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ON A DUALITY THEOREM OF WAKAMATSU

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Abstract

Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module, with $S = \mathrm{End}({}_RU)$ satisfying the condition that each finitely presented left R-module X with $\mathrm{Ext}^i_R(X, U) = 0$ for any $i \geq 1$ is U-torsionless. If M is a finitely presented left R-module such that $\mathrm{Ext}^i_R(M, U) = 0$ for any $i \geq 0$ with $i \neq n$ (where n is a nonnegative integer), then $\mathrm{Ext}^n_S(\mathrm{Ext}^n_R(M, U), U) \cong M$ and $\mathrm{Ext}^i_S(\mathrm{Ext}^n_R(M, U), U) = 0$ for any $i \geq 0$ with $i \neq n$. A duality is thus induced between the category of finitely presented holonomic left R-modules and the category of finitely presented holonomic right S-modules.

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1. Introduction

For a ring R, we use Mod R (respectively Mod R^{op}) to denote the category of left (respectively right) R-modules, and use mod R (respectively mod R^{op}) to denote the category of finitely presented left (respectively right) R-modules.

We define $\mathrm{gen}^*({}_RR) = \{X \in \mathrm{mod}\ R \mid$ there exists an exact sequence $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to X \to 0$ in mod R, with P_i projective for any $i \geq 0\}$ (see [6]). For a module ${}_RU$ in Mod R (respectively mod R), we use $\mathrm{add}_R U$ to denote the full subcategory of Mod R (respectively mod R) that consists of all modules isomorphic to direct summands of finite sums of copies of ${}_RU$; we also let ${}^{\perp}_RU$ denote the full subcategory of Mod R (respectively mod R) that consists of all ${}_RC$ with $\mathrm{Ext}^i_R({}_RC, {}_RU) = 0$ for any $i \geq 1$. The module ${}_RU$ is called *self-orthogonal* if ${}_RU \in {}^{\perp}_RU$.

Definition 1.1 [6]. A self-orthogonal module ${}_RU$ in $\mathrm{gen}^*({}_RR)$ is called a *generalized tilting module* if there exists an exact sequence

$$0 \to {}_RR \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

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such that: (1) $U_i \in \mathrm{add}_R U$ for any $i \geq 0$; and (2) after applying the functor $\mathrm{Hom}_R(-, U)$, the sequence is still exact.

For a module ${}_RU$ in Mod R (respectively mod R) and a nonnegative integer n, we define $\mathcal{H}_n({}_RU) = \{X \in \mathrm{Mod}\ R$ (respectively mod R) $\mid \mathrm{Ext}^i_R(X, U) = 0$ for any $i \geq 0$ with $i \neq n\}$. A module is called *holonomic* (with respect to ${}_RU$) if it is in $\mathcal{H}_n({}_RU)$ (see [6]). In [6, Proposition 8.1], Wakamatsu proved the following result.

THEOREM 1.2. *Let R be a left noetherian ring, S a right noetherian ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If the injective dimensions of U_S and ${}_RU$ are both finite, then for any nonnegative integer n, the functor* $\mathrm{Ext}^n(-, {}_RU_S)$ *induces a duality $\mathcal{H}_n({}_RU)^{op} \approx \mathcal{H}_n(U_S)$.*

Recall that a bimodule ${}_RU_S$ is called a *faithfully balanced bimodule* if the natural maps $R \to \mathrm{End}(U_S)$ and $S \to \mathrm{End}({}_RU)^{op}$ are isomorphisms. By [6, Corollary 3.2], we have that ${}_RU_S$ is a faithfully balanced and self-orthogonal bimodule with ${}_RU \in \mathrm{gen}^*({}_RR)$ and $U_S \in \mathrm{gen}^*(S_S)$ if and only if ${}_RU$ is a generalized tilting module with $S = \mathrm{End}({}_RU)$, and if and only if U_S is a generalized tilting module with $R = \mathrm{End}(U_S)$. With this observation in mind, we point out that Theorem 1.2 was, in fact, also obtained by Miyashita in [4, Theorem 6.1]. The aim of this paper is to prove the above result in a more general situation. The following theorem is the main result in this paper.

THEOREM 1.3. *Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If both ${}^{\perp}_RU$ and ${}^{\perp}U_S$ have the U-torsionless property, then for any nonnegative integer n, the functor* $\mathrm{Ext}^n(-, {}_RU_S)$ *induces a duality $\mathcal{H}_n({}_RU)^{op} \approx \mathcal{H}_n(U_S)$.*

Recall from [2] that ${}^{\perp}_RU$ (respectively ${}^{\perp}U_S$) is said to have the *U-torsionless property* if each module in ${}^{\perp}_RU$ (respectively ${}^{\perp}U_S$) is U-torsionless. By [3, Theorem 2.2], it is easy to verify that under the assumptions of Theorem 1.3, if the injective dimensions of U_S and ${}_RU$ are both finite, then both ${}^{\perp}_RU$ and ${}^{\perp}U_S$ have the U-torsionless property.

2. Preliminaries

In this section, we give some definitions and collect some elementary facts which will be useful in the rest of the paper.

Let both U and A be in Mod R (respectively Mod S^{op}). We denote either one of $\mathrm{Hom}_R({}_RA, {}_RU)$ and $\mathrm{Hom}_S(A_S, U_S)$ by A^*. For a homomorphism f between R-modules (respectively S^{op}-modules), we put $f^* = \mathrm{Hom}(f, U)$.

Let ${}_RU_S$ be an $(R$–$S)$-bimodule. For A in Mod R (respectively Mod S^{op}), let $\sigma_A : A \to A^{**}$, defined by $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$, be the canonical evaluation homomorphism; A is called *U-torsionless* if σ_A is a monomorphism, and *U-reflexive* if σ_A is an isomorphism. Under the assumption that $R = \mathrm{End}(U_S)$ (respectively $S = \mathrm{End}({}_RU)$), it is easy to see that any projective module in mod R (respectively mod S^{op}) is U-reflexive.

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DEFINITION 2.1 [2]. Let R and S be rings, and let ${}_RU_S$ be an $(R–S)$-bimodule. A full subcategory $\mathcal{X}$ of Mod R is said to have the *U-torsionless property* (respectively the *U-reflexive property*) if each module in $\mathcal{X}$ is U-torsionless (respectively U-reflexive). The notion of a full subcategory $\mathcal{X}$ of Mod S^{op} having the *U-torsionless property* (respectively *U-reflexive property*) can be defined analogously.

A ring R is called a *left coherent ring* if every finitely generated submodule of a finitely presented left R-module is finitely presented. The notion of a *right coherent ring* can be defined analogously (see [5]).

Let ${}_RU_S$ be an $(R–S)$-bimodule. Recall from [1] that a module M in Mod R (respectively mod R) is said to have *generalized Gorenstein dimension zero* (with respect to ${}_RU_S$), denoted by $\text{G-dim}_U(M) = 0$, if the following conditions are satisfied: (1) $M \in {}^{\perp}_RU$ and $\text{Ext}^i_S(M^*, U_S) = 0$ for any $i \geq 1$; and (2) M is U-reflexive. We use $\mathcal{G}_U$ to denote the full subcategory of Mod R (respectively mod R) consisting of the modules with generalized Gorenstein dimension zero. The following result gives some characterizations of ${}^{\perp}_RU$ having the U-torsionless property.

PROPOSITION 2.2. *Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module with $S = \text{End}({}_RU)$. Then the following statements are equivalent.*

(1) *${}^{\perp}_RU$ has the U-torsionless property.*
(2) *${}^{\perp}_RU$ has the U-reflexive property.*
(3) *${}^{\perp}_RU = \mathcal{G}_U$.*

PROOF. This conclusion was proved in [2, Proposition 2.3] in the case where R is a left noetherian ring and S is a right noetherian ring. The argument remains valid in the setting here, so we omit it. □

Let U_S be a module in Mod S^{op}. For a positive integer n, an exact sequence $X_0 \to X_1 \to \cdots \to X_n$ in Mod S^{op} is called *dual exact* (with respect to U_S) if the induced sequence $X_n^* \to \cdots \to X_1^* \to X_0^*$ is also exact.

PROPOSITION 2.3. *Let both U and N be in* Mod S^{op}*, and let n be a positive integer. Then the following statements are equivalent.*

(1) $\text{Ext}^i_S(N, U) = 0$ *for any* $1 \leq i \leq n$.
(2) *Any exact sequence* $0 \to K \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to N \to 0$ *in* Mod S^{op}*, with Q_i in ${}^{\perp}U_S$ for any $0 \leq i \leq n-1$, is dual exact (with respect to U_S).*
(3) *Any exact sequence* $Q_{n+1} \to Q_n \to \cdots \to Q_1 \to Q_0 \to N \to 0$ *in* Mod S^{op}*, with Q_i in ${}^{\perp}U_S$ for any $0 \leq i \leq n+1$, is dual exact (with respect to U_S).*

PROOF. (1) $\Rightarrow$ (2). The case $n = 1$ is clear. Now suppose $n \geq 2$ and that

$$0 \to K \to Q_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \to N \to 0$$

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is an exact sequence in Mod S^{op}, with Q_i in ${}^{\perp}U_S$ for any $0 \leq i \leq n-1$. Then $\text{Ext}_S^1(\text{Im}\, d_i, U) \cong \text{Ext}_S^{i+1}(N, U) = 0$ for any $1 \leq i \leq n-1$. It follows that the induced sequence

$$0 \to N^* \to Q_0^* \xrightarrow{d_1^*} Q_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} Q_{n-1}^* \to K^* \to 0$$

is exact.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Suppose $n = 1$ and that there exists an exact sequence

$$Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \to N \to 0,$$

with Q_i in ${}^{\perp}U_S$ for any $0 \leq i \leq 2$, which is dual exact (with respect to U_S). Put $K = \text{Im}\, d_1$ and assume that $d_1 = \mu\pi$, where $\pi : Q_1 \to K$ is an epimorphism and $\mu : K \to Q_0$ is a monomorphism.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^* & \longrightarrow & Q_0^* & \xrightarrow{\mu^*} & K^* & \longrightarrow & \text{Ext}_S^1(N, U) \longrightarrow 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & N^* & \longrightarrow & Q_0^* & \xrightarrow{d_1^*} & Q_1^* & \xrightarrow{d_2^*} & Q_2^*
\end{array}$$

Since $0 \to K^* \xrightarrow{\pi^*} Q_1^* \xrightarrow{d_2^*} Q_2^*$ is exact, $\text{Im}\, \mu^* \cong \text{Im}(\pi^*\mu^*) \cong \text{Im}\, d_1^* \cong \text{Ker}\, d_2^* \cong \text{Im}\, \pi^* \cong K^*$. So μ^* is an epimorphism and hence $\text{Ext}_S^1(N, U) = 0$. Then, by using induction on n, we obtain our conclusion. □

3. Main results

In this section, R and S are any rings and ${}_RU_S$ is an $(R$–$S)$-bimodule satisfying the conditions that $\text{End}(U_S) = R$ and U_S is self-orthogonal. Later in this section we shall prove Theorem 1.3, but in order to do this, we first need some lemmas.

For a module M in Mod R, we use $l.\text{pd}_R(M)$ to denote the projective dimension of M.

LEMMA 3.1. *Let n be a positive integer and let $M \in \text{gen}^*({}_RR) \cap \mathcal{H}_n({}_RU)$. If $l.\text{pd}_R(M) \leq n$, then $\text{Ext}_S^n(\text{Ext}_R^n(M, U), U) \cong M$ and $\text{Ext}_R^n(M, U) \in \mathcal{H}_n(U_S)$.*

PROOF. Let $M \in \text{gen}^*({}_RR) \cap \mathcal{H}_n({}_RU)$ with $l.\text{pd}_R(M) \leq n$. Suppose that

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

is an exact sequence in mod R such that P_i is projective for any $0 \leq i \leq n$. Then we have an exact sequence

$$0 \to P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to \text{Ext}_R^n(M, U) \to 0 \qquad (1)$$

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with $P_i^* \in \operatorname{add} U_S$ for any $0 \leq i \leq n$. Since $\operatorname{End}(U_S) = R$, P_i is U-reflexive for any $0 \leq i \leq n$. We then get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & & & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{P_n}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{P_{n-1}}} & & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{P_1}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{P_0}} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & [\operatorname{Ext}^n_R(M, U)]^* & \longrightarrow & P_n^{**} & \xrightarrow{d_n^{**}} & P_{n-1}^{**} & \xrightarrow{d_{n-1}^{**}} & \cdots \longrightarrow & P_1^{**} & \xrightarrow{d_1^{**}} & P_0^{**} & \longrightarrow & \operatorname{Ext}^n_S(\operatorname{Ext}^n_R(M, U), U) & \longrightarrow & 0
\end{array}$$

So $[\operatorname{Ext}^n_R(M, U)]^* = 0$ and f is an isomorphism; hence $M \cong \operatorname{Ext}^n_S(\operatorname{Ext}^n_R(M, U), U)$.

From the exactness of the bottom row in the above diagram, we know that the exact sequence

$$P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to \operatorname{Ext}^n_R(M, U) \to 0$$

(which is part of the exact sequence (1)) is dual exact (with respect to U_S). Since U_S is self-orthogonal, $P_i^* \in {}^{\perp}U_S$ for any $0 \leq i \leq n$. It follows from Proposition 2.3 that $\operatorname{Ext}^i_S(\operatorname{Ext}^n_R(M, U), U) = 0$ for any $1 \leq i \leq n-1$. On the other hand, from the exact sequence (1) we get that $\operatorname{Ext}^{n+i}_S(\operatorname{Ext}^n_R(M, U), U) \cong \operatorname{Ext}^i_S(P_0^*, U) = 0$ for any $i \geq 1$, and that $\operatorname{Ext}^n_R(M, U) \in \operatorname{mod} S^{op}$ provided $U_S \in \operatorname{mod} S^{op}$. Consequently, we conclude that $\operatorname{Ext}^n_R(M, U) \in \mathcal{H}_n(U_S)$. □

LEMMA 3.2. *Assume that each module in* $\operatorname{gen}^*({}_R R) \cap {}^{\perp}_R U$ *is* U*-reflexive, and let* n *be a positive integer. If* M *is a module in* $\operatorname{gen}^*({}_R R)$ *satisfying the condition that* $\operatorname{Ext}^{n+i}_R(M, U) = 0$ *for any* $i \geq 1$*, then* $[\operatorname{Ext}^n_R(M, U)]^* = 0$.

PROOF. Suppose that $M \in \operatorname{gen}^*({}_R R)$ with $\operatorname{Ext}^{n+i}_R(M, U) = 0$ for any $i \geq 1$, and that

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

is an exact sequence in $\operatorname{mod} R$ such that P_i is projective for any $i \geq 0$. Then $\operatorname{Ext}^1_R(\operatorname{Coker} d_n, U) \cong \operatorname{Ext}^n_R(M, U)$ and $\operatorname{Ext}^i_R(\operatorname{Im} d_n, U) \cong \operatorname{Ext}^{n+i}_R(M, U) = 0$ for any $i \geq 1$ (that is, $\operatorname{Im} d_n \in {}^{\perp}_R U$). It is clear that $\operatorname{Im} d_n \in \operatorname{gen}^*({}_R R)$; so $\operatorname{Im} d_n \in \operatorname{gen}^*({}_R R) \cap {}^{\perp}_R U$ and hence $\operatorname{Im} d_n$ is U-reflexive by assumption.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & & & \operatorname{Im} d_n & \longrightarrow & P_{n-1} & \longrightarrow & \operatorname{Coker} d_n \longrightarrow 0 \\
 & & & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{\operatorname{Im} d_n}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{P_{n-1}}} & & \\
0 & \longrightarrow & [\operatorname{Ext}^1_R(\operatorname{Coker} d_n, U)]^* & \longrightarrow & (\operatorname{Im} d_n)^{**} & \longrightarrow & P_{n-1}^{**} & &
\end{array}$$

Therefore $[\operatorname{Ext}^1_R(\operatorname{Coker} d_n, U)]^* = 0$ and $[\operatorname{Ext}^n_R(M, U)]^* = 0$. □

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LEMMA 3.3. *Assume that* ${}_R^{\perp}U = \mathcal{G}_U$, *and let* n *be a positive integer. If* $M \in \mathrm{gen}^*({}_RR) \cap \mathcal{H}_n({}_RU)$, *then* $\mathrm{Ext}_S^n(\mathrm{Ext}_R^n(M, U), U) \cong M$ *and* $\mathrm{Ext}_S^i(\mathrm{Ext}_R^n(M, U), U) = 0$ *for any* $i \geq 0$ *with* $i \neq n$.

PROOF. If $l.\mathrm{pd}_R(M) \leq n$, then the conclusion follows from Lemma 3.1. Now suppose that $l.\mathrm{pd}_R(M) \geq n + 1$ and that

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

is an exact sequence in mod R, with P_i projective for any $0 \leq i \leq n$. Since $M \in \mathcal{H}_n({}_RU)$, we get a complex which is exact except at the index n:

$$0 \to P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} P_{n-1}^* \xrightarrow{d_n^*} P_n^* \xrightarrow{d_{n+1}^*} \cdots$$

with $P_i^* \in \mathrm{add}\, U_S$ for any $i \geq 0$. Thus, $\mathrm{Ext}_R^n(M, U) \cong \mathrm{Ker}\, d_{n+1}^*/\mathrm{Im}\, d_n^*$. Put $B = P_n^*/\mathrm{Im}\, d_n^*$ and $Y = \mathrm{Im}\, d_{n+1}^* (\cong P_n^*/\mathrm{Ker}\, d_{n+1}^*)$. Then we get an exact sequence

$$0 \to \mathrm{Ext}_R^n(M, U) \to B \to Y \to 0. \tag{2}$$

Because $M \in \mathrm{gen}^*({}_RR) \cap \mathcal{H}_n({}_RU)$, both $\mathrm{Im}\, d_n$ and $\mathrm{Im}\, d_{n+1}$ are in ${}_R^{\perp}U$. It follows easily that $(\mathrm{Im}\, d_{n+1})^* \cong \mathrm{Im}\, d_{n+1}^* (= Y)$. By assumption, ${}_R^{\perp}U = \mathcal{G}_U$, so $\mathrm{Im}\, d_{n+1} \in \mathcal{G}_U$ and $\mathrm{Ext}_S^i(Y, U) = 0$ for any $i \geq 1$. From the exact sequence (2), we obtain the isomorphism

$$\mathrm{Ext}_S^i(B, U) \cong \mathrm{Ext}_S^i(\mathrm{Ext}_R^n(M, U), U)$$

for any $i \geq 1$.

On the other hand, we have an exact sequence

$$0 \to P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to B \to 0.$$

Using an argument similar to that in the proof of Lemma 3.1, we deduce that $M \cong \mathrm{Ext}_S^n(B, U)$ and $\mathrm{Ext}_S^i(B, U) = 0$ for any $i \geq 1$ with $i \neq n$. Thus $M \cong \mathrm{Ext}_S^n(\mathrm{Ext}_R^n(M, U), U)$ and $\mathrm{Ext}_S^i(\mathrm{Ext}_R^n(M, U), U) = 0$ for any $i \geq 1$ with $i \neq n$. In addition, $[\mathrm{Ext}_R^n(M, U)]^* = 0$ by Lemma 3.2. The proof is therefore complete. □

LEMMA 3.4. *Assume that* ${}_R^{\perp}U = \mathcal{G}_U$, *and let* n *be a nonnegative integer. If* $M \in \mathrm{gen}^*({}_RR) \cap \mathcal{H}_n({}_RU)$, *then* $\mathrm{Ext}_S^n(\mathrm{Ext}_R^n(M, U), U) \cong M$ *and* $\mathrm{Ext}_S^i(\mathrm{Ext}_R^n(M, U), U) = 0$ *for any* $i \geq 0$ *with* $i \neq n$.

PROOF. Since ${}_R^{\perp}U = \mathcal{G}_U$ by assumption, the case for $n = 0$ is trivial. The conclusion for $n \geq 1$ follows from Lemma 3.3. □

The following theorem is the main result of this section.

THEOREM 3.5. *Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If ${}_R^{\perp}U$ has the U-torsionless property and $M \in \mathcal{H}_n({}_RU)$ for some $n \geq 0$, then $\mathrm{Ext}^n_S(\mathrm{Ext}^n_R(M, U), U) \cong M$ and $\mathrm{Ext}^n_R(M, U) \in \mathcal{H}_n(U_S)$.*

PROOF. Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. Then $\mathrm{gen}^*({}_RR) = \mathrm{mod}\ R$ and $\mathrm{gen}^*(S_S) = \mathrm{mod}\ S^{op}$. By [6, Corollary 3.2], ${}_RU_S$ is faithfully balanced and self-orthogonal, with ${}_RU \in \mathrm{mod}\ R$ and $U_S \in \mathrm{mod}\ S^{op}$. If ${}_R^{\perp}U$ has the U-torsionless property, then ${}_R^{\perp}U = \mathcal{G}_U$ by Proposition 2.2. Therefore, our result follows from Lemma 3.4. □

Theorem 1.3 now follows immediately from Theorem 3.5 and its dual result.

Let A be a left R-module; A is called FP-*injective* if $\mathrm{Ext}^1_R(X, A) = 0$ for any finitely presented left R-module X. The *left* FP-*injective dimension* of A, denoted by $l.\mathrm{FP}\text{-}\mathrm{id}_R(A)$, is defined as $\inf\{n \geq 0 \mid \mathrm{Ext}^{n+1}_R(X, A) = 0$ for any finitely presented left R-module $X\}$. The notion of *right* FP-*injective dimension* of a right R-module B, denoted by $r.\mathrm{FP}\text{-}\mathrm{id}_R(B)$, is defined analogously (see [5]).

Let N be in Mod S^{op} and suppose that

$$0 \to N \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} I_i \xrightarrow{\delta_{i+1}} \cdots$$

is an exact sequence in Mod S^{op}, with I_i FP-injective for any $i \geq 0$. Such an exact sequence is called an FP-*injective resolution* of N. Recall from [3] that an FP-injective resolution is called *ultimately closed* if there is a positive integer n such that $\mathrm{Im}\ \delta_n = \bigoplus_{j=0}^m W_j$, where each W_j is a direct summand of $\mathrm{Im}\ \delta_{i_j}$ with $i_j < n$. It is easy to see that $r.\mathrm{FP}\text{-}\mathrm{id}_S(U) \leq n$ if and only if there exists an exact sequence $0 \to U_S \to E_0 \to E_1 \to \cdots \to E_n \to 0$ in Mod S^{op} with E_i FP-injective for any $0 \leq i \leq n$. It is clear that such an FP-injective resolution is ultimately closed.

Assume that R is a left coherent ring and that $U_S \in \mathrm{mod}\ S^{op}$. By [3, Theorem 2.4], if U_S has an ultimately closed FP-injective resolution (in particular, if $r.\mathrm{FP}\text{-}\mathrm{id}_S(U) < \infty$), then any module in ${}_R^{\perp}U \cap \mathrm{mod}\ R$ is U-reflexive. The following result is therefore an immediate consequence of Theorem 1.3.

COROLLARY 3.6. *Let R be a left coherent ring, S a right coherent ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If both ${}_RU$ and U_S have ultimately closed FP-injective resolutions (in particular, if both $r.\mathrm{FP}\text{-}\mathrm{id}_S(U)$ and $l.\mathrm{FP}\text{-}\mathrm{id}_R(U)$ are finite), then for any nonnegative integer n, the functor $\mathrm{Ext}^n(-, {}_RU_S)$ induces a duality $\mathcal{H}_n({}_RU)^{op} \approx \mathcal{H}_n(U_S)$.*

Notice that a left (respectively right) noetherian ring is a left (respectively right) coherent ring, and that the notions of finitely presented modules and FP-injective modules coincide with those of finitely generated modules and injective modules over noetherian rings; thus Theorem 1.2, due to Wakamatsu and Miyashita, is a special case of Corollary 3.6.

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