Notes on Number Theory III

ON THE SUM OF CONSECUTIVE PRIMES

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Let $A: a_1 < a_2 < \ldots$ be a sequence of positive integers and let $f(A:n) = f(n)$ denote the number of representations of n as the sum of one or more consecutive a's. Although $f(n)$ will usually oscillate rapidly, an asymptotic estimate for its average value

$$(1) \qquad F(A;x) = F(x) = \frac{1}{x} \sum_{n=1}^{x} f(n)$$

can be obtained for many sequences A. For the case $a_i = i$, i.e. A the set of positive integers, LeVeque [1] proved

$$(2) \qquad F(x) = \frac{1}{2} \log x + \gamma + \frac{1}{2} \log 2 - \frac{1}{2} + 0(x^{-1/2}) .$$

A direct consequence of (2) is

$$(3) \qquad F(x) \sim \frac{1}{2} \log x .$$

More generally, LeVeque showed that if A is any arithmetic progression of positive terms with common difference d then

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(4) $$F(x) = \frac{1}{2} \log x + \gamma - \frac{1}{2} \log \frac{d}{2} - \frac{1}{2} + 0(x^{-1/2})$$

which again implies (3). Using a variation of the argument used later in this note we can show that (3) is valid for every sequence of positive asymptotic density. It is curious that the density itself does not enter into the leading term of the asymptotic estimate though of course, as suggested by (4), it will make its appearance if a more accurate estimate is required.

Our main object in this note, however, is to treat the case $a_i = p_i$, i.e. A the sequence of primes. For this case we will show that

(5) $$F(x) \sim \log 2 .$$

We will conclude with some remarks and related open questions.

Suppose then that A is the set of primes. Every set of consecutive primes whose sum is $\leq x$ will contribute 1 to the sum $f(1) + f(2) + \ldots + f(x)$. The number of such sets of r primes is clearly at most $\pi(x/r)$ and at least $\pi(x/r)-r$. Hence

(6) $$\sum_{r=1}^{k} (\pi(\frac{x}{r})-r) \leq f(1) + f(2) + \ldots + f(x) \leq \sum_{r=1}^{k} \pi(\frac{x}{r}) ,$$

where k is determined by

(7) $$p_1 + p_2 + \ldots + p_k \leq x < p_1 + p_2 + \ldots + p_{k+1} .$$

From (7) and the well known $p_r \asymp r \log r$ * we find

(8) $$k \asymp \sqrt{\frac{x}{\log x}} .$$

* We use $f \asymp g$ to denote that f/g is bounded above and below by positive numbers, for large values of the argument.

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This implies $\sum_{r=1}^{k} r = o(x)$. Hence, if we can show that

$\sum_{r=1}^{k} \pi(\frac{x}{r}) \sim x \log 2$ it will follow that $f(1) + f(2) + \ldots + f(x) \sim x \log 2$.

However, using (8) and the prime number theorem, the following estimates can easily be justified:

$$\sum_{r=1}^{k} \pi(\frac{x}{r}) \sim \sum_{r=1}^{k} \frac{x}{r \log x/r} \sim \int_{1}^{k} \frac{x\,dr}{r \log x/r} =$$

$$[-x \log(\log x/r)]_1^k = x(\log\log x - \log\log \frac{x}{k}) \sim$$

$$x(\log\log x - \log\log \sqrt{x \log x}) \sim x \log 2 ,$$ so that

(5) is established.

Since the average value of $f(n)$ is $\log 2$ it follows that $f(n) = 0$ infinitely often. The following problems, among others, suggest themselves:

1. Is $f(n) = 1$ infinitely often?

2. Is $f(n) = k$ solvable for every k ?

3. Do the numbers n for which $f(n) = k$ have a density for every k ?

4. Is $\overline{\lim}\, f(n) = \infty$?

REFERENCE

1. W. J. LeVeque. On representation as a sum of consecutive integers. Can. Jour. Math. 4 (1950) 399-405.

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