A is effectively a von Neumann algebraic property: it is equivalent to the same property for the bi-commutant A''. Noting that the unitary representations π of Γ which are weakly contained in λ_Γ correspond to $*$-representations of $C_{\mathrm{red}}^*(\Gamma)$, we see that C^*-just-infiniteness is defined in terms of von Neumann algebras generated by all $*$-representations of $C_{\mathrm{red}}^*(\Gamma)$.

Before getting to the proof of Theorem 1.5, let us prove some general properties of C^*-just-infinite groups.

PROPOSITION 2.14. *Let Γ be a C^*-just-infinite group. Then, Γ is either amenable or C^*-simple.*

Proof. Assume that Γ is not C^*-simple. Let $\pi : C^*_{\mathrm{red}}(\Gamma) \to B(H_\pi)$ be a non-faithful representation, and let $I = \ker(\pi)$. In particular, π defines a unitary representation of Γ which is weakly contained in λ. Define $\mathsf{A} = I + \mathbb{C}1$. Then, A is a Γ-invariant unital C^*-subalgebra of $C^*_{\mathrm{red}}(\Gamma)$ and $\pi(\mathsf{A}) = \mathbb{C}1$. Thus, by the assumption, there exists a Γ-invariant state on $B(H_\pi)$. Since π is weakly contained in λ, there is also a Γ-invariant state on $B(\ell^2(\Gamma))$ [**Bek90**, Corollary 5.3]. Restricting this state to $\ell^\infty(\Gamma)$ yields an invariant mean on $\ell^\infty(\Gamma)$, and so Γ is amenable. $\qquad\square$

Example 2.15. It follows from Proposition 2.14 that there are simple non-amenable groups Γ that are not C^*-just-infinite. Indeed, the existence of non-C^*-simple groups with the unique trace property was proven by Le Boudec in [**LB17**]. And as remarked after [**LB17**, Theorem C], those examples include groups admitting simple subgroups of index two. Obviously, any such subgroup is also non-C^*-simple and non-amenable, and therefore also non-C^*-just-infinite by Proposition 2.14.

2.4. *Proof of Theorem 1.5.* The proof of Theorem 1.5 is structured similarly to Margulis' proof of the normal subgroup theorem, consisting of two parts: the amenability half and the property (T) half.

Proof of Theorem 1.5. Let $\mathsf{A} \subset C^*_{\mathrm{red}}(\Gamma)$ be a non-trivial Γ-invariant unital C^*-subalgebra, and let π be a unitary representation of Γ which is weakly contained in λ. *Amenability half.* We prove that $\pi(\mathsf{A})'$ admits a Γ-invariant state.

Let P be a minimal parabolic subgroup, and K a maximal compact subgroup of G such that $G = KP$. Let ν_P be the unique K-invariant probability on G/P, and μ a fully supported probability on Γ such that $(G/P, \nu_P)$ is the Poisson boundary of the (Γ, μ)-random walk [**Fur67**].

We first show that it suffices to prove the claim for $\pi = \lambda$. Note that the action $G \curvearrowright G/P$ is faithful; indeed, since $\ker(G \curvearrowright G/P)$ is contained in P, it is a normal amenable subgroup of G, which is also a subproduct of G by semisimplicity. Since each factor of G is non-amenable, it follows that $\ker(G \curvearrowright G/P)$ is trivial. In particular, Γ also acts faithfully on G/P. Thus, since the action $\Gamma \curvearrowright G/P$ is strongly proximal [**Fur67**], it follows that Γ has trivial amenable radical [**Fur03**, Proposition 7]. This implies Γ is C^*-simple by [**BCdlH94**, Theorem 1] (cf. [**BKKO17**, Corollary 6.10]). Hence, π is weakly equivalent to λ, which implies that the map $\pi(g) \mapsto \lambda(g)$ extends to a $*$-homomorphism $\psi : C^*_\pi(\Gamma) \to C^*_{\mathrm{red}}(\Gamma)$. By Arveson's extension theorem [**Arv69**, Theorem 1.2.3], we may extend ψ to a ucp map from $B(H_\pi)$ to $B(\ell^2(\Gamma))$ which we will still denote by ψ. Since $C^*_\pi(\Gamma)$ lies in the multiplicative domain of ψ, it follows that ψ is Γ-equivariant and hence $\psi(\pi(\mathsf{A})') \subset \mathsf{A}'$. Therefore, it suffices to show that A' admits a Γ-invariant state.

Define $\theta_\mu = \sum_{g \in \Gamma} \mu(g)\mathrm{Ad}_{\rho(g)}$, where $\rho : \Gamma \to B(\ell^2(\Gamma))$ is the right regular representation. Then, θ_μ is a Γ-equivariant ucp map on $B(\ell^2(\Gamma))$, and we denote by $\widetilde{\mathcal{H}}_\mu$ the non-commutative Poisson boundary of (Γ, μ) in the sense of Izumi [**Izu04**],

that is, the space $\{\mathfrak{a} \in B(\ell^2(\Gamma)) : \theta_\mu(\mathfrak{a}) = \mathfrak{a}\}$ of fixed points of θ_μ. We also denote by $\mathcal{H}_\mu = \widetilde{\mathcal{H}}_\mu \cap \ell^\infty(\Gamma)$ the space of bounded μ-harmonic functions on Γ.

Any point-weak* limit $\mathcal{E}_\mu$ of the sequence of maps $(1/n) \sum_{k=1}^n \theta_{\mu^k}$ defines a Γ-equivariant ucp idempotent from $B(\ell^2(\Gamma))$ onto $\widetilde{\mathcal{H}}_\mu$, and endowed with the Choi–Effros product defined by $\mathcal{E}_\mu$, the space $\widetilde{\mathcal{H}}_\mu$ becomes a von Neumann algebra. Note that $\widetilde{\mathcal{H}}_\mu$ is also canonically equipped with a Γ-action since it is a Γ-invariant subspace of $B(\ell^2(\Gamma))$.

Since $C^*_{\mathrm{red}}(\Gamma)$ lies in the multiplicative domain of $\mathcal{E}_\mu$, it follows that $\mathcal{E}_\mu(\mathsf{A}') \subset \mathsf{A}' \cap \widetilde{\mathcal{H}}_\mu$. We will show that the latter intersection admits a Γ-invariant state, hence its composition with $\mathcal{E}_\mu$ yields a Γ-invariant state on A', and this will conclude the proof of this part.

By [**Izu04**, Theorem 4.1], there is a Γ-equivariant von Neumann algebra isomorphism $\Phi \colon \Gamma \ltimes L^\infty(G/P) \to \widetilde{\mathcal{H}}_\mu$ that restricts to the identity map on $C^*_{\mathrm{red}}(\Gamma)$, and to the Poisson transform $\mathcal{P}_{\nu_P} \colon L^\infty(G/P) \to \mathcal{H}_\mu$ defined by $\mathcal{P}_{\nu_P}(f)(g) = \int_{G/P} f(gx) \, d\nu_P(x)$ for $f \in L^\infty(G/P)$ and $g \in \Gamma$.

Since Φ is identity on $C^*_{\mathrm{red}}(\Gamma)$, it maps the relative commutant B of A in the crossed product $\Gamma \ltimes L^\infty(G/P)$ onto $\mathsf{A}' \cap \widetilde{\mathcal{H}}_\mu$. Thus, it now suffices to show that the Γ-von Neumann algebra B admits a Γ-invariant state.

Let $\mathcal{E}_0 \colon \Gamma \ltimes L^\infty(G/P) \to L^\infty(G/P)$ be the canonical conditional expectation, which is Γ-equivariant and faithful. Then, the composition $\varrho(\cdot) := \int_{G/P} \mathcal{E}_0(\cdot)(x) \, d\nu_P(x)$ defines a faithful ultraweakly continuous μ-stationary state on $\Gamma \ltimes L^\infty(G/P)$. We will show that the restriction of ϱ to B is Γ-invariant.

First, we see that $\Gamma \curvearrowright \mathsf{B}$ is ergodic, that is, the Γ-invariant subalgebra $\{\mathfrak{b} \in \mathsf{B} : g\mathfrak{b} = \mathfrak{b}$ for all $g \in \Gamma\}$ of B is trivial. One can show this, for instance, by using the freeness of $\Gamma \curvearrowright G/P$ to conclude a unique stationarity as in [**HK**, Example 4.13], and then apply [**BBHP20**, Proposition 2.7(3)] (alternatively, [**DP20**, Theorem A] can be used to show the claim). However, it was pointed out to us by Rémi Boutonnet that the ergodicity in the above follows from the following general fact, which we include for the future easy reference; we thank him for providing us with the proof.

LEMMA 2.16. (Boutonnet) *Let Γ be an icc group with $\mu \in \mathrm{Prob}(\Gamma)$ whose support generates Γ. If (X, ν) is any ergodic (Γ, μ)-space, then the action of Γ on the crossed product $\Gamma \ltimes L^\infty(X)$ by conjugation is ergodic.*

Proof. Denote by $\mathcal{E} \colon \Gamma \ltimes L^\infty(X) \to L^\infty(X)$ the canonical conditional expectation. Let $\mathfrak{a} \in \Gamma \ltimes L^\infty(X)$ be Γ-fixed, and set $\mathfrak{b} = \mathfrak{a} - \mathcal{E}(\mathfrak{a})$. Then, $\mathfrak{b}$ is Γ-fixed and $\mathcal{E}(\mathfrak{b}) = 0$. For $g \in \Gamma$, let $\phi_g := \mathcal{E}(\mathfrak{b}\lambda_{g^{-1}}) \in L^\infty(X)$ be the Fourier coefficient of $\mathfrak{b}$ at g. Since $\mathfrak{b}$ is Γ-fixed, the map $g \mapsto |\phi_g|^2$ is Γ-equivariant with respect to the conjugation action of Γ on itself, and so the function $\zeta(g) := \int_X |\phi_g|^2 \, d\nu$ is μ-harmonic (for the conjugate action). Moreover, $\sum_{g \in \Gamma} \zeta(g) = \int_X \mathcal{E}(\mathfrak{b}^*\mathfrak{b}) \, d\nu < \infty$, and so $\zeta \in \ell^1(\Gamma)$. Thus, ζ attains its maximum values on a finite conjugate-invariant subset of Γ, that is, $\{e\}$ by the icc assumption. Since $\zeta \geq 0$ and $\zeta(e) = 0$, it follows that $\zeta = 0$. So $\mathfrak{b} = 0$ and $\mathfrak{a} = \mathcal{E}(\mathfrak{a})$. The function $\mathcal{E}(\mathfrak{a}) \in L^\infty(X)$ is Γ-invariant, and hence constant by ergodicity of $\Gamma \curvearrowright (X, \nu)$. $\square$

Returning to the proof of the theorem, we can now invoke [**BH21**, Theorem B] to conclude that either the restriction $\varrho|_\mathsf{B}$ is Γ-invariant, or that there exists a

closed subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant von Neumann algebra embedding $\vartheta : L^\infty(G/Q, \nu_Q) \to \mathsf{B}$, such that $\varrho \circ \vartheta = \nu_Q$, where ν_Q is the pushforward of ν_P under the canonical map $G/P \to G/Q$. We will rule out the latter possibility.

For the sake of contradiction, assume otherwise, that such ϑ exists. Then the composition $\mathcal{E}_0 \circ \vartheta$ is an ultraweakly continuous Γ-equivariant ucp map from $L^\infty(G/Q, \nu_Q)$ into $L^\infty(G/P, \nu_P)$. The canonical embedding is the unique such map (see, e.g., [**BS06**, Theorem 2.14]), so $\mathcal{E}_0 \circ \vartheta = \mathrm{id}\,|_{L^\infty(G/Q)}$. Since $\mathcal{E}_0$ is a faithful conditional expectation, it follows that $\vartheta = \mathrm{id}\,|_{L^\infty(G/Q)}$ by [**Ham85**, Lemma 3.3]. In particular, $L^\infty(G/Q) \subset \mathsf{B} \cap L^\infty(G/P)$, and therefore, $\Phi(L^\infty(G/Q)) \subset \mathsf{A}' \cap \mathcal{H}_\mu$. As remarked above, Φ restricts to the Poisson transform on $L^\infty(G/P)$, so $\mathsf{A} \subset (\mathcal{P}_{\nu_Q}(L^\infty(G/Q)))'$.

Since A is non-trivial and $\mathsf{A} \cap \ell^\infty(\Gamma) = \mathbb{C}1$, there exists $\mathfrak{a} \in \mathsf{A}$ and $g, h \in \Gamma$ with $g \neq h$ such that $\langle \mathfrak{a}\delta_g, \delta_h \rangle \neq 0$. Denoting by $\mathfrak{m}_\phi \in B(\ell^2(\Gamma))$ the multiplication operator associated to a function $\phi \in \ell^\infty(\Gamma)$, $\mathfrak{a}$ commutes with $\rho(g)\mathfrak{m}_{\mathcal{P}_{\nu_Q}(f)}\rho(g^{-1}) = \mathfrak{m}_{\mathcal{P}_{g\nu_Q}(f)}$ for every $g \in \Gamma$ and $f \in C(G/Q)$. The action $G \curvearrowright G/Q$ is strongly proximal, so given $x \in G/Q$, there exits a net (g_i) in Γ such that $g_i \nu_Q \xrightarrow{\text{weak}*} \delta_x$ in $\mathrm{Prob}(G/Q)$. This implies $\mathfrak{m}_{\mathcal{P}_{g_i\nu_Q}(f)} \xrightarrow{\text{WOT}} \mathfrak{m}_{\mathcal{P}_{\delta_x}(f)}$ in $B(\ell^2(\Gamma))$, and therefore $\mathfrak{a}$ commutes with $\mathfrak{m}_{\mathcal{P}_{\delta_x}(f)}$ for every $f \in C(G/Q)$ and $x \in G/Q$. Hence,

$$f(gx)\langle \mathfrak{a}\delta_g, \delta_h \rangle = \langle \mathfrak{a}\mathfrak{m}_{\mathcal{P}_{\delta_x}(f)}\delta_g, \delta_h \rangle = \langle \mathfrak{m}_{\mathcal{P}_{\delta_x}(f)}\mathfrak{a}\delta_g, \delta_h \rangle$$
$$= \langle \mathfrak{a}\delta_g, \mathfrak{m}_{\mathcal{P}_{\delta_x}(\bar{f})}\delta_h \rangle = f(hx)\langle \mathfrak{a}\delta_g, \delta_h \rangle$$

for all $f \in C(G/Q)$ and $x \in G/Q$, which implies that gh^{-1} lies in $\ker(\Gamma \curvearrowright G/Q)$. However, it follows from similar reasoning as in the beginning of the proof that $\Gamma \curvearrowright G/Q$ is faithful. This contradicts $g \neq h$, and therefore completes the proof of the first half.

Property (T) half. The assumptions imply that G has property (T) and since Γ is a lattice in G, Γ also has property (T) (see, e.g., [**BdlHV08**, Theorem 1.7.1]). Recall that π is a unitary representation of Γ weakly contained in λ.

Let $M = \pi(\mathsf{A})'$, and let $(M, H_\mathsf{s}, \mathsf{J}_\mathsf{s}, \mathsf{P}_\mathsf{s})$ be its standard form in the sense of [**Haa75**]. By [**Haa75**, Theorem 3.2], the action $\Gamma \curvearrowright M$ is implemented by a unitary representation σ of Γ on H_s. As usual, we endow $B(H_\mathsf{s})$ with the Γ-action defined by unitaries $\sigma_g, g \in \Gamma$.

By the amenability half, there exists a Γ-invariant state on M. Since M has separable predual, a standard argument yields a sequence (ω_n) of ultraweakly continuous states on M such that $\|\omega_n \circ \sigma_g - \omega_n\| = \|g\omega_n - \omega_n\| \xrightarrow{n \to \infty} 0$ for every $g \in \Gamma$.

By [**Haa75**, Lemma 2.10], the map $\xi \mapsto \omega_\xi|_M$ is a Γ-equivariant homeomorphism from P_s onto M_*^+, where ω_ξ is the vector state defined by the vector ξ. Hence, there exists an almost σ-invariant sequence in H_s, and property (T) implies the existence of a $\sigma(\Gamma)$-invariant unit vector ξ_0 in H_s. The vector state ω_{ξ_0} restricts to an ultraweakly continuous Γ-invariant state on M. This completes the proof. $\square$

We note that the proof of Theorem 1.5 also yields the following theorem.

THEOREM 2.17. *Every group Γ as in the statement of Theorem 1.5 is λ_Γ-just-infinite.*

Remark 2.18. Margulis' NST and its generalizations, such as Stuck–Zimmer's theorem [**SZ94**], are considered as rigidity results for Γ. However, there are just-infinite groups that should not really be considered as rigid objects in the same spirit as higher rank lattices.

Examples 2.5 and 2.15 show that the non-commutative just-infiniteness properties we consider in this section are indeed stronger properties than the usual group-theoretic notion, and therefore in this sense, Theorems 1.5 and 2.17 entail sharper results compared to the NST for lattices Γ as in the statements.

Remark 2.19. Another fact that follows from the arguments in the proof of Theorem 1.5 is that, given a discrete property (T) group Γ and a unitary representations π of Γ, the weak π-just-infiniteness of Γ depends only on the weak equivalence class of π.

3. *Proofs of Theorems 1.1, 1.2, and 1.3*

Proof of Theorem 1.1. Let N be a non-trivial Γ-invariant von Neumann subalgebra of $\mathcal{L}(\Gamma)$. By Corollary 2.9 and Theorem 2.17, N is a factor. As shown in [**Bru18**, Lemma 4.3.2 and Remark 4.2.8], there is a non-trivial normal subgroup $\Lambda \leq \Gamma$ such that the restriction of the Γ-action on N to Λ is inner. This gives a projective unitary representation $\sigma : \Lambda \to \mathcal{U}(\mathsf{N})$. Since N is a factor, we also get a projective unitary representation $\sigma' : \Lambda \to \mathcal{U}(\mathsf{N}' \cap \mathcal{L}(\Gamma))$ such that $\lambda_\Gamma(g) = \sigma(g)\sigma'(g)$ for all $g \in \Lambda$. Furthermore, the latter decomposition is unique. In particular, both $\sigma(\Lambda)''$ and $\sigma'(\Lambda)''$ are Γ-invariant, hence factors by Corollary 2.9.

Let $\tilde{\mathsf{N}}$ be the von Neumann algebra generated by $\sigma(\Lambda) \cup \sigma'(\Lambda)$. Then, $\mathcal{L}(\Lambda) \subseteq \tilde{\mathsf{N}} \subseteq \mathcal{L}(\Gamma)$. Since Λ has finite index in Γ (by Margulis' NST), it follows that $\mathcal{L}(\Lambda)$ has finite index in $\tilde{\mathsf{N}}$. Thus, as observed in the proof of [**Bru18**, Lemma 4.3.6], it cannot be the case that the trace vanishes on $\sigma(g)$ and $\sigma'(g)$ for all non-neutral $g \in \Lambda$.

The proof of [**BH21**, Theorem C] can be modified to conclude that if π is a projective unitary representation of Λ generating a finite factor M, then either M is finite dimensional or else, $\tau(\pi(g)) = 0$ for all $g \in \Lambda$, $g \neq e$, where τ is the trace of M.

Thus, either $\sigma(\Lambda)''$ or $\sigma'(\Lambda)''$ is finite dimensional, and hence trivial by Proposition 2.7 since they are Γ-invariant. We therefore conclude that either N or its relative commutant $\mathsf{N}' \cap \mathcal{L}(\Gamma)$ contains $\mathcal{L}(\Lambda)$, and hence has finite index in $\mathcal{L}(\Gamma)$. This implies that one of the these von Neumann algebras is finite dimensional [**Jon83**, Corollary 2.2.3], and hence trivial since they are Γ-invariant. Since we assumed N is non-trivial, it follows that $\mathsf{N}' \cap \mathcal{L}(\Gamma)$ is trivial. We may now invoke [**CD20**, Theorem 3.15] to conclude the result. $\square$

Remark 3.1. We thank the anonymous referee for pointing out to us that the proof of Theorem 1.1 could be simplified by using the recent results of [**BBH21**]. Indeed, since $\mathsf{N} \subset \mathcal{L}(\Gamma)$ is a non-trivial Γ-invariant subfactor, [**BBH21**, Theorem 6.2] implies that the inclusion $\mathsf{N}\mathfrak{p} \subset \mathfrak{p}\mathcal{L}(\Gamma)\mathfrak{p}$ has finite Jones index for some non-zero projection $\mathfrak{p} \in \mathsf{N}' \cap \mathcal{L}(\Gamma)$. This further implies that $\mathfrak{p}(\mathsf{N}' \cap \mathcal{L}(\Gamma))\mathfrak{p}$ is finite dimensional [**Jon83**, Corollary 2.2.3] and so is $\mathsf{N}' \cap \mathcal{L}(\Gamma)$. Then by Corollary 2.9 and Theorem 2.17, it follows that $\mathsf{N}' \cap \mathcal{L}(\Gamma)$ is trivial.

Proof of Theorem 1.3. Let $\mathcal{E}$ be an ultraweakly continuous Γ-equivariant conditional expectation on $\mathcal{L}(\Gamma)$. Then the image of $\mathcal{E}$ is a Γ-invariant von Neumann subalgebra of

$\mathcal{L}(\Gamma)$, and hence of the form $\mathcal{L}(\Lambda)$ for some normal subgroup $\Lambda \leq \Gamma$, by Theorem 1.1. Since Γ is icc, the canonical trace τ is the unique trace on $\mathcal{L}(\Gamma)$. Since the characteristic function $\mathbf{1}_\Lambda$ extends to the unique trace preserving conditional expectation from $\mathcal{L}(\Gamma)$ onto $\mathcal{L}(\Lambda)$, it suffices to show that $\mathcal{E}$ is trace preserving. For this, we observe that since $\mathcal{E}$ is Γ-equivariant, $\tau\mathcal{E}$ is Γ-invariant and ultraweakly continuous, and hence a trace on $\mathcal{L}(\Gamma)$. By the uniqueness of τ, we conclude that $\tau\mathcal{E} = \tau$, thereby completing the proof. $\qquad\square$

Proof of Theorem 1.2. Let π be a weakly mixing unitary representation of Γ. From [**BH21**, Corollary D], it follows that π weakly contains λ_Γ and $C^*_\pi(\Gamma)$ has a unique trace τ, which is the lift of the canonical trace on $C^*_{\mathrm{red}}(\Gamma)$. Let I be the kernel of the quotient map $C^*_\pi(\Gamma) \to C^*_{\mathrm{red}}(\Gamma)$. Since τ factors to a faithful trace on $C^*_{\mathrm{red}}(\Gamma)$, it follows that $I = \{\mathfrak{a} \in C^*_\pi(\Gamma) : \tau(\mathfrak{a}^*\mathfrak{a}) = 0\}$.

Assume $\mathcal{E}$ is a Γ-equivariant conditional expectation from $C^*_\pi(\Gamma)$ onto a C^*-subalgebra $\mathsf{A} \subseteq C^*_\pi(\Gamma)$. Since $\mathcal{E}$ is Γ-equivariant, A is Γ-invariant.

Furthermore, the Γ-equivariance of $\mathcal{E}$ implies that $\tau\mathcal{E}$ is Γ-invariant, and hence a trace on $C^*_\pi(\Gamma)$. Since τ is the unique trace on $C^*_\pi(\Gamma)$, we conclude $\tau\mathcal{E} = \tau$.

Given $\mathfrak{a} \in I$, we get $\tau(\mathcal{E}(\mathfrak{a})^*\mathcal{E}(\mathfrak{a})) \leq \tau(\mathcal{E}(\mathfrak{a}^*\mathfrak{a})) = \tau(\mathfrak{a}^*\mathfrak{a}) = 0$, which implies $\mathcal{E}(I) \subset I$. Thus, $\mathcal{E}$ factors to a map on $C^*_{\mathrm{red}}(\Gamma)$, which is also a Γ-equivariant conditional expectation, we still denote it by $\mathcal{E}$. Since $\mathcal{E}$ is trace-preserving, it is faithful on $C^*_{\mathrm{red}}(\Gamma)$, and hence $\mathcal{E}$ extends to an ultraweakly continuous trace-preserving conditional expectation $\mathcal{E} : \mathcal{L}(\Gamma) \to \mathsf{A}''$ (see [**Pop00**, Lemma 4.5]). By Theorem 1.3, we have that $\mathcal{E}|_{\lambda_\Gamma(\Gamma)} = \mathbf{1}_{\lambda_\Gamma(\Lambda)}$ for some normal subgroup $\Lambda \leq \Gamma$. This implies $\mathcal{E}|_{\pi(\Gamma)} = \mathbf{1}_{\pi(\Lambda)}$. $\qquad\square$

We conclude this section with an example of an icc group Γ for which $C^*_{\mathrm{red}}(\Gamma)$ contains a Γ-invariant unital C^*-subalgebra that is not the image of any Γ-equivariant conditional expectation.

Example 3.2. Let Γ be as in Example 2.15, and let A be the C^*-subalgebra of $C^*_{\mathrm{red}}(\Gamma)$ constructed from a non-zero proper ideal I as in the proof of Proposition 2.14. Assume $\mathcal{E} : C^*_{\mathrm{red}}(\Gamma) \to \mathsf{A}$ is a Γ-equivariant conditional expectation. Since $C^*_{\mathrm{red}}(\Gamma)$ has a unique trace, it follows, as seen in the proof of Theorem 1.5 above, that $\mathcal{E}$ is faithful and trace preserving, and hence extends to the canonical trace preserving conditional expectation from $\mathcal{L}(\Gamma)$ onto A''. Since $\mathcal{L}(\Gamma)$ is a factor, and I is non-zero, it follows that $\mathsf{A}'' = \mathcal{L}(\Gamma)$. In particular, $\mathcal{E} = \mathrm{id}$, and therefore, $\mathsf{A} = C^*_{\mathrm{red}}(\Gamma)$. This implies the quotient of $C^*_{\mathrm{red}}(\Gamma)$ by I has dimension one, and so admits a trace, which contradicts the uniqueness of the canonical trace on $C^*_{\mathrm{red}}(\Gamma)$. Hence, A cannot be the image of any Γ-equivariant conditional expectation on $C^*_{\mathrm{red}}(\Gamma)$.

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