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ON THE RANGE OF AN INTEGRAL TRANSFORMATION

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ABSTRACT. The range of the $\mathcal{Y}_\nu$ transformation, defined by

$$(\mathcal{Y}_\nu f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} Y_\nu(xt) f(t)\, dt,$$

is characterized on the spaces $\mathcal{L}_{\mu,p}$ defined by the norm

$$\|f\|_{\mu,p} = \Big(\int_0^\infty |x^\mu f(x)|^p \frac{dx}{x} \Big)^{\frac{1}{p}} < \infty, \quad 1 < p < \infty,$$

for $\mu = \frac{1}{2} - \nu$.

1. **Introduction.** Denote by $\mathbf{C}_0$ the collection of complex-valued functions continuous and compactly supported in $(0, \infty)$ and by $[X, Y]$ the collection of bounded linear transformations from the Banach space X to the Banach space Y, $[X, X]$ being abbreviated to $[X]$. Also, if $1 < p < \infty$, let $\gamma(p) = \max(1/p, 1/p')$, where $p' = p/(p-1)$. The integral transformation we will study in this article is the $\mathcal{Y}_\nu$ transformation defined for $f \in \mathbf{C}_0$ by

$$(\mathcal{Y}_\nu f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} Y_\nu(xt) f(t)\, dt, \tag{1}$$

where Y_ν is the Bessel function of the second kind. We have studied this transformation earlier on the spaces $\mathcal{L}_{\mu,p}$ defined to consist of those complex-valued Lebesgue measurable functions on $(0, \infty)$ such that $\|f\|_{\mu,p} < \infty$, where

$$\|f\|_{\mu,p} = \Big(\int_0^\infty |x^\mu f(x)|^p \frac{dx}{x} \Big)^{\frac{1}{p}}, \quad 1 < p < \infty. \tag{2}$$

In [6] it was shown that if $1 < p < \infty$ and $\gamma(p) \le \mu < \frac{3}{2} - |\nu|$, then $\mathcal{Y}_\nu \in [\mathcal{L}_{\mu,p}, \mathcal{L}_{1-\mu,q}]$ for all $q \ge p$ such that $q' \ge 1/\mu$. It was also shown there that, except when $\mu = \frac{1}{2} - \nu$, the range of $\mathcal{Y}_\nu$ on such $\mathcal{L}_{\mu,p}$ was the same as the range of the Hankel transformation H_ν on $\mathcal{L}_{\mu,p}$, that is $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p}) = H_\nu(\mathcal{L}_{\mu,p})$, where H_ν is defined for $\nu > -1$ and $f \in \mathbf{C}_0$ by

$$(H_\nu f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt) f(t)\, dt, \tag{3}$$

and J_ν is the Bessel function of the first kind. Since $H_\nu(\mathcal{L}_{\mu,p})$ was given a fairly simple characterization in [5, Theorem 2], this characterization also applies to $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p})$ except when $\mu = \frac{1}{2} - \nu$. When $\mu = \frac{1}{2} - \nu$, in which case $-\frac{1}{2} < \nu \le 0$, it is known that $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p}) \subseteq$

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$H_\nu(\mathcal{L}_{\mu,p})$, but it was shown in [1, Theorem 5.1 and Corollary] that $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p}) \neq H_\nu(\mathcal{L}_{\mu,p})$, and $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p})$ has not been characterized except when $\nu = 0$, in which case $p = 2$. In this article we shall characterize $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p})$ when $\mu = \frac{1}{2} - \nu$. A similar program was carried out for the $\mathcal{H}_\nu$ or Struve transformation on $\mathcal{L}_{-(\nu+\frac{1}{2}),p}$ in [4, Section 3] and for the extended Hankel transformation $H_\nu, \nu < -1$ when $\mu = \frac{3}{2} - \nu - 2l$ in [3, Section 6].

In Section 2 we prove a preliminary lemma, and in section three we give the characterization of $\mathcal{Y}_\nu(\mathcal{L}_{\mu,p})$ when $\mu = \frac{1}{2} - \nu$.

Our notation will be that of [6], and we shall use the operator M_α whose definition and properties are given in [6, Section 1] and the Mellin transformation $\mathcal{M}$ whose properties are also given there. Also, we shall use the operators P_-, Q_+ and Q_- which are defined in [2, Sections 2 and 3] and whose properties are given there. One further notation that we use is $\int^{\to\infty}$ which is explained in [7, Section 1.7].

2. **A preliminary lemma.** In [6, Theorem 4.2] it was shown that on $\mathcal{L}_{\mu,p}$, where $1 < p < \infty$, $\gamma(p) \leq \mu < \frac{3}{2} - |\nu|$,

$$\mathcal{Y}_\nu = -M_{\frac{1}{2}-\nu} H_- M_{\nu-\frac{1}{2}} H_\nu,$$

where H_- is the odd Hilbert transformation, for whose theory see [6, Section 3]. However, for our work here we shall need another representation of $\mathcal{Y}_\nu$, given by the following lemma.

LEMMA 1. *On $\mathcal{L}_{\frac{1}{2}-\nu,p}$, where $1 < p < \infty$, $\gamma(p) \leq \frac{1}{2} - \nu < \frac{3}{2} - |\nu|$, $\nu \neq 0$,*

$$\mathcal{Y}_\nu = -M_{\frac{1}{2}-\nu} P_- M_{\nu-\frac{1}{2}} H_\nu M_{\nu-\frac{1}{2}} Q_+ M_{\frac{1}{2}-\nu}. \tag{4}$$

PROOF. From [6, Theorem 4.1], $\mathcal{Y}_\nu \in [\mathcal{L}_{\frac{1}{2}-\nu,p}, \mathcal{L}_{\frac{1}{2}+\nu,p}]$. From the properties of the operators M_α, Q_+, H_ν, and P_- in the references cited in the Introduction, $M_{\frac{1}{2}-\nu}$ maps $\mathcal{L}_{\frac{1}{2}-\nu,p}$ boundedly onto $\mathcal{L}_{0,p}$, which is mapped boundedly into into itself by Q_+, and $M_{\nu-\frac{1}{2}}$ maps this boundedly onto $\mathcal{L}_{\frac{1}{2}-\nu,p}$; H_ν now maps this boundedly onto $\mathcal{L}_{\frac{1}{2}+\nu,p}$ which is mapped boundedly onto $\mathcal{L}_{1,p}$ by $M_{\nu-\frac{1}{2}}$, and this last space is mapped boundedly into itself by P_-; finally $M_{\frac{1}{2}-\nu}$ maps $\mathcal{L}_{1,p}$ onto $\mathcal{L}_{\frac{1}{2}+\nu,p}$ boundedly. Thus both sides of (4) are in $[\mathcal{L}_{\frac{1}{2}-\nu,p}, \mathcal{L}_{\frac{1}{2}+\nu,p}]$, and thus since obviously $\mathbf{C}_0$ is dense in $\mathcal{L}_{\mu,p}$, and since $\frac{1}{2} = \gamma(2) \leq \gamma(p) \leq \frac{1}{2} - \nu$, it suffices to prove (4) for $p = 2$. For this we use the Mellin transformation $\mathcal{M}$, using [6, (1.10),(2.3) and (4.4)] and [2, Theorems 2.2 and 3.1]. Thus, if $f \in \mathcal{L}_{\frac{1}{2}-\nu,2}$ and $\operatorname{Re} s = \frac{1}{2} - \nu$,

$$\begin{aligned}
&(\mathcal{M} M_{\frac{1}{2}-\nu} P_- M_{\nu-\frac{1}{2}} H_\nu M_{\nu-\frac{1}{2}} Q_+ M_{\frac{1}{2}-\nu} f)(s) \\
&\quad = (\mathcal{M} P_- M_{\nu-\frac{1}{2}} H_\nu M_{\nu-\frac{1}{2}} Q_+ M_{\frac{1}{2}-\nu} f)\Big(s + \frac{1}{2} - \nu\Big) \\
&\quad = \frac{s - \frac{1}{2} - \nu}{s + \frac{1}{2} - \nu} (\mathcal{M} M_{\nu-\frac{1}{2}} H_\nu M_{\nu-\frac{1}{2}} Q_+ M_{\frac{1}{2}-\nu} f)\Big(s + \frac{1}{2} - \nu\Big) \\
&\quad = \frac{s - \frac{1}{2} - \nu}{s + \frac{1}{2} - \nu} (\mathcal{M} H_\nu M_{\nu-\frac{1}{2}} Q_+ M_{\frac{1}{2}-\nu} f)(s)
\end{aligned}$$

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$$= \frac{s-\frac{1}{2}-\nu}{s+\frac{1}{2}-\nu} m_\nu(s)(\mathcal{M}M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f)(1-s)$$
$$= \frac{s-\frac{1}{2}-\nu}{s+\frac{1}{2}-\nu} m_\nu(s)(\mathcal{M}Q_+M_{\frac{1}{2}-\nu}f)\Big(\frac{1}{2}+\nu-s\Big)$$
$$= \frac{s-\frac{1}{2}-\nu}{s+\frac{1}{2}-\nu} m_\nu(s)\frac{-\frac{1}{2}+\nu-s}{\frac{1}{2}+\nu-s}\tan\frac{\pi(\frac{1}{2}+\nu-s)}{2}(\mathcal{M}M_{\frac{1}{2}-\nu}f)\Big(\frac{1}{2}+\nu-s\Big)$$
$$= m_\nu(s)\cot\frac{\pi(s+\frac{1}{2}-\nu)}{2}(\mathcal{M}f)(1-s) = -(\mathcal{M}\mathcal{Y}_\nu f)(s),$$

where $m_\nu(s)$ is the multiplier associated with the Hankel transformation of order ν, that is

$$m_\nu(s) = 2^{s-\frac{1}{2}}\frac{\Gamma\Big(\frac{1}{2}(\nu+s+\frac{1}{2})\Big)}{\Gamma\Big(\frac{1}{2}(\nu-s+\frac{3}{2})\Big)},$$

and the result follows.

3. **Characterization of the range.** The following theorem gives the characterization of $\mathcal{Y}_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$.

THEOREM 2. *A function* $g \in \mathcal{Y}_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, *where* $1 < p < \infty$, $\gamma(p) \le \frac{1}{2}-\nu < \frac{3}{2}-|\nu|$ *if and only if:*

(a) $g \in H_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$;

(b) $\int_1^{\to\infty} t^{\nu-\frac{1}{2}} g(t)\,dt$ *converges;*

(c) $\phi \in H_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, *where*

$$\phi(x) = x^{-\frac{1}{2}-\nu}\int_x^{\to\infty} t^{\nu-\frac{1}{2}}g(t)\,dt, x > 0.$$

PROOF. Without loss of generality, we may suppose $\nu \ne 0$; for the case $\nu = 0$ was dealt with in [1, Theorem 5.2], and the conditons given there are the same as those of this theorem if one notes that when $\nu = 0$, $p = 2$, that $\mathcal{L}_{\frac{1}{2},2} = L_2(0,\infty)$, and that $H_0\big(L_2(0,\infty)\big) = L_2(0,\infty)$.

Suppose then that $\nu \ne 0$ and that $g \in \mathcal{Y}_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, say $g = \mathcal{Y}_\nu f$ where $f \in \mathcal{L}_{\frac{1}{2}-\nu,p}$. Then (a) follows from [6, Theorem 4.1] and (b) follows from [1, Theorem 5.1]. From Lemma 1, $g = -M_{\frac{1}{2}-\nu}P_-M_{\nu-\frac{1}{2}}H_\nu M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f = M_{\frac{1}{2}-\nu}P_-M_{\nu-\frac{1}{2}}H_\nu\psi$, where $\psi = -M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f$. Hence, since, as was shown in the proof of Lemma 1, $M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu} \in [\mathcal{L}_{\frac{1}{2}-\nu,p}]$, $\psi \in \mathcal{L}_{\frac{1}{2}-\nu,p}$. But then $M_{\frac{1}{2}-\nu}(P_-)^{-1}M_{\nu-\frac{1}{2}}g = H_\nu\psi$. Since $g \in H_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, $g \in \mathcal{L}_{\frac{1}{2}+\nu,p}$ and thus $M_{\nu-\frac{1}{2}}g \in \mathcal{L}_{0,p}$. Hence, from [2, Theorem 2.4],

$$(M_{\frac{1}{2}-\nu}(P_-)^{-1}M_{\nu-\frac{1}{2}}g)(x) = x^{\frac{1}{2}-\nu}\Big(x^{-1}\int_x^{\to\infty} t^{\nu-\frac{1}{2}}g(t)\,dt + x^{\nu-\frac{1}{2}}g(x)\Big)$$
$$= \phi(x) + g(x).$$

Hence $\phi = H_\nu\psi - g$, and since $g \in H_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, $\phi \in H_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$.

Conversely, suppose g satisfies (a), (b) and (c). Then ψ_1 exists so that $g = H_\nu\psi_1$. But from [2, Theorem 3.1], since on $\mathcal{L}_{0,p}$, $Q_+Q_- = Q_-Q_+ = I$, Q_+ maps $\mathcal{L}_{0,p}$ one-to-one onto itself, and hence $f_1 \in \mathcal{L}_{\frac{1}{2}-\nu,p}$ exists so that $\psi_1 = M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f_1$, and thus $g = H_\nu M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f_1$. Similarly $f_2 \in \mathcal{L}_{\frac{1}{2}-\nu,p}$ exists so that $\phi = H_\nu M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f_2$. From Lemma 1,

$$\mathcal{Y}_\nu f_1 = -M_{\frac{1}{2}-\nu}P_-M_{\nu-\frac{1}{2}}H_\nu M_{\nu-\frac{1}{2}}Q_+M_{\frac{1}{2}-\nu}f_1 = -M_{\frac{1}{2}-\nu}P_-M_{\nu-\frac{1}{2}}g,$$

or, using the definition of P_- in [2, Section 2]

$$(\mathcal{Y}_\nu f_1)(x) = -x^{\frac{1}{2}-\nu}\Big(x^{\nu-\frac{1}{2}}g(x) - \int_x^\infty t^{\nu-\frac{3}{2}}g(t)\,dt\Big) = -g(x) + x^{\frac{1}{2}-\nu}\int_x^\infty t^{\nu-\frac{3}{2}}g(t)\,dt.$$

Similarly

$$\begin{aligned}(\mathcal{Y}_\nu f_2)(x) &= -\phi(x) + x^{\frac{1}{2}-\nu}\int_x^\infty t^{\nu-\frac{3}{2}}\phi(t)\,dt\\ &= -x^{-\frac{1}{2}-\nu}\int_x^{\to\infty} t^{\nu-\frac{1}{2}}g(t)\,dt + x^{\frac{1}{2}-\nu}\int_x^\infty t^{-2}dt\int_t^{\to\infty} u^{\nu-\frac{1}{2}}g(u)\,du.\end{aligned}$$

Thus, integrating by parts,

$$\begin{aligned}(\mathcal{Y}_\nu f_2)(x) &= -x^{-\frac{1}{2}-\nu}\int_x^{\to\infty} t^{\nu-\frac{1}{2}}g(t)\,dt\\ &\quad + x^{\frac{1}{2}-\nu}\Big(-t^{-1}\int_t^{\to\infty} u^{\nu-\frac{1}{2}}g(u)\,du\,\Big|_x^\infty - \int_x^\infty t^{\nu-\frac{3}{2}}g(t)\,dt\Big)\\ &= -x^{\frac{1}{2}-\nu}\int_x^\infty t^{\nu-\frac{3}{2}}g(t)\,dt = -g(x) - (\mathcal{Y}_\nu f_1)(x),\end{aligned}$$

and thus,

$$g(x) = -(\mathcal{Y}_\nu f_1)(x) - (\mathcal{Y}_\nu f_2)(x) = (\mathcal{Y}_\nu f)(x)$$

where $f = -f_1 - f_2$. Clearly $f \in \mathcal{L}_{\frac{1}{2}-\nu,p}$, and thus $g \in \mathcal{Y}_\nu(\mathcal{L}_{\frac{1}{2}-\nu,p})$, and the theorem is proved.

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