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FUNCTIONS WITH A FINITE NUMBER OF NEGATIVE SQUARES

BY
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1. **Introduction.** Let f be a complex-valued function defined on the real line R with the property that $f(-x)=\overline{f(x)}$ for every $x \in R$. If k is a nonnegative integer, f is said to *have k negative squares*, or to be *indefinite of order k*, if the Hermitian form

$$\sum_{i,j=1}^{n} f(x_i-x_j)\xi_i\bar{\xi}_j \tag{1}$$

has at most k negative squares for any choice of n and $x_1, \ldots, x_n$ in R, and for some choice of $x_1, \ldots, x_n$ the form has exactly k negative squares. Krein [7] proved that if f is a continuous function with k negative squares then there is a nondecreasing function σ and a polynomial Q of degree k such that

$$f(x) = h_\rho(x)+\int_{-\infty}^{\infty} \frac{e^{i\lambda x}-S_\rho(x, \lambda)}{[Q_0(\lambda)]^2}\, d\sigma(\lambda), \tag{2}$$

where

(i) if $\bar{Q}(\lambda)=\overline{Q(\bar{\lambda})}$ then $h_\rho(x)$ is a solution of the differential equation $\bar{Q}(-id/dx)\times Q(-id/dx)h_\rho(x)=0$,

(ii) if $\alpha_1, \ldots, \alpha_r$ are the distinct real roots of $Q(\lambda)$ with multiplicities $m_1, \ldots, m_r$, then

$$Q_0(\lambda) = \prod_{i=1}^{r} (\lambda-\alpha_i)^{m_i},$$

(iii) if $\rho>\max\{|\alpha_1|, \ldots, |\alpha_r|\}$ then $S_\rho(x, \lambda)$ is a regularizing correction which is equal to 0 for $|\lambda|>\rho$ and for $|\lambda|\leq\rho$ it is equal to $[Q_0(\lambda)]^2$ times the sum of the principal parts of the function $e^{i\lambda x}[Q_0(\lambda)]^{-2}$ over all its poles.

Earlier Krein ([8], [6]) had given integral representations for the case $k=1$. The definition of a function with k negative squares reduces to that of a positive definite function in the case $k=0$, and Krein's integral representation (2) becomes Bochner's theorem.

Cooper [3] has generalized Bochner's theorem in a different direction. If F is a set of functions on R he called a function f *positive definite for F* if the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\varphi(x)\overline{\varphi(y)}\, dx\, dy$$

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exists as a Lebesgue integral and is nonnegative for every $\varphi \in F$. The class of functions positive definite for L^1 turns out to be identical, up to sets of measure zero, with the usual positive definite functions. However if we take F to be C_c, the continuous functions with compact support, then Cooper's definition gives rise to a much larger class of functions. The analogue of Bochner's theorem is that every function positive definite for C_c is the Fourier-Stieltjes transform of a positive measure, possibly unbounded, in the sense of Cesàro summability almost everywhere.

Our aim in this paper is to enlarge the class of functions with a finite number of negative squares in a sense similar to that in which Cooper's positive definite functions extend those of Bochner. In the form (1) $f(x_i-x_j)$ is replaced by the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\varphi_i(x)\overline{\varphi_j(y)}\,dx\,dy$$

for φ_i in some function class F, and the integral representation (2) continues to hold in a summability sense.

Krein's definition of a function with k negative squares can be formulated just as easily on a group, and our definition makes sense on a locally compact group. Indeed in §2 we give the elementary properties of these functions on a locally compact group. However when it comes to proving integral representation theorems in §3 we must restrict our attention to the real line. In fact, even for the ordinary functions with a finite number of negative squares as defined by the form (1), integral representation theorems are known only for the groups R^n[4] and the integers ([6], Theorem 5.2).

2. **Elementary properties of the class $P_k(F)$.** Let f be a complex-valued function, defined on a locally compact group G, such that $f(x^{-1})=\overline{f(x)}$ for every $x \in G$. If F is a linear space of complex-valued functions on G, f is said to be *indefinite of order k for F* if the Hermitian form

$$Q(\xi,\xi)=\int_G\int_G f(y^{-1}x)\varphi(x)\overline{\varphi(y)}\,dx\,dy \tag{3}$$

where

$$\xi=[\xi_1,\ldots,\xi_n],\qquad \varphi=\sum_{i=1}^{n}\xi_i\varphi_i,$$

exists as a Haar integral over the product group $G\times G$ and has at most k negative squares for any choice of n and $\varphi_1,\ldots,\varphi_n$ in F, and for some choice of $\varphi_1,\ldots,\varphi_n$ in F the form has exactly k negative squares. Notice that Q can also be written in terms of the scalar product

$$(\varphi,\psi)=\int_G\int_G f(y^{-1}x)\varphi(x)\overline{\psi(y)}\,dx\,dy \qquad (\varphi,\psi\in F)$$

as

$$Q(\xi,\xi)=\sum_{i,j=1}^{n}(\varphi_i,\varphi_j)\xi_i\bar{\xi}_j.$$

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Let $P_k(F)$ be the class of functions which are indefinite of order k for F. $P_0(F)=P(F)$, the class of functions positive definite for F, in the notation of [12]. If F has the property that for any compact subset C of G, F contains a function which is strictly positive on C and has compact support, then it is not difficult to show that every function in $P_k(F)$ is locally summable with respect to the left Haar measure on G. The function classes F that we deal with will always enjoy this property.

It is clear that $F_1 \subset F_2$ implies that $P_k(F_1) \supset P_k(F_2)$. More precise information about how $P_k(F)$ varies with F is provided by the following two theorems which generalize theorems given in [12]. Since their proofs are similar we prove only the second of them.

THEOREM 2.1. *$P_k(C_c)=P_k(L_c^p)$ for any $p\geq 2$, where L_c^p denotes the class of function in $L^p(G)$ with compact support.*

THEOREM 2.2. *Let $1\leq p\leq 2$ and $q=p/2(p-1)$. If $f\in P_k(L_c^2)$ and f is in L^q locally then $f\in P_k(L_c^p)$.*

Proof. We denote integration with respect to the left Haar measure on G by dx. Let φ, $\psi\in L_c^p$. Then the adjoint $\tilde{\psi}(x)=\overline{\psi(x^{-1})}\Delta(x^{-1})$, where Δ is the modular function of G, is also in L_c^p. Thus $\tilde{\psi} * \varphi \in L_c^r$, where $r^{-1}=2p^{-1}-1=1-q^{-1}$ [5, p. 296]. Since f is locally in L^q, the integral

$$\int_G\int_G f(y^{-1}x)\varphi(x)\overline{\psi(y)}\,dx\,dy=\int_G f(x)\int_G \varphi(yx)\overline{\psi(y)}\,dy\,dx$$
$$=\int_G f(x)\tilde{\psi} * \varphi(x)\,dx$$

exists. If $\{\varphi_1,\ldots,\varphi_n\}\subset L_c^p$ the integrals $(\varphi_i,\varphi_j)=\int f(x)\tilde{\varphi}_j * \varphi_i(x)\,dx$ exist. For each $i=1,\ldots,n$ choose a sequence $\{\varphi_i^m\}_{m=1}^\infty \subset L_c^2$ such that $\|\varphi_i-\varphi_i^m\|_p\to 0$ as $m\to\infty$ and the supports of the φ_i^m are contained in a common compact set. Define $\varphi^*(x)=\varphi(x^{-1})$. Then we also have

$$\|\varphi_i^*-(\varphi_i^m)^*\|_p\to 0 \quad\text{and}\quad \|\tilde{\varphi}_i-\tilde{\varphi}_i^m\|_p\to 0 \quad\text{as}\quad m\to\infty.$$

Hence

$$\|\tilde{\varphi}_j * \varphi_i-\tilde{\varphi}_j^m * \varphi_i\|_r \leq \|\tilde{\varphi}_j\|_p\,\|\varphi_i-\varphi_i^m\|_p^{2-p}\,\|\varphi_i^*-(\varphi_i^m)^*\|_p^{p-1}$$
$$+\|\tilde{\varphi}_j-\tilde{\varphi}_j^m\|_p\,\|\varphi_i^m\|_p^{2-p}\,\|(\varphi_i^m)^*\|_p^{p-1}\to 0 \quad\text{as}\quad m\to\infty.$$

(Note that in the general case formula (ii) of [5, p. 296] is replaced by

$$\|f*g\|_r\leq \|f\|_p\,\|g\|_q^{q/r}\,\|g^*\|_q^{q(p-1)/p}.)$$

Thus for $i,j=1,\ldots,n$ we have $(\varphi_i^m,\varphi_j^m)\to(\varphi_i,\varphi_j)$ as $m\to\infty$. Since $\varphi_i^m\in L_c^2$ each of the matrices $[(\varphi_i^m,\varphi_j^m)]$ has at most k negative eigenvalues. But the eigenvalues of a matrix are continuous functions of its elements, and so the matrix $[(\varphi_i, \varphi_j)]$ has at most k negative eigenvalues. Furthermore since $L_c^2 \subset L_c^p$ and $f \in P(L_c^2)$ there must exist $\varphi_1, \ldots, \varphi_n \in L_c^p$ such that the form (3) has exactly k negative squares. Therefore $f \in P_k(L_c^p)$.

The converse of Theorem 2.2 can be proved for the case $p=1$ by using the uniform boundedness theorem twice. The shorter proof given here uses Cohen's theorem on factorization in group algebras.

THEOREM 2.3. *If $f \in P_k(L_c^1)$ then f is essentially bounded on any compact set.*

Proof. If $g \in L_c^1$ then there exist functions $\xi, \varphi \in L_c^1$ such that $g=\xi * \varphi$ since L_c^1 is a subalgebra of L^1 [2]. Thus if $\psi=\bar{\xi}$ we have $\int fg=\int f(\tilde{\psi} * \varphi)<\infty$ for every $g \in L_c^1$. This proves that $f \in L^\infty$ on any compact set ([1, p. 85]).

We conclude this section by showing that the functions with k negative squares treated by Krein are special cases of the functions that we are considering.

THEOREM 2.4. *A continuous function f has k negative squares if and only if it is in $P_k(C_c)$.*

Proof. First suppose that G contains $x_1, \ldots, x_n$ such that the form

$$\sum_1^n f(x_j^{-1}x_i)\xi_i\bar{\xi}_j$$

has exactly r negative squares. Then there is a nonsingular linear transformation $\xi_i=\sum_{j=1}^n a_{ij}\eta_j$ such that

$$\sum_{i,j=1}^n f(x_j^{-1}x_i)\xi_i\bar{\xi}_j = -\sum_{i=1}^r |\eta_i|^2+\sum_{i=r+1}^n |\eta_i|^2.$$

Thus, by taking $\eta_i^{(p)}=\delta_{ip}$, we obtain

$$\sum_{i,j=1}^n f(x_j^{-1}x_i)a_{ip}\overline{a_{jq}} = -\delta_{pq}, \qquad p, q \le r.$$

Let V_i be a neighbourhood of x_i, ψ_i a positive continuous function with support in V_i such that $\int \psi_i(x)\,dx=1$, and $\varphi_p=\sum_{i=1}^n a_{ip}\psi_i$. By taking the V_i sufficiently small we can make $(\varphi_p, \varphi_q)=\iint f(y^{-1}x)\varphi_p(x)\overline{\varphi_q(y)}\,dx\,dy$ arbitrarily close to $\sum f(x_j^{-1}x_i)a_{ip}\bar{a}_{jq}$. Since the eigenvalues of a matrix are continuous functions of its elements we conclude that the matrix $[(\varphi_p, \varphi_q)]_{p,q=1}^r$ has r negative eigenvalues if the V_i are chosen sufficiently small.

Now suppose that there exist $\varphi_1, \ldots, \varphi_n$ in $C_c(G)$ such that the form $\sum_1^n (\varphi_i, \varphi_j)\xi_i\bar{\xi}_j$ has exactly r negative squares. As in the above paragraph there is a nonsingular matrix $[b_{ij}]_{i,j=1}^n$ such that $(\psi_j, \psi_j)=-\delta_{ij}$ for $i, j\le r$ if $\psi_j=\sum_{i=1}^n b_{ij}\varphi_i$.

Let K be a compact set which contains the supports of $\psi_1, \ldots, \psi_r$. Since the functions $f(y^{-1}x)\psi_i(x)\overline{\psi_j(y)}$ are uniformly continuous on $K\times K$ we can partition K into disjoint sets $E_1, \ldots, E_m$ so that the sums

$$\sum_{p\ q=1}^{m} f(x_q^{-1}x_p)\psi_i(x_p)\overline{\psi_j(x_q)}m(E_p)m(E_q),$$

where m is the left Haar measure, differ from the integrals

$$\int_G \int_G f(y^{-1}x)\psi_i(x)\overline{\psi_j(y)}\, dx\, dy = -\delta_{ij}, \qquad i, j \leq r,$$

by less than any preassigned amount. This shows that the form $\sum_1^m f(x_q^{-1}x_p)\xi_p\bar{\xi}_q$ has r negative squares.

The theorem follows from combining the results of the two preceding paragraphs.

3. **Representation theorems.** In this section the only group that we shall deal with is the real line. If C_c^∞ is the class of infinitely differentiable functions with compact support then $P_k(C_c^\infty)=P_k(C_c)=P_k(L_c^p)$, $p\geq 2$. We shall find it convenient in the proofs to take the function class F to be C_c^∞, and for this reason the theorems are stated for $P_k(C_c^\infty)$. The Fourier transform of a function $\varphi \in L^1$ is denoted by

$$\Phi(\lambda) = \hat{\varphi}(\lambda) = \int_{-\infty}^{\infty} \varphi(x)e^{-i\lambda x}\, dx.$$

THEOREM 3.1. *If $f \in P_k(C_c^\infty)$ then there is a polynomial Q of degree at most k such that*

$$\iint \overline{f(x-y)}Q\left(-i\frac{d}{dx}\right)\varphi(x)\overline{Q\left(-i\frac{d}{dy}\right)\varphi(y)}\, dx\, dy \geq 0 \tag{4}$$

holds for every $\varphi \in C_c^\infty$.

Proof. The indefinite scalar product

$$(\varphi, \psi) = \iint f(x-y)\varphi(x)\overline{\psi(y)}\, dx\, dy \qquad (\varphi, \psi \in C_c^\infty) \tag{5}$$

clearly satisfies axioms I, II, IV and V of the six axioms for a Π_k space of Iohvidov and Krein [6]. Let H be the isotropic subspace of C_c^∞, i.e., $H=\{\psi \in C_c^\infty; (\psi, \varphi)=0$ for every $\varphi \in C_c^\infty\}$. The factor space C_c^∞/H can be written as a direct sum $\Pi_+\oplus\Pi_-$ where Π_- is a k-dimensional negative subspace. If $\bar{\Pi}_+$ is a completion of the positive subspace Π_+ then $\Pi_k=\bar{\Pi}_+\oplus\Pi_-$ satisfies all six axioms of Iohvidov and Krein.

The operator $A\varphi=id\varphi/dx$ is symmetric on C_c^∞ with respect to the scalar product (5), and since H is invariant under A we can regard A as a well-defined symmetric operator on the dense subspace C_c^∞/H of Π_k. Since A is defined on all of Π_- we

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can write A as a matrix $[A_{ij}]_{i,j=1}^2$ with respect to the direct sum $\Pi_k=\bar{\Pi}_+\oplus\Pi_-$, where A_{22} is a symmetric operator on Π_- and A_{11} is a symmetric operator densely defined on the Hilbert space $\bar{\Pi}_+$. By means of Naimark's theorem on self-adjoint extensions of the second kind [9] A_{11} can be extended to a self-adjoint operator A'_{11} on a Hilbert space $\Pi'_+\supset\bar{\Pi}_+$. Define $\Pi'_k=\Pi'_+\oplus\Pi_-$ and extend $A_{21}:\bar{\Pi}_+\to\Pi_-$ to $A'_{21}:\Pi'_+\to\Pi_-$ by defining it to be zero except on $\bar{\Pi}_+$. If $A'_{12}:\Pi_-\to\Pi'_+$ is defined to coincide with A_{12} then the operator $A'=[A'_{ij}]_{i,j=1}^2$ on Π'_k is a self-adjoint extension of A. A theorem of Pontryagin [10] proves the existence of a k-dimensional subspace L of Π'_k which is invariant under A' and is such that $(x, x)\le 0$ for every $x\in L$. Let Q be the minimal polynomial of the operator A' restricted to L; then $Q(A')x=0$ for every $x\in L$. Thus if $\bar{Q}(\lambda)=\overline{Q(\bar{\lambda})}$ we have

$$(\bar{Q}(A')\Pi'_k, L)=(\Pi'_k, Q(A')L)=0.$$

But the orthogonal complement of L is a nonnegative subspace and hence $(\bar{Q}(A')x, \bar{Q}(A')x)\ge 0$ for every $x\in\Pi'_k$. In particular $(\bar{Q}(A)\bar{\varphi}, \bar{Q}(A)\bar{\varphi})\ge 0$ for every $\varphi\in C_c^\infty$ and thus

$$\iint f(x-y)\overline{Q(-id/dx)\varphi(x)}\,Q(-id/dy)\,\varphi(y)\,dx\,dy$$
$$=\iint f(x-y)\bar{Q}(id/dx)\overline{\varphi(x)}\,\overline{\bar{Q}(id/dy)\overline{\varphi(y)}}\,dx\,dy\ge 0.$$

The following theorem can be proved either by the method of directed functionals as used in [7] or by using the Bochner-Schwartz theorem as in [11]. We follow Shah Tao-Shing in the latter approach.

THEOREM 3.2. *There is a positive measure σ and a function h_ρ satisfying the differential equation*

$$\bar{Q}(-id/dx)Q(-id/dx)h_\rho(x)=0$$

such that, for every $\varphi\in C_c^\infty$,

$$\int\overline{f(x)}\varphi(x)\,dx=\int\overline{h_\rho(x)}\varphi(x)\,dx+\int\left\{\int(e^{-i\lambda x}-S_\rho(-x,\lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}. \tag{6}$$

Proof. Since f is locally summable we can consider it as a distribution so that the inequality (4) can be rewritten as

$$\iint\left[\bar{Q}\left(-i\frac{d}{dy}\right)Q\left(i\frac{d}{dx}\right)\overline{f(x-y)}\right]\varphi(x)\overline{\varphi(y)}\,dx\,dy\ge 0$$

or

$$\int\left[\bar{Q}\left(i\frac{d}{dx}\right)Q\left(i\frac{d}{dx}\right)\overline{f(x)}\right]\varphi*\tilde{\varphi}(x)\,dx\ge 0.$$

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Thus the distribution $\bar{Q}(id/dx)Q(id/dx)\bar{f}$ is positive definite and hence by the Bochner-Schwartz theorem it is the Fourier transform of a positive tempered measure μ, i.e.,

$$\int \left[\bar{Q}\left(i\frac{d}{dx}\right)Q\left(i\frac{d}{dx}\right)\overline{f(x)}\right]\varphi(x)\,dx = \int \hat{\varphi}(\lambda)\,d\mu(\lambda) \qquad (\varphi \in C_c^\infty)$$

and

$$\int \frac{d\mu(\lambda)}{(1+|\lambda|^2)^p} < \infty \quad \text{for some } p \geq 0.$$

If Q_0 is defined as in §1 and $Q(\lambda)=Q_0(\lambda)Q_1(\lambda)$ then the polynomial Q_1 has no real zeros and the measure σ defined by

$$d\sigma(\lambda) = \frac{d\mu(\lambda)}{|Q_1(\lambda)|^2}$$

is a positive tempered measure. To verify that the functional T_ρ defined by

$$(T_\rho, \varphi) = \int \left\{\int (e^{-i\lambda x} - S_\rho(-x, \lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \qquad (\varphi \in C_c^\infty)$$

is a distribution, let $\varphi_n \to 0$ in C_c^∞. Thus the support of each φ_n is contained in a common interval $[-a, a]$ and $d^r\varphi_n/dx^r \to 0$ uniformly for every r. T_ρ can be written as $U_\rho + V_\rho$ where

$$(U_\rho, \varphi) = \int_{-\rho}^{\rho}\left\{\int (e^{-i\lambda x} - S_\rho(-x, \lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2},$$

$$(V_\rho, \varphi) = \int_{|\lambda|>\rho} \hat{\varphi}(\lambda)\,\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}.$$

Since $d^{2p}\varphi_n/dx^{2p} \to 0$ uniformly there is a constant A such that $|d^{2p}\varphi_n(x)/dx^{2p}| \leq A$ for every n and x. This implies that $|\hat{\varphi}_n(\lambda)| \leq 2aA\,|\lambda|^{-2p}$ for every n. But since μ is p-tempered we have

$$\int_{|\lambda|>\rho} \frac{2aA\,d\sigma(\lambda)}{|\lambda|^{2p}\,[Q_0(\lambda)]^2} < \infty$$

and so the dominated convergence theorem implies that $(V_\rho, \varphi_n) \to 0$. Also $(U_\rho, \varphi_n) \to 0$ by the bounded convergence theorem and thus T_ρ is a distribution.

$S_\rho(x, \lambda)$ is of the form $\sum_{j=1}^{r} \exp(i\alpha_j x)P_j(x)$, where P_j is a polynomial of degree $2m_j - 1$, and is therefore a solution of the differential equation $\bar{Q}(-id/dx)\times Q(-id/dx)S_\rho(x, \lambda)=0$. It follows that, for every $\varphi \in C_c^\infty$,

$$\int \overline{S_\rho(x, \lambda)}\,\bar{Q}\left(-i\frac{d}{dx}\right)Q\left(-i\frac{d}{dx}\right)\varphi(x)\,dx = 0.$$

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Therefore

$$\begin{aligned}(\bar{Q}(-id/dx)Q(-id/dx)T_\rho, \varphi) &= (T_\rho, \bar{Q}(-id/dx)Q(-id/dx)\varphi)\\ &= \int\left\{\int e^{-i\lambda x}\bar{Q}\left(-i\frac{d}{dx}\right)Q\left(-i\frac{d}{dx}\right)\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}\\ &= \int |Q(\lambda)|^2\,\hat{\varphi}(\lambda)\,\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}\\ &= \int \hat{\varphi}(\lambda)\,d\mu(\lambda)\\ &= \int\left[\bar{Q}\left(i\frac{d}{dx}\right)Q\left(i\frac{d}{dx}\right)\overline{f(x)}\right]\varphi(x)\,dx\\ &= (\bar{Q}(-id/dx)Q(-id/dx)f, \varphi).\end{aligned}$$

Consequently the distribution $h_\rho = f - T_\rho$ satisfies the differential equation $\bar{Q}(-id/dx)Q(-id/dx)h_\rho(x)=0$. But the only solutions of this equation are ordinary functions. The equation $f = h_\rho + T_\rho$ then becomes (6).

So far, all that we know about the measure σ is that it is positive and tempered:

$$\int \frac{d\sigma(\lambda)}{(1+|\lambda|^2)^p} < \infty \quad \text{for some } p \geq 0.$$

However in order to deduce integral representation theorems for f from Theorem 3.2 we need more precise information about σ. Specifically, for what values of p is σ p-tempered? If f is continuous it is known [7] that σ is p-tempered for all $p \geq m$, where m is the degree of Q_0. The following theorem shows that in the general case σ is p-tempered for $p > m + \frac{1}{2}$.

THEOREM 3.3.

$$\int_{\rho<|\lambda|<a} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} = o(a) \quad as \quad a \to \infty.$$

Proof. Consider the function

$$\psi_a(x) = \begin{cases} \frac{4a}{\pi}\left(1 - \frac{2a}{\pi}|x|\right) & \text{if } |x| < \frac{\pi}{2a} \\ 0 & \text{if } |x| \geq \frac{\pi}{2a}\end{cases}$$

and its Fourier transform

$$\Psi_a(\lambda) = \frac{32a^2}{\pi^2\lambda^2}\left(\sin\frac{\pi\lambda}{4a}\right)^2.$$

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These functions have the following properties: $0 \le \psi_a(x) \le 4a/\pi$, ψ_a has support in $[-\pi/2a, \pi/2a]$, $\Psi_a(\lambda) > 1$ for $|\lambda| \le a$, and $\Psi_a(\lambda) \ge 0$ everywhere. We can find a function $\varphi_a \in C_c^\infty$ such that φ_a and Φ_a have these same properties. First notice that $\psi_a = g_a * g_a$ where $g_a(x)$ is equal to $2\sqrt{(2a)/\pi}$ for $|x| \le \pi/4a$ and is zero elsewhere. If we approximate g_a by $h_a \in C_c^\infty$ in the L^1 norm then $\varphi_a = h_a * \tilde{h}_a$ approximates ψ_a in L^1 and hence Φ_a approximates Ψ_a in L^∞. Thus we can choose $h_a \in C_c^\infty$ so that $0 \le h_a(x) \le g_a(x)$ and $\Phi_a(\lambda) > 1$ for $|\lambda| \le a$. Then φ_a and Φ_a possess the desired properties.

By setting $\varphi = \varphi_a$ in (6) we obtain

$$\begin{aligned}\int_{\rho<|\lambda|<a} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} &\le \int_{\rho<|\lambda|<a} \Phi_a(\lambda) \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \\ &\le \int_{\rho<|\lambda|} \Phi_a(\lambda) \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \\ &= -\int_{-\rho}^{\rho} \left\{ \int_{-\infty}^{\infty} (e^{-i\lambda x} - S_\rho(-x, \lambda))\varphi_a(x)\, dx \right\} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \\ &\quad + \int_{-\infty}^{\infty} \overline{(f(x)} - \overline{h_\rho(x))}\varphi_a(x)\, dx \\ &= \int_{-\pi/2a}^{\pi/2a} (\overline{f(x)} - \overline{h_\rho(x)} - \overline{u(x)})\varphi_a(x)\, dx\end{aligned}$$

where

$$u(x) = \int_{-\rho}^{\rho} \frac{e^{i\lambda x} - S_\rho(x, \lambda)}{[Q_0(\lambda)]^2}\, d\sigma(\lambda).$$

Thus

$$\frac{1}{a} \int_{\rho<|\lambda|<a} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \le \frac{4}{\pi} \int_{-\pi/2a}^{\pi/2a} |f(x) - h_\rho(x) - u(x)|\, dx$$

which approaches 0 as $a \to \infty$.

COROLLARY. *The measure σ is p-tempered for every $p > m + \frac{1}{2}$ where m is the degree of Q_0.*

Proof. Let

$$g(t) = \int_\rho^t \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}.$$

If $\varepsilon > 0$ then

$$\int_\rho^a \frac{d\sigma(\lambda)}{\lambda^{1+\varepsilon}[Q_0(\lambda)]^2} = \frac{g(a)}{a^{1+\varepsilon}} - \frac{g(\rho)}{\rho^{1+\varepsilon}} + (1+\varepsilon)\int_\rho^a \frac{g(t)}{t} \frac{dt}{t^{1+\varepsilon}}$$

converges as $a \to \infty$ since $g(t) = o(t)$. Hence if $p > m + \frac{1}{2}$,

$$\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{(1+|\lambda|^2)^p} < \infty.$$

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The next theorem shows that Krein's representation theorem holds almost everywhere if f is bounded in a neighbourhood of 0.

THEOREM 3.4. *Let $f \in P_k(C_c^\infty)$ and suppose that f is bounded in some neighbourhood of* 0. *Then*

$$\int_{|\lambda|>\rho} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} < \infty$$

and, for almost all x,

$$f(x) = h_\rho(x) + \int_{-\infty}^{\infty} \frac{e^{i\lambda x} - S_\rho(x, \lambda)}{[Q_0(\lambda)]^2} \, d\sigma(\lambda). \tag{7}$$

Proof. In the proof of Theorem 3.3 it was shown that

$$\int_{\rho<|\lambda|<a} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \le \frac{4a}{\pi} \int_{-\pi/2a}^{\pi/2a} |f(x) - h_\rho(x) - u(x)| \, dx.$$

By hypothesis there are positive constants δ and A such that $|f(x)| \le A$ for $|x| \le \delta$. Since h_ρ and u are both continuous, $B = \sup\{|h_\rho(x) + u(x)|; |x| \le \delta\} < \infty$. Then for $a > \pi/2\delta$ we have

$$\int_{\rho<|\lambda|<a} \frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2} \le 4(A+B).$$

This proves the first assertion and allows us to reverse the order of integration in (6) and obtain, after replacing φ by $\bar{\varphi}$,

$$\int f(x)\varphi(x) \, dx = \int h_\rho(x)\varphi(x) \, dx + \int \left\{ \int \frac{e^{i\lambda x} - S_\rho(x, \lambda)}{[Q_0(\lambda)]^2} \, d\sigma(\lambda) \right\} \varphi(x) \, dx.$$

Since this equation holds for every $\varphi \in C_c^\infty$ it follows that (7) holds almost everywhere.

If f is unbounded at 0 the integral representation (7) continues to hold provided that the integral is interpreted in a summability sense.

THEOREM 3.5. *If $f \in P_k(C_c^\infty)$ then there is a positive measure σ and a polynomial Q of degree at most k such that the equation*

$$f(x) = h_\rho(x) + \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{e^{i\lambda x} - S_\rho(x, \lambda)}{[Q_0(\lambda)]^2} \Phi_n(\lambda) \, d\sigma(\lambda)$$

holds in the following cases:

(i) *almost everywhere (and at points of continuity) for summability by Weierstrass means,* i.e., $\Phi_n(\lambda) = \exp(-\lambda^2/n^2)$, *if there is a positive constant c such that $f(x)\exp(-cx^2)$ is in $L^1(-\infty, \infty)$,*

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(ii) *in* L^1 *over compact sets (and at points of continuity) if* $\Phi_n(\lambda)=0(\lambda^{-1-\varepsilon})$ *as* $\lambda\to\pm\infty$ *for some* $\varepsilon>0$ *and* $\Phi_n=\hat{\varphi}_n$ *where* φ_n *is continuous,* $\varphi_n\geq 0$, $\int\varphi_n(x)\,dx=1$, *and support* (φ_n) *decreases to* 0,

(iii) *in the sense of distributions, i.e.,* $\int f_n\varphi\to\int f\varphi$, *where*

$$f_n(x) = h_\rho(x)+\int_{-\infty}^{\infty}\frac{e^{i\lambda x}-S_\rho(x,\lambda)}{[Q_0(\lambda)]^2}\Phi_n(\lambda)\,d\sigma(\lambda),$$

if $\Phi_n(\lambda)=0(\lambda^{-1-\varepsilon})$ *as* $\lambda\to\pm\infty$, $\Phi_n(\lambda)\to 1$ *as* $n\to\infty$, *and* $|\Phi_n(\lambda)|\leq 1$.

Proof. Let φ be any continuous function such that $\int f\varphi$ exists, $\varphi(x)=0(x^{-2m-2})$ as $x\to\pm\infty$, and $\Phi(\lambda)=0(\lambda^{-1-\varepsilon})$ as $\lambda\to\pm\infty$ for some $\varepsilon>0$. (This will be the case if Φ is the summability function in (i) or (ii).) We shall show that

$$\text{(8)}\quad \int\overline{f(x)}\varphi(x)\,dx=\int\overline{h_\rho(x)}\varphi(x)\,dx+\int\left\{\int(e^{-i\lambda x}-S_\rho(-x,\ \lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}.$$

First notice that the second integral on the right-hand side exists because it can be written as

$$\int_{-\rho}^{\rho}\left\{\int_{-\infty}^{\infty}(e^{-i\lambda x}-S_\rho(-x,\lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}+\int_{|\lambda|>\rho}\Phi(\lambda)\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}.$$

Since $\Phi(\lambda)=0(\lambda^{-1-\varepsilon})$ the latter integral exists by the corollary to Theorem 3.3; the former integral exists because $\varphi(x)=0(x^{-2m-2})$ and

$$S_\rho(x,\lambda)=\sum_{j=1}^{r}\exp(i\alpha_j x)\sum_{p=0}^{2m_j-1}\frac{[ix(\lambda-\alpha_j)]^p}{p!},\qquad |\lambda|\leq\rho.$$

There is a sequence $\{\psi_n\}\subset C_c^\infty$ such that $\psi_n(x)\to\varphi(x)$ and $|\psi_n(x)|\leq|\varphi(x)|$. The equation (8) holds with φ replaced by ψ_n; therefore by the dominated convergence theorem it holds for φ.

Now set $\varphi(x)=\varphi_n(t+x)$ in (8), where $\Phi_n=\hat{\varphi}_n$ is the summability function in (i) or (ii). Thus

$$\text{(9)}\quad \int(f(x)-h_\rho(x))\varphi_n(t-x)\,dx=\int_{-\rho}^{\rho}\left\{\int_{-\infty}^{\infty}(e^{i\lambda x}-S_\rho(x,\lambda))\varphi_n(t-x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}$$
$$+\int_{|\lambda|>\rho}\Phi_n(\lambda)e^{it\lambda}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}$$

since $\overline{f(x)}=f(-x)$ and $\overline{h_\rho(x)}=h_\rho(-x)$. The left-hand side of (9) converges to $f(t)-h_\rho(t)$ if t is a point of continuity of f, and in case (i)

$$(\varphi_n(x)=\tfrac{1}{2}\pi^{-1/2}n\exp(-n^2x^2/4))$$

the convergence holds almost everywhere [13, p. 31, Theorem 16]. For every value of t the first term on the right-hand side converges to

$$\int_{-\rho}^{\rho}\frac{e^{i\lambda t}-S_\rho(t,\lambda)}{[Q_0(\lambda)]^2}\,d\sigma(\lambda)$$

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by the bounded convergence theorem. Thus if t is a point of continuity of f, or a point of the Lebesgue set in case (i), we have

$$f(t) = h_\rho(t)+\int_{-\rho}^{\rho}\frac{e^{i\lambda t}-S_\rho(t, \lambda)}{[Q_0(\lambda)]^2}\,d\sigma(\lambda)+\lim_{n\to\infty}\int_{|\lambda|>\rho}\Phi_n(\lambda)e^{i\lambda t}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}$$
$$= h_\rho(t)+\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}-S_\rho(t, \lambda)}{[Q_0(\lambda)]^2}\Phi_n(\lambda)\,d\sigma(\lambda)$$

since $\Phi_n(\lambda)\to 1$ uniformly on $[-\rho, \rho]$. Again this equation also holds in case (ii) in the sense of convergence in L^1 over compact sets.

In case (iii) we have, using the assumption that $\Phi_n(\lambda)=0(\lambda^{-1-\varepsilon})$,

$$\int f_n\varphi = \int h_\rho\varphi+\int\left\{\int\frac{e^{i\lambda x}-S_\rho(x, \lambda)}{[Q_0(\lambda)]^2}\Phi_n(\lambda)\,d\sigma(\lambda)\right\}\varphi(x)\,dx$$
$$=\int h_\rho\varphi+\int\left\{\int(e^{i\lambda x}-S_\rho(x, \lambda))\varphi(x)\,dx\right\}\Phi_n(\lambda)\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}$$

for $\varphi\in C_c^\infty$. By the dominated convergence theorem this converges to

$$\int h_\rho\varphi+\int\left\{\int(e^{i\lambda x}-S_\rho(x, \lambda))\varphi(x)\,dx\right\}\frac{d\sigma(\lambda)}{[Q_0(\lambda)]^2}$$

which is equal to $\int f\varphi$.

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