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ALL NON-ARCHIMEDEAN NORMS ON $K[X_1, \ldots, X_r]$

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Abstract. If K is a field with a non-trivial non-Archimedean absolute value (multiplicative norm) $| \, |$, we describe all non-Archimedean K-algebra norms on the polynomial algebra $K[X_1, \ldots, X_r]$ which extend $| \, |$.

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1. Introduction. Let K be a field with a non-trivial non-Archimedean absolute value (multiplicative norm) $| \, |$. In this paper, we study K-algebra non-Archimedean norms on $K[X_1, \ldots, X_r]$ which extend $| \, |$. Some problems connected with the norms on p-adic vector spaces were solved by I. S. Cohen [**5**] and A. F. Monna [**8**], and then O. Goldmann and N. Iwahori were concerned in [**6**] with the intrinsic structure that is carried by the set of all norms on a given finite dimensional vector space over a locally compact field. When $r = 1$, the case of K-algebra non-Archimedean norms on $K[X]$ which are multiplicative and extend $| \, |$ has been treated in [1–3]. In Section 2 below we consider generalizations of the Gauss valuation. We investigate the case when a K-vector space norm is a K-algebra norm and we also address the question of when two norms are equivalent. In Section 3 we then discuss possible types of norms on $K[X_1, \ldots, X_r]$ which extend a given non-trivial non-Archimedean absolute value on K. The completion of $K[X_1, \ldots, X_r]$ with respect to a non-Archimedean Gauss norm is given in Section 4.

There are many applications of non-Archimedean multiplicative norms on $K[X_1, \ldots, X_r]$ in algebraic geometry where a basic tool is to describe all the absolute values on $K(X_1, \ldots, X_r)$ which extend $| \, |$. In [**7**] F.-V. Kuhlmann determined which value groups and which residue fields can possibly occur in this case. In the case $r = 1$ the r.t. extensions $| \, |_L$ of $| \, |$ to $L = K(X)$ have been considered by M. Nagata [**9**], who

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conjectured that $L_{|\,|_L}$ is a simple transcendental extension of a finite algebraic extension of $K_{|\,|}$. This problem has been affirmatively solved (see for example [**1**]). Some results on the corresponding problem for $K(X_1, \dots, X_r)$ are given in Section 5.

2. Gauss norms on $K[X_1, \dots, X_r]$. Let K be a field with a non-trivial non-Archimedean absolute value (multiplicative norm) $|\ |$, i.e. $|\ | : K \to [0, \infty)$ such that for all $\alpha, \beta \in K$

A1. $|\alpha| = 0 \Leftrightarrow \alpha = 0$;

A2. $|\alpha\beta| = |\alpha|\,|\beta|$;

A3. $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$;

A4. there exists $\gamma \in K$ different from zero such that $|\gamma| \neq 1$.

Then $(K, |\ |)$ is called a *valued field.*

In what follows we work with the polynomial algebra $K[X_1, \dots, X_r]$, and study the K-algebra norms $||\ || : K[X_1, ..., X_r] \to [0, \infty)$ which extend $|\ |$, i.e. $||\ ||$ satisfies, for all $P, Q \in K[X_1, \dots, X_r]$, the conditions A1, A3 and for all $\alpha \in K$ and $P, Q \in K[X_1, \dots, X_r]$

N1. $||\alpha P|| = |\alpha|\,||P||$;

N2. $||PQ|| \leq ||P||||Q||$;

N3. $||\alpha|| = |\alpha|$.

If $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, we put $N(\mathbf{n}) = n_1 + \cdots + n_r$. We order the elements of $\mathbb{N}^r$ in the following manner: $\mathbf{i} < \mathbf{j}$ if either $N(\mathbf{i}) < N(\mathbf{j})$ or $N(\mathbf{i}) = N(\mathbf{j})$ and $\mathbf{i}$ is less than $\mathbf{j}$ with respect to the lexicographical order. Hence it follows that for each $\mathbf{j}$ there are only a finite number of $\mathbf{i}$ such that $\mathbf{i} \leq \mathbf{j}$. For simplicity, for any $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$, we denote $\mathbf{X}^{\mathbf{m}} = X_1^{m_1} \cdots X_r^{m_r}$ and $a_{\mathbf{m}} = a_{m_1,\dots,m_r}$. We also denote $\mathbf{X} = (X_1, \dots, X_r)$.

If

$$P = \sum_{\mathbf{j} \leq \mathbf{n}} a_{\mathbf{j}} \mathbf{X}^{\mathbf{j}} \in K[\mathbf{X}], \tag{1}$$

denote

$$E(P) = \{\mathbf{j} \in \mathbb{N}^r : \mathbf{j} \leq \mathbf{n}, a_{\mathbf{j}} \neq 0\}$$

and $\mathbf{d}(P) = \mathbf{n}$ is the greatest element of $E(P)$ with respect to the lexicographical order. If $a_{\mathbf{d}(P)} = 1$ the polynomial P is called *monic*.

Let $(K, |\ |)$ be a valued field as above and $||\ ||$ a K-algebra norm on $K[\mathbf{X}]$ which extends $|\ |$. In what follows we define a non-Archimedean norm on the polynomial algebra $K[\mathbf{X}]$ which is a generalization of the Gauss valuation.

We start with the following simple lemma.

LEMMA 1. *Suppose that K is a field and $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{N}^r}$ a sequence of polynomials from $K[\mathbf{X}]$ such that, for every $\mathbf{j}$, $\mathbf{d}(P_{\mathbf{j}}) = \mathbf{j}$ and ordered with respect to the order defined on $\mathbb{N}^r$. Then every $Q \in K[\mathbf{X}]$ can be represented uniquely in the form*

$$Q = \sum_{\mathbf{j} \leq \mathbf{d}(Q)} b_{\mathbf{j}} P_{\mathbf{j}}, \tag{2}$$

where $b_{\mathbf{j}} \in K$.

Proof. If $Q = \sum_{\mathbf{j}\in E(Q)} c_{\mathbf{j}}\mathbf{X}^{\mathbf{j}}$, $P_{\mathbf{d}(Q)} = \sum_{\mathbf{j}\in E(P_{\mathbf{d}(Q)})} a_{\mathbf{j}}\mathbf{X}^{\mathbf{j}}$, then $Q = c_{\mathbf{d}(Q)}a^{-1}_{\mathbf{d}(Q)}P_{\mathbf{d}(Q)} + Q_{\mathbf{i}}$, where $\mathbf{i} = \mathbf{d}(Q_{\mathbf{i}})$ and $\mathbf{i} < \mathbf{d}(Q)$. By putting $b_{\mathbf{d}(Q)} = c_{\mathbf{d}(Q)}a^{-1}_{\mathbf{d}(Q)}$ the statement follows easily by induction with respect to $\mathbf{d}(Q)$. □

We denote

$$E_{\mathcal{F}}(Q) = \{\mathbf{j} \in \mathbb{N}^r : b_{\mathbf{j}} \neq 0\ ,\ \text{in (2)}\}.$$

Suppose that $(K, |\ |)$ is a valued field, $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ a sequence of polynomials from $K[\mathbf{X}]$ such that, for every $\mathbf{j}$, $\mathbf{d}(P_{\mathbf{j}}) = \mathbf{j}$, ordered with respect to the order defined on $\mathbb{N}^r$ and $\mathcal{N} = \{\delta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ a sequence of positive real numbers such that $\delta_{(0,0,\dots0)} = 1$. We call $\mathcal{F}$ and $\mathcal{N}$ *admissible sequences of polynomials and positive numbers*, respectively.

For every $Q \in K[\mathbf{X}]$ written in the form (2) we define

$$\|Q\|_{\mathcal{F},\mathcal{N}} = \max_{\mathbf{j}\leq\mathbf{d}(Q)} \{|b_{\mathbf{j}}|\delta_{\mathbf{j}}\}, \tag{3}$$

with $\mathbf{j} \in E_{\mathcal{F}}(Q)$. If $P_{\mathbf{s}}, P_{\mathbf{t}} \in \mathcal{F}$, then by Lemma 1

$$P_{\mathbf{s}}P_{\mathbf{t}} = \sum_{\mathbf{j}\leq\mathbf{s}+\mathbf{t}} \gamma_{\mathbf{j}}(\mathbf{s},\mathbf{t})P_{\mathbf{j}}, \quad \gamma_{\mathbf{j}}(\mathbf{s},\mathbf{t}) \in K, \tag{4}$$

where $\gamma_{\mathbf{j}}(\mathbf{s},\mathbf{t}) = \gamma_{\mathbf{j}}(\mathbf{t},\mathbf{s})$, for every $\mathbf{j}$. Then we set

$$\rho_{\mathbf{s},\mathbf{t}} = \max_{\mathbf{j}\leq\mathbf{s}+\mathbf{t}}\{|\gamma_{\mathbf{j}}(\mathbf{s},\mathbf{t})|\delta_{\mathbf{j}}\}. \tag{5}$$

PROPOSITION 1. *Suppose that $(K, |\ |)$ is a valued field, $\mathcal{F}$ and $\mathcal{N}$ admissible sequence of polynomials and real numbers, respectively. Then $\|\ \|_{\mathcal{F},\mathcal{N}}$, defined by* (3) *is a K-vector space non-Archimedean norm on $K[\mathbf{X}]$ which extends $|\ |$. Moreover $\|\ \|_{\mathcal{F},\mathcal{N}}$, is a K-algebra norm on $K[\mathbf{X}]$ if and only if*

$$\rho_{\mathbf{s},\mathbf{t}} \leq \delta_{\mathbf{s}}\delta_{\mathbf{t}}, \tag{6}$$

for every $\mathbf{s}, \mathbf{t}$.

Proof. The first statement is easily verified. For the second part we consider $P, Q \in K[\mathbf{X}]$, where $P = \sum_{\mathbf{i}\leq\mathbf{d}(P)} a_{\mathbf{i}}P_{\mathbf{i}}$ and Q is given by (2). Then, by (4),

$$\begin{aligned} PQ &= \sum_{\mathbf{u}\leq\mathbf{d}(PQ)}\left(\sum_{\mathbf{v}+\mathbf{w}=\mathbf{u}} a_{\mathbf{v}}b_{\mathbf{w}}P_{\mathbf{v}}P_{\mathbf{w}}\right) = \sum_{\mathbf{u}\leq\mathbf{d}(PQ)}\left(\sum_{\mathbf{v}+\mathbf{w}=\mathbf{u}} a_{\mathbf{v}}b_{\mathbf{w}}\left(\sum_{\mathbf{j}\leq\mathbf{u}}\gamma_{\mathbf{j}}(\mathbf{v},\mathbf{w})P_{\mathbf{j}}\right)\right) \\ &= \sum_{\mathbf{u}\leq\mathbf{d}(PQ)}\left(\sum_{\mathbf{j}\leq\mathbf{u}}\left(\sum_{\mathbf{v}+\mathbf{w}=\mathbf{u}} a_{\mathbf{v}}b_{\mathbf{w}}\gamma_{\mathbf{j}}(\mathbf{v},\mathbf{w})\right)P_{\mathbf{j}}\right) = \sum_{\mathbf{j}\leq\mathbf{d}(PQ)} c_{\mathbf{j}}P_{\mathbf{j}}, \end{aligned}$$

where

$$c_{\mathbf{j}} = \sum_{\mathbf{j}\leq\mathbf{u}\leq\mathbf{d}(PQ)}\left(\sum_{\mathbf{v}+\mathbf{w}=\mathbf{u}} a_{\mathbf{v}}b_{\mathbf{w}}\gamma_{\mathbf{j}}(\mathbf{v},\mathbf{w})\right) \tag{7}$$

and since only a finite number of $a_{\mathbf{v}}$, $b_{\mathbf{w}}$ are different from zero, all the sums are finite. Then, if (6) holds,

$$\begin{aligned}\|PQ\|_{\mathcal{F},\mathcal{N}} &\leq \max_{\mathbf{j}\leq \mathbf{d}(PQ)} \left\{ \max_{\mathbf{j}\leq \mathbf{u}\leq \mathbf{d}(PQ)} \left\{ \left| \sum_{\mathbf{v}+\mathbf{w}=\mathbf{u}} a_{\mathbf{v}} b_{\mathbf{w}} \gamma_{\mathbf{j}}(\mathbf{v},\mathbf{w}) \right| \right\} \delta_{\mathbf{j}} \right\} \\ &\leq \max_{\mathbf{j}\leq \mathbf{d}(PQ)} \left\{ \max_{\mathbf{j}\leq \mathbf{u}\leq \mathbf{d}(PQ)} \left\{ \max_{\mathbf{v}+\mathbf{w}=\mathbf{u}} \left\{ \left| a_{\mathbf{v}} b_{\mathbf{w}} \gamma_{\mathbf{j}}(\mathbf{v},\mathbf{w}) \right| \right\} \right\} \delta_{\mathbf{j}} \right\} \\ &\leq \max_{\mathbf{u}\leq \mathbf{d}(PQ)} \left\{ \max_{\mathbf{v}+\mathbf{w}=\mathbf{u}} \left\{ |a_{\mathbf{v}} b_{\mathbf{w}}|\, \rho_{\mathbf{v},\mathbf{w}} \right\} \right\} \leq \max_{\mathbf{u}\leq \mathbf{d}(PQ)} \left\{ \max_{\mathbf{v}+\mathbf{w}=\mathbf{u}} \left\{ |a_{\mathbf{v}} b_{\mathbf{w}}|\, \delta_{\mathbf{v}} \delta_{\mathbf{w}} \right\} \right\} \\ &\leq \max_{\mathbf{i}\leq \mathbf{d}(P)} \{ |a_{\mathbf{i}}|\, \delta_{\mathbf{i}} \} \max_{\mathbf{j}\leq \mathbf{d}(Q)} \left\{ \left| b_{\mathbf{j}} \right| \delta_{\mathbf{j}} \right\} = \|P\|_{\mathcal{F},\mathcal{N}} \|Q\|_{\mathcal{F},\mathcal{N}}.\end{aligned}$$

This completes the proof of the proposition. □

We call the norm given by (3) the *Gauss norm* on $K[\mathbf{X}]$ defined by $\mathcal{F}$ and $\mathcal{N}$. If $\| \, \|_{\mathcal{F},\mathcal{N}}$ is a K-algebra norm on $K[\mathbf{X}]$, then by (5) and (6) it follows that

$$\delta_{\mathbf{n}} \leq \min_{\mathbf{i}+\mathbf{j}=\mathbf{n}} \left\{ \frac{\delta_{\mathbf{i}} \delta_{\mathbf{j}}}{|\gamma_{\mathbf{n}}(\mathbf{i},\mathbf{j})|} \right\}. \tag{8}$$

If

$$P_{\mathbf{j}} = \sum_{\mathbf{i}\leq \mathbf{j}} a_{\mathbf{i},\mathbf{j}} \mathbf{X}^{\mathbf{i}}, \tag{9}$$

then

$$P_{\mathbf{s}} P_{\mathbf{t}} = \sum_{\mathbf{j}\leq \mathbf{s}+\mathbf{t}} c_{\mathbf{j}} \mathbf{X}^{\mathbf{j}},$$

where

$$c_{\mathbf{j}} = \sum_{\mathbf{u}+\mathbf{v}=\mathbf{j}} a_{\mathbf{u},\mathbf{s}} a_{\mathbf{v},\mathbf{t}},$$

and all the sums are finite. We consider $\mathbf{i}_1$ the greatest element of $E(P_{\mathbf{s}}P_{\mathbf{t}} - \gamma_{\mathbf{s}+\mathbf{t}}(\mathbf{s},\mathbf{t})P_{\mathbf{s}+\mathbf{t}})$. Thus, by (4),

$$\gamma_{\mathbf{i}_1}(\mathbf{s},\mathbf{t}) = c_{\mathbf{i}_1} - a_{\mathbf{i}_1,\mathbf{s}+\mathbf{t}}. \tag{10}$$

By induction with respect to the defined order it follows that

$$\gamma_{\mathbf{j}}(\mathbf{s},\mathbf{t}) = T_{\mathbf{j}} - a_{\mathbf{j},\mathbf{s}+\mathbf{t}}, \ \mathbf{j} = \mathbf{i}_2, \mathbf{i}_3, \ldots, \tag{11}$$

where $\mathbf{i}_0 = \mathbf{s}+\mathbf{t} > \mathbf{i}_1 > \mathbf{i}_2 > \ldots$, $\mathbf{i}_k$ is the greatest element of $E(P_{\mathbf{s}}P_{\mathbf{t}} - \sum_{\mathbf{i}_z > \mathbf{i}_{k-1}} \gamma_{\mathbf{i}_z}(\mathbf{s},\mathbf{t})P_{\mathbf{i}_z})$, $T_{\mathbf{j}}$ is a polynomial with integral coefficients in $a_{\mathbf{v},\mathbf{w}}$ with either $\mathbf{w} < \mathbf{s}+\mathbf{t}$ or $\mathbf{w} = \mathbf{s}+\mathbf{t}$ and $\mathbf{v} > \mathbf{j}$.

Now for $k \in \{1, 2, \ldots, r\}$ we consider $\mathbf{e}_k = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^r$. If $\mathbf{n} \in \mathbb{N}^r$, $N(\mathbf{n}) > 1$ we denote $\mathbf{n}_- \in \mathbb{N}^r$, the greatest element such that $\mathbf{n} = \mathbf{n}_- + \mathbf{e}_k$, for some $k \in \{1, 2, \ldots, r\}$. In this case we denote $\mathbf{e}_k = \mathbf{e}(\mathbf{n})$.

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The following result shows that for every admissible $\mathcal{N} = \{\delta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ such that

$$C = \inf_{\mathbf{j},k} \left\{ \frac{\delta_{\mathbf{j}+\mathbf{e}_k}}{\delta_{\mathbf{j}}} \right\} > 0, \tag{12}$$

and satisfying (8) one can construct Gauss norms on $K[X]$ of the form $\| \ \|_{\mathcal{F},\mathcal{N}}$. A trivial case is when we take $P_{\mathbf{j}}(\mathbf{X}) = \mathbf{X}^{\mathbf{j}}$, but also we can find Gauss norms such that $P_{\mathbf{s}+\mathbf{t}} \neq P_{\mathbf{s}} P_{\mathbf{t}}$.

We put $\mu_{(0,0,\ldots,0)} = 1$ and for any $\mathbf{n}$ with $N(\mathbf{n}) > 1$,

$$\mu_{\mathbf{n}} = \min_{\mathbf{i}+\mathbf{j}=\mathbf{n}} \{\delta_{\mathbf{i}}\delta_{\mathbf{j}}\}, \ \ \tau_{\mathbf{n}} = \min_{N(\mathbf{m})=N(\mathbf{n})-1} \{\mu_{\mathbf{n}}, C\mu_{\mathbf{m}}\}. \tag{13}$$

PROPOSITION 2. *Suppose that* $(K, | \ |)$ *is a valued field and* $\mathcal{N} = \{\delta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}}$ *an admissible sequence of real numbers verifying* (8) *and* (12). *Then there exist infinitely many sequences of admissible polynomials* $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ *such that* $\| \ \|_{\mathcal{F},\mathcal{N}}$ *defined by* (3) *is a Gauss norm of* K*-algebra on* $K[\mathbf{X}]$.

Proof. We construct sequences of monic polynomials $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ such that $\| \ \|_{\mathcal{F},\mathcal{N}}$ defined by (3) is a Gauss norm of K-algebra on $K[\mathbf{X}]$. We put $P_{(0,\ldots,0)} = 1$ and if $\mathbf{j} = \mathbf{e}_r = (0, \ldots, 0, 1) \in \mathbb{N}^r$, $P_{\mathbf{j}} = a_{\mathbf{j}} + \mathbf{X}^{\mathbf{j}}$, with an arbitrary $a_{\mathbf{j}} \in K$. Generally, if $N(\mathbf{j}) = 1$, we take an arbitrary monic polynomial $P_{\mathbf{j}} = \sum_{\mathbf{i}\leq\mathbf{j}} a_{\mathbf{i},\mathbf{j}}\mathbf{X}^{\mathbf{i}}$, where $a_{\mathbf{i},\mathbf{j}} \in K$. If $\mathbf{j} = (0, \ldots, 0, 2)$ we take the monic polynomial $P_{\mathbf{j}} = \sum_{\mathbf{i}\leq\mathbf{j}} a_{\mathbf{i},\mathbf{j}}\mathbf{X}^{\mathbf{i}}$, with $E(P_{\mathbf{j}}) \setminus \{\mathbf{j}\}$ a subset of the union of all $E(P_{\mathbf{i}})$ with $\mathbf{i} < \mathbf{j}$. Then by (4), we can write $P_{\mathbf{e}_r}^2 = P_{\mathbf{j}} + \sum_{\mathbf{v}<\mathbf{j}} \gamma_{\mathbf{v}}(\mathbf{e}_r, \mathbf{e}_r)P_{\mathbf{v}}$ and by (11) we can find the coefficients $a_{\mathbf{i},\mathbf{j}}$ such that

$$|\gamma_{\mathbf{i}}(\mathbf{e}_r, \mathbf{e}_r)|\delta_{\mathbf{i}} < \tau_{\mathbf{j}}, \ \ \mathbf{i} < \mathbf{j}.$$

By choosing arbitrary the coefficients $a_{\mathbf{i},\mathbf{j}}$ when $\mathbf{i}$ is not in $E_{\mathcal{F}_{\mathbf{j}}}(P_{\mathbf{e}_r}^2)$, where $\mathcal{F}_{\mathbf{j}} = \{P_{\mathbf{i}}\}_{\mathbf{i}\leq\mathbf{j}}$, we find $E(P_{\mathbf{j}})$. In the same manner we can construct all the polynomials $P_{\mathbf{j}} = \sum_{\mathbf{i}\leq\mathbf{j}} a_{\mathbf{i},\mathbf{j}}\mathbf{X}^{\mathbf{i}}$, with $N(\mathbf{j}) = 2$. Then by induction, we consider $\mathbf{n} \in \mathbb{N}^r$, and suppose that for all $\mathbf{s}$ with $N(\mathbf{s}) \leq N(\mathbf{n}) - 1$ and $\mathbf{t} \in E_{\mathcal{F}_{\mathbf{s}}}(P_{\mathbf{s}})$ we have

$$|\gamma_{\mathbf{t}}(\mathbf{e}_k, \mathbf{s}_-)|\delta_{\mathbf{t}} \leq \tau_{\mathbf{s}_-+\mathbf{e}_k}, \ \ |\gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j})|\delta_{\mathbf{t}} \leq \delta_{\mathbf{i}}\delta_{\mathbf{j}}, \ \ \mathbf{i}+\mathbf{j} = \mathbf{s}, \ k \in \{1, 2, \ldots, r\}. \tag{14}$$

By (11) we can choose the coefficients of $P_{\mathbf{n}}$ such that the first condition of (14) holds for $\mathbf{s} = \mathbf{n}$. To verify the second condition we consider $\mathbf{i} + \mathbf{j} = \mathbf{n}$, with $N(\mathbf{i})$ and $N(\mathbf{j})$ less than $N(\mathbf{n})$. Then, without loss of generality, we may suppose that $\mathbf{e}(\mathbf{j}) = \mathbf{e}(\mathbf{n})$ and we obtain

$$\begin{aligned}
P_{\mathbf{e}(\mathbf{n})}P_{\mathbf{n}_-} &= P_{\mathbf{e}(\mathbf{n})} \left(P_{\mathbf{i}}P_{\mathbf{j}_-} - \sum_{\mathbf{t}<\mathbf{n}_-} \gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-)P_{\mathbf{t}} \right) \\
&= P_{\mathbf{i}} \sum_{\mathbf{t}\leq\mathbf{j}} \gamma_{\mathbf{t}}(\mathbf{e}(\mathbf{n}), \mathbf{j}_-)P_{\mathbf{t}} - \sum_{\mathbf{t}<\mathbf{n}_-} \gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-)P_{\mathbf{e}(\mathbf{n})}P_{\mathbf{t}} \\
&= P_{\mathbf{i}}P_{\mathbf{j}} + \sum_{\mathbf{t}<\mathbf{j}} \gamma_{\mathbf{t}}(\mathbf{e}(\mathbf{n}), \mathbf{j}_-) \sum_{\mathbf{u}\leq\mathbf{i}+\mathbf{t}} \gamma_{\mathbf{u}}(\mathbf{i}, \mathbf{t})P_{\mathbf{u}} - \sum_{\mathbf{t}<\mathbf{n}_-} \gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-) \sum_{\mathbf{u}\leq\mathbf{e}(\mathbf{n})+\mathbf{t}} \gamma_{\mathbf{u}}(\mathbf{e}(\mathbf{n}), \mathbf{t})P_{\mathbf{u}} \\
&= P_{\mathbf{i}}P_{\mathbf{j}} + \sum_{\mathbf{t}<\mathbf{j}} \sum_{\mathbf{u}\leq\mathbf{i}+\mathbf{t}} \gamma_{\mathbf{t}}(\mathbf{e}(\mathbf{n}), \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{i}, \mathbf{t})P_{\mathbf{u}} - \sum_{\mathbf{t}<\mathbf{n}_-} \sum_{\mathbf{u}\leq\mathbf{e}(\mathbf{n})+\mathbf{t}} \gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{e}(\mathbf{n}), \mathbf{t})P_{\mathbf{u}}.
\end{aligned}$$

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Hence, for a fixed $\mathbf{u}$,

$$\gamma_{\mathbf{u}}(\mathbf{e}(\mathbf{n}), \mathbf{n}_-) = \gamma_{\mathbf{u}}(\mathbf{i}, \mathbf{j}) + \sum_{\substack{\mathbf{t} < \mathbf{j} \\ \mathbf{u} \le \mathbf{i} + \mathbf{t}}} \gamma_{\mathbf{t}}(\mathbf{e}(\mathbf{n}), \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{i}, \mathbf{t}) - \sum_{\substack{\mathbf{t} < \mathbf{n}_- \\ \mathbf{u} \le \mathbf{e}(\mathbf{n}) + \mathbf{t}}} \gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{e}(\mathbf{n}), \mathbf{t}). \tag{15}$$

Now by (14) it follows that

$$|\gamma_{\mathbf{t}}(\mathbf{e}(\mathbf{n}), \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{i}, \mathbf{t})| \le \frac{\tau_{\mathbf{j}}}{\delta_{\mathbf{t}}}\frac{\delta_{\mathbf{i}}\delta_{\mathbf{t}}}{\delta_{\mathbf{u}}} \le \frac{\delta_{\mathbf{i}}\delta_{\mathbf{j}}}{\delta_{\mathbf{u}}}, \ |\gamma_{\mathbf{t}}(\mathbf{i}, \mathbf{j}_-)\gamma_{\mathbf{u}}(\mathbf{e}(\mathbf{n}), \mathbf{t})| \le \frac{\delta_{\mathbf{i}}\delta_{\mathbf{j}_-}}{\delta_{\mathbf{t}}}\frac{\tau_{\mathbf{t}+\mathbf{e}(\mathbf{n})}}{\delta_{\mathbf{u}}} \le \frac{\delta_{\mathbf{i}}\delta_{\mathbf{j}_-}}{\delta_{\mathbf{t}}}\frac{C\mu_{\mathbf{t}}}{\delta_{\mathbf{u}}} \le \frac{\delta_{\mathbf{i}}\delta_{\mathbf{j}}}{\delta_{\mathbf{u}}}.$$

Hence one has (14) for $\mathbf{s} = \mathbf{n}$ and by Proposition 1, it follows that we can find infinitely many sequences of monic polynomials $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{N}^r}$ such that $\| \ \|_{\mathcal{F},\mathcal{N}}$ defined by (3) is a Gauss norm of K-algebra on $K[\mathbf{X}]$. □

Next, we study when two Gauss norms are equivalent.

PROPOSITION 3. *Suppose that* $(K, | \ |)$ *is a valued field and* $\| \ \|_{\mathcal{F}_\alpha,\mathcal{N}_\alpha}$, $\alpha = 1, 2$, *where* $\mathcal{F}_\alpha = \{P_{\mathbf{j},\alpha}\}_{\mathbf{j} \in \mathbb{N}^r}$, $\mathcal{N}_\alpha = \{\delta_{\mathbf{j},\alpha}\}_{\mathbf{j} \in \mathbb{N}^r}$, *are two Gauss norms on* $K[\mathbf{X}]$. *If by* (2)

$$P_{\mathbf{j},\alpha} = \sum_{\mathbf{i} \le \mathbf{j}} c_{\mathbf{i},\mathbf{j}}^{(\alpha)} P_{\mathbf{i},3-\alpha}, \ \alpha = 1, 2, \tag{16}$$

then the norms are equivalent if and only if there exist positive constants C_1, C_2 *such that*

$$\delta_{\mathbf{j},1} \ge C_1 \left|c_{\mathbf{i},\mathbf{j}}^{(1)}\right| \delta_{\mathbf{i},2}, \ \ C_2\delta_{\mathbf{j},2} \ge \left|c_{\mathbf{i},\mathbf{j}}^{(2)}\right| \delta_{\mathbf{i},1}, \ \text{ for any } \mathbf{i}, \mathbf{j}, \ \text{ with } \mathbf{i} \le \mathbf{j}. \tag{17}$$

Proof. If the norms are equivalent, then there exist positive constants C_1, C_2 such that for every $Q \in K[\mathbf{X}]$

$$C_1\|Q\|_{\mathcal{F}_2,\mathcal{N}_2} \le \|Q\|_{\mathcal{F}_1,\mathcal{N}_1} \le C_2\|Q\|_{\mathcal{F}_2,\mathcal{N}_2}.$$

Consequently, we obtain

$$\delta_{\mathbf{j},1} = \left\|P_{\mathbf{j},1}\right\|_{\mathcal{F}_1,\mathcal{N}_1} \ge C_1 \left\| \sum_{\mathbf{i} \le \mathbf{j}} c_{\mathbf{i},\mathbf{j}}^{(1)} P_{\mathbf{i},2} \right\|_{\mathcal{F}_2,\mathcal{N}_2} = C_1 \max_{\mathbf{i} \le \mathbf{j}} \left\{ \left|c_{\mathbf{i},\mathbf{j}}^{(1)}\right| \delta_{\mathbf{i},2} \right\}.$$

Conversely, suppose that (17) holds. If $Q \in K[\mathbf{X}]$, then

$$Q = \sum_{\mathbf{j} \le \mathbf{d}(Q)} b_{\mathbf{j}} P_{\mathbf{j},2} = \sum_{\mathbf{j} \le \mathbf{d}(Q)} b_{\mathbf{j}} \left(\sum_{\mathbf{i} \le \mathbf{j}} c_{\mathbf{i},\mathbf{j}}^{(2)} P_{\mathbf{i},1} \right) = \sum_{\mathbf{j} \le \mathbf{d}(Q)} \left(\sum_{\mathbf{i} \ge \mathbf{j}} c_{\mathbf{j},\mathbf{i}}^{(2)} b_{\mathbf{i}} \right) P_{\mathbf{j},1}.$$

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Hence it follows that

$$\|Q\|_{\mathcal{F}_1,\mathcal{N}_1} = \max_{\mathbf{j}\le \mathbf{d}(Q)} \left\{ \left| \sum_{\mathbf{i}\ge\mathbf{j}} c^{(2)}_{\mathbf{j},\mathbf{i}} b_{\mathbf{i}} \right| \delta_{\mathbf{j},1} \right\} \le \max_{\mathbf{j}\le \mathbf{d}(Q)} \left\{ \max_{\mathbf{i}\ge\mathbf{j}} \left\{ \left| c^{(2)}_{\mathbf{j},\mathbf{i}} b_{\mathbf{i}} \right| \right\} \delta_{\mathbf{j},1} \right\}$$
$$\le C_2 \max_{\mathbf{i}\le \mathbf{d}(Q)} \left\{ |b_{\mathbf{i}}|\, \delta_{\mathbf{i},2} \right\} = C_2 \|Q\|_{\mathcal{F}_2,\mathcal{N}_2}.$$

□

REMARK 1. Consider $\| \;\|_{\mathcal{F},\mathcal{N}}$, a Gauss K-algebra norm on $K[\mathbf{X}]$ defined by $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{N} = \{\delta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$. If for every $\mathbf{j} \in \mathbb{N}^r$, $c_{\mathbf{j}}$ is an element different from zero from K and $P^*_{\mathbf{j}} = c_{\mathbf{j}} P_{\mathbf{j}}$, $\delta^*_{\mathbf{j}} = |c_{\mathbf{j}}|\delta_{\mathbf{j}}$, then by Proposition 3 it follows easily that the Gauss norm defined by $\mathcal{F}^* = \{P^*_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{N}^* = \{\delta^*_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ is a K-algebra norm on $K[\mathbf{X}]$ and the norms $\| \;\|_{\mathcal{F}^*,\mathcal{N}^*}$, $\| \;\|_{\mathcal{F},\mathcal{N}}$ are equivalent. Hence it follows that up to an equivalence we can consider a Gauss norm defined by a family of monic polynomials.

EXAMPLE 1. Suppose $(K, | \;|)$ is a valued field and $\mathbf{S} = \{(\beta_{k,1}, \ldots, \beta_{k,r})\}_{k\ge1}$ is a fixed sequence of elements of $\overset{\circ}{K}{}^r$, where $\overset{\circ}{K} = \bar{B}_K(0, 1) = \{x \in K; |x| \le 1\}$. We take $\mathcal{F}_1 = \{\mathbf{X}^{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{F}_2 = \{P_{\mathbf{j},2}\}_{\mathbf{j}\in\mathbb{N}^r}$, where

$$P_{\mathbf{j},2} = \prod_{0<k\le j_1} (X_1 - \beta_{k,1}) \prod_{0<k\le_2} (X_2 - \beta_{k,2}) \ldots \prod_{0<k\le_r} (X_r - \beta_{k,r}).$$

Then it follows easily that all $c^{(\alpha)}_{\mathbf{i},\mathbf{j}}$, $\alpha = 1, 2$, defined by (16) belong to $\overset{\circ}{K}$.

We put $\mathcal{N}_1 = \mathcal{N}_2 = \{\delta_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^\mathbf{r}}$ where, for every $\mathbf{j}, \mathbf{s}, \mathbf{t} \in \mathbb{N}^\mathbf{r}$ with $\mathbf{j} \le \mathbf{s} + \mathbf{t}$,

$$\delta_{\mathbf{j}} \le \delta_{\mathbf{s}}\delta_{\mathbf{t}}.$$

For example we may take either $\delta_{\mathbf{j}} = a^{N(\mathbf{j})}$ with $a > 1$, for all $\mathbf{j}$, or $\delta_{\mathbf{j}} = (N(\mathbf{j}) + 1)^p$ with p a fixed positive integer, for all $\mathbf{j}$. Since all $\gamma_{\mathbf{j},\alpha}$, $\alpha = 1, 2$, defined by (4) belong to $\overset{\circ}{K}$, by Proposition 1 it easily follows that $\| \;\|_{\mathcal{F}_1,\mathcal{N}_1}$ and $\| \;\|_{\mathcal{F}_2,\mathcal{N}_2}$ are K-algebra norms on $K[\mathbf{X}]$ and (17) holds with $C_1 = C_2 = 1$. Hence the norms are equivalent.

Let $(K, | \;|)$ be a valued field and $\| \;\|$ a non-Archimedean norm on $K[\mathbf{X}]$ which extends $| \;|$. If $\mathbf{j} \in \mathbb{N}^r$, put

$$M^{(\mathbf{j})} = \{Q \in K[\mathbf{X}] \text{ monic}, \mathbf{d}(Q) = \mathbf{j}\}, \; M^{(\mathbf{j})}_{\| \|} = \left\{ \|Q\|; Q \in M^{(\mathbf{j})} \right\}. \tag{18}$$

On $K[\mathbf{X}]$ there are non-Archimedean norms which are not Gauss norms (see Remark 3). The following result gives a criterion for a non-Archimedean norm on $K[\mathbf{X}]$ to be a Gauss norm.

PROPOSITION 4. *Let $(K, | \;|)$ be a valued field and let $\| \;\|$ be a K-algebra non-Archimedean norm on $K[\mathbf{X}]$ which extends $| \;|$. Then $\| \;\|$ is a Gauss norm defined by a family of monic polynomials if and only if for every $\mathbf{j} \in \mathbb{N}^r$, there exists $P_{\mathbf{j}} \in M^{(\mathbf{j})}$ such that $\|P_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\| \|}$. In this case $\| \;\|$ is defined by $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{N} = \{\|P_{\mathbf{j}}\|\}_{\mathbf{j}\in\mathbb{N}^r}$.*

Proof. If $\| \;\|$ is a Gauss norm defined by a family of monic polynomials $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, then by (3) it follows that for every $\mathbf{j} \in \mathbb{N}^r$, and $Q \in M^{(\mathbf{j})}$, $\|Q\| \ge \delta_{\mathbf{j}}$. Since $\|P_{\mathbf{j}}\| = \delta_{\mathbf{j}}$, it follows that $\|P_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\| \|}$.

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Conversely, if for every $\mathbf{j} \in \mathbb{N}^r$ there exists $P_{\mathbf{j}} \in M^{(\mathbf{j})}$ such that $\|P_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\| \|}$, then we can take $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{N} = \{\|P_{\mathbf{j}}\|\}_{\mathbf{j}\in\mathbb{N}^r}$. Since $\|P_{\mathbf{s+t}}\| \le \|P_{\mathbf{s}}P_{\mathbf{t}}\| \le \|P_{\mathbf{s}}\|\|P_{\mathbf{t}}\|$ and $\|P_{\mathbf{i}_1}\| \le \frac{\|P_{\mathbf{s+t}}-P_{\mathbf{s}}P_{\mathbf{t}}\|}{|\gamma_{\mathbf{i}_1}(\mathbf{s},\mathbf{t})|}$, where $\mathbf{i}_1$ is the greatest element of $E(P_{\mathbf{s}}P_{\mathbf{t}} - P_{\mathbf{s+t}})$, by induction with respect to the given order it follows that $\mathcal{F}$ and $\mathcal{N}$ verify the conditions of Proposition 1. We take $Q \in K[\mathbf{X}]$ and prove by induction on $\mathbf{q} = \mathbf{d}(Q)$, with respect to the given order that $\|Q\| = \|Q\|_{\mathcal{F},\mathcal{N}}$. It is enough to consider the case when Q is a monic polynomial. If $\mathbf{q} = (0, \dots, 0, 1)$ we can write $P_{\mathbf{q}} = \mathbf{X}^{\mathbf{q}} - a$ and $Q = \mathbf{X}^{\mathbf{q}} - b$, $a, b \in K$. Since

$$Q = P_{\mathbf{q}} + a - b, \tag{19}$$

we obtain

$$\|Q\| \le \max\left\{\|P_{\mathbf{q}}\|, |a-b|\right\} = \|Q\|_{\mathcal{F},\mathcal{N}}. \tag{20}$$

If $\|P_{\mathbf{q}}\| \ne |a-b|$, by (19) it follows that $\|Q\| = \|Q\|_{\mathcal{F},\mathcal{N}}$. Otherwise, by the definition of $P_{\mathbf{q}}$ and by (20) we obtain $\|P_{\mathbf{q}}\| \le \|Q\| \le \|Q\|_{\mathcal{F},\mathcal{N}} = \|P_{\mathbf{q}}\|$ and $\|Q\| = \|Q\|_{\mathcal{F},\mathcal{N}}$, for $\mathbf{q} = (0, \dots, 0, 1)$.

Now suppose that $\|P\| = \|P\|_{\mathcal{F},\mathcal{N}}$, for all the polynomials with $\mathbf{d}(P) < \mathbf{q}$ and let $Q \in K[\mathbf{X}]$ such that $\mathbf{d}(Q) = \mathbf{q}$. Then

$$Q = b_{\mathbf{q}}P_{\mathbf{q}} + Q_{\mathbf{i}}, \tag{21}$$

where $b_{\mathbf{q}} \in K$ and $\mathbf{d}(Q_{\mathbf{i}}) = \mathbf{i} < \mathbf{q}$. Hence

$$\|P_{\mathbf{q}}\| \le \frac{1}{|b_{\mathbf{q}}|}\|Q\| \le \max\left\{\|P_{\mathbf{q}}\|, \frac{1}{|b_{\mathbf{q}}|}\|Q_{\mathbf{i}}\|\right\}. \tag{22}$$

Thus,

$$|b_{\mathbf{q}}|\|P_{\mathbf{q}}\| \le \|Q\| \le \max\left\{\|b_{\mathbf{q}}P_q\|, \|Q_{\mathbf{i}}\|_{\mathcal{F},\mathcal{N}}\right\} = \|Q\|_{\mathcal{F},\mathcal{N}}. \tag{23}$$

If $\|Q_{\mathbf{i}}\|_{\mathcal{F},\mathcal{N}} = \|b_{\mathbf{q}}P_{\mathbf{q}}\|$, by (23) it follows that $\|Q\| = \|Q\|_{\mathcal{F},\mathcal{N}}$. Otherwise by by (21) we obtain $\|Q\| = \|Q\|_{\mathcal{F},\mathcal{N}}$ and the proposition is proved. □

Now we prove that in the case of p-adic fields all non-Archimedean norms on $K[\mathbf{X}]$ which extend $| \ |$ are Gauss norms.

Corollary 1. *Suppose K is a locally compact field and $\| \ \|$ is a K-algebra non-Archimedean norm on $K[\mathbf{X}]$ which extends $| \ |$. Then $\| \ \|$ is a Gauss norm.*

Proof. By Proposition 4 it follows that it is enough to show that for $\mathbf{j} \in \mathbb{N}^r$, there exists $P_{\mathbf{j}} \in M^{(\mathbf{j})}$ such that $\|P_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\| \|}$. Thus for a fixed $\mathbf{j} \in \mathbb{N}^r$ we choose a sequence $\{P_{\mathbf{j},i}\}_{i\in\mathbb{N}}$ of elements from $M^{(\mathbf{j})}$ such that for every i, $\|P_{\mathbf{j},i}\| \ge \|P_{\mathbf{j},i+1}\|$, and $\lim_{i\to\infty} \|P_{\mathbf{j},i}\| = \inf M^{(\mathbf{j})}_{\| \|}$. If $P_{\mathbf{j},i} = \sum_{\mathbf{t}\le\mathbf{j}} a_{\mathbf{j},i,\mathbf{t}}\mathbf{X}^{\mathbf{t}}$, we distinguish two cases:

(i)*The set of coefficients of all polynomials $P_{\mathbf{j},i}$ is bounded in K*. Then, since K is locally compact, for every $\mathbf{t}$ there exists a subsequence $\{a_{\mathbf{j},i_m,\mathbf{t}}\}_{m\in\mathbb{N}}$ of $\{a_{\mathbf{j},i,\mathbf{t}}\}_{i\in\mathbb{N}}$ which converges to an element $a_{\mathbf{j},\mathbf{t}} \in K$. If we put $P_{\mathbf{j}} = \sum_{\mathbf{t}\le\mathbf{j}} a_{\mathbf{j},\mathbf{t}}\mathbf{X}^{\mathbf{t}}$, it follows easily that $P_{\mathbf{j}} \in M^{(\mathbf{j})}$ and $\|P_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\| \|}$.

(ii)*The above set of coefficients is unbounded.* If $\bar{B}_K(0, 1) = \{x \in K; |x| \le 1\}$ then its maximal ideal $B_K(0, 1) = \{x \in K; |x| < 1\}$ is a principal ideal generated by an element π. We take b_i, the smallest positive integer such that $f_i = \pi^{b_i}P_{\mathbf{j},i} \in \bar{B}_K(0, 1)[\mathbf{X}]$.

Choosing, if it is necessary, a subsequence we may assume that $\lim_{i\to\infty} b_i = \infty$. Since $\bar{B}_K(0, 1)$ is a compact set, there exists a subsequence $\{f_{i_s}\}_{s\in\mathbb{N}}$ which converges to a polynomial $f \in \bar{B}_K(0, 1)[\mathbf{X}]$. From our choice of b_i it follows that f_i is primitive for any i. Hence it follows that f is primitive, in particular $f \neq 0$. Since $\|P_{\mathbf{j},i}\| \leq \|P_{\mathbf{j},1}\|$ for each i, we obtain that $f = \lim_{i\to\infty} \|f_i\| = 0$, a contradiction which implies the corollary. □

3. Types of non-Archimedean norms on $K[\mathbf{X}]$. In order to describe all the non-Archimedean norms on $K[\mathbf{X}]$ which extend $|\ |$ we first establish the following lemma.

LEMMA 2. *Suppose that $(K, |\ |)$ is a valued field and $\{\|\ \|_i\}_{i\in I}$ is a family of non-Archimedean norms on $K[\mathbf{X}]$ which extend $|\ |$ such that for any $Q_1, Q_2, Q_3 \in K[\mathbf{X}]$ there exists an $i_0 \in I$ verifying*

$$\inf_{i\in I} \left\{\|Q_j\|_i\right\} = \|Q_j\|_{i_0},\ j = 1, 2, 3.$$

Then, if for all $R \in K[\mathbf{X}]$ we define

$$\|R\| = \inf_{i\in I} \{\|R\|_i\}, \tag{24}$$

we obtain a non-Archimedean norm on $K[\mathbf{X}]$ which extends $|\ |$. Furthermore, if for every $i \in I$, $\|\ \|_i$ is a K-algebra norm, then also the norm given by (24) *is a K-algebra norm.*

Proof. If $Q, R \in K[\mathbf{X}]$, we consider for example $Q_1 = Q + R,\ Q_2 = Q,\ Q_3 = R$. Then there is an $i_0 \in I$ such that

$$\|Q + R\| = \|Q + R\|_{i_0} \leq \max\left\{\|Q\|_{i_0}, \|R\|_{i_0}\right\} = \max\{\|Q\|, \|R\|\}.$$

The other required properties are similarly proved. □

Let $\|\ \|$ be a non-Archimedean norm on $K[\mathbf{X}]$ which extends $|\ |$. For every $\mathbf{j} \in \mathbb{N}^r$ we construct a sequence of polynomials $\Pi_{\mathbf{j}} = \{P_{\mathbf{j},i}\}_{i\in\mathbb{N}}$, with $P_{\mathbf{j},i} \in M^{(\mathbf{j})}$ in the following way. If there exists $Q_{\mathbf{j}} \in M^{(\mathbf{j})}$ such that $\|Q_{\mathbf{j}}\| = \inf M^{(\mathbf{j})}_{\|\ \|}$, we fix this polynomial and for every $i \in \mathbb{N}$ we put $P_{\mathbf{j},i} = Q_{\mathbf{j}}$, otherwise we take $\{P_{\mathbf{j},i}\}_{i\in\mathbb{N}}$ such that $\|P_{\mathbf{j},i+1}\| < \|P_{\mathbf{j},i}\|$, for any i, and $\lim_{i\to\infty} \|P_{\mathbf{j},i}\| = \inf M^{(\mathbf{j})}_{\|\ \|}$. We consider

$$\Sigma = \left\{\sigma = \{s_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}^r}, s_{\mathbf{i}} \in \mathbb{N}\right\}, \tag{25}$$

and for every $\sigma = \{s_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$,

$$\mathcal{F}_\sigma = \left\{P_{\mathbf{j},s_{\mathbf{j}}}\right\}_{\mathbf{j}\in\mathbb{N}^r},\ \mathcal{N}_\sigma = \left\{\|P_{\mathbf{j},s_{\mathbf{j}}}\|\right\}_{\mathbf{j}\in\mathbb{N}^r}, P_{\mathbf{j},s_{\mathbf{j}}} \in \Pi_{\mathbf{j}}. \tag{26}$$

REMARK 2. If $\inf M^{(\mathbf{j})}_{\|\ \|}$ is not attained, for each $\mathbf{j}$, then for each $P \in M^{(\mathbf{j})}$ there exists $Q \in M^{(\mathbf{j})}$ such that $\|Q\| < \|P\|$. Then $\|P\| = |a|\|(P - Q)/a\|$, where $a \in K$ and $(P - Q)/a \in M^{(\mathbf{i})}$ with $\mathbf{i} < \mathbf{j}$. Hence, by induction it follows that the values of the norm coincide with the valuation group $|K^*|$.

We are ready to prove the following result.

THEOREM 1. *Let $(K, |\ |)$ be a valued field and let $\|\ \|$ be a non-Archimedean norm on $K[\mathbf{X}]$ which extends $|\ |$. If, for every $\mathbf{j} \in \mathbb{N}^r$, there exists $P_{\mathbf{j}} \in M^{(\mathbf{j})}$ such that $\|P_{\mathbf{j}}\| =$*

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$\inf M^{(\mathbf{j})}_{\| \ \|}$, *then* $\| \ \|$ *is a Gauss norm defined by* $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$, $\mathcal{N} = \{\|P_{\mathbf{j}}\|\}_{\mathbf{j}\in\mathbb{N}^r}$, *where* $P_{\mathbf{j}}$ *can be chosen in* $\Pi_{\mathbf{j}}$. *Otherwise, the set of* K*-vector space norms* $\{\| \ \|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma}\}_{\sigma\in\Sigma}$ *verifies the conditions from Lemma* 2 *and* $\| \ \|$ *is equal to the norm defined by* (24).

Proof. The first case follows by Proposition 4, where $P_{\mathbf{j}}$ can be chosen in $\Pi_{\mathbf{j}}$.

In the second case, we prove that $\{\| \ \|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma}\}_{\sigma\in\Sigma}$ verifies the conditions from Lemma 2. We take the monic polynomials $Q_j \in K[\mathbf{X}]$ with $\mathbf{q}_j = \mathbf{d}(Q_j)$, $j = 1, 2, 3$ and put

$$\theta_j = \inf_{\sigma\in\Sigma} \left\{\|Q_j\|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma}\right\},\ j = 1, 2, 3.$$

If $\|Q_j\| = \inf M^{(\mathbf{q}_j)}_{\| \ \|}$, we choose $P_{\mathbf{q}_j,s_{\mathbf{q}_j}}$ with $s_{\mathbf{q}_j} = 0$, otherwise we can take $P_{\mathbf{q}_j,s_{\mathbf{q}_j}} \in \Pi_{\mathbf{q}_j}$ such that $\|Q_j\| > \|P_{q_j,s_{q_j}}\|$. Then $Q_j = P_{\mathbf{q}_j,s_{\mathbf{q}_j}} + a_{\mathbf{q}_j^{(1)},j} Q_{\mathbf{q}_j^{(1)},j}$, where $Q_{\mathbf{q}_j^{(1)},j}$ is monic, $\mathbf{d}(Q_{\mathbf{q}_j^{(1)},j}) = \mathbf{q}_j^{(1)} < \mathbf{q}_j$ and

$$\|Q_j\| = \max\left\{\|P_{\mathbf{q}_j,s_{\mathbf{q}_j}}\|, \|a_{\mathbf{q}_j^{(1)},j} Q_{\mathbf{q}_j^{(1)},j}\|\right\}. \tag{27}$$

Now we choose polynomials $P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}}$ such that either $s_{\mathbf{q}_j^{(1)}} = 0$, if $\|Q_{\mathbf{q}_j^{(1)},j}\| = \inf M^{(\mathbf{q}_j^{(1)})}_{\| \ \|}$ or $\|Q_{\mathbf{q}_j^{(1)},j}\| > \|P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}}\|$, otherwise. Hence $Q_{\mathbf{q}_j^{(1)},j} = P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}} + \tilde{a}_{\mathbf{q}_j^{(2)}} Q_{\mathbf{q}_j^{(2)},j}$, where $Q_{\mathbf{q}_j^{(2)},j}$ is monic and $\mathbf{d}(Q_{\mathbf{q}_j^{(2)},j}) = \mathbf{q}_j^{(2)} < \mathbf{q}_j^{(1)}$. Thus

$$\|Q_{\mathbf{q}_j^{(1)},j}\| = \max\left\{\|P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}}\|, \|\tilde{a}_{\mathbf{q}_j^{(2)}} Q_{\mathbf{q}_j^{(2)},j}\|\right\} \tag{28}$$

and

$$Q_j = P_{\mathbf{q}_j,s_{\mathbf{q}_j}} + a_{\mathbf{q}_j^{(1)},j} P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}} + a_{\mathbf{q}_j^{(2)}} Q_{\mathbf{q}_j^{(2)},j},$$

where $a_{\mathbf{q}_j^{(2)}} = a_{\mathbf{q}_j^{(1)}} \tilde{a}_{\mathbf{q}_j^{(2)}}$. In this way after a finite number of steps we obtain

$$Q_j = P_{\mathbf{q}_j,s_{\mathbf{q}_j}} + a_{\mathbf{q}_j^{(1)},j} P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}} + a_{\mathbf{q}_j^{(2)},j} P_{\mathbf{q}_j^{(2)},s_{\mathbf{q}_j^{(2)}}} + \cdots + a_{(0,\ldots,0),j}.$$

Hence

$$\|Q_j\| \le \max\left\{\|P_{\mathbf{q}_j,s_{\mathbf{q}_j}}\|, \|a_{\mathbf{q}_j^{(1)},j} P_{\mathbf{q}_j^{(1)},s_{\mathbf{q}_j^{(1)}}}\|, \|a_{\mathbf{q}_j^{(2)},j} P_{\mathbf{q}_j^{(2)},s_{\mathbf{q}_j^{(2)}}}\|, \ldots, |a_{(0,\ldots,0),j}|\right\}. \tag{29}$$

By using (27) and (28), it follows that one has equality in (29). Moreover, one can choose the same polynomials $P_{\mathbf{i},s_{\mathbf{i}}}$ for all the polynomials Q_j, $j = 1, 2, 3$. Now we choose $\sigma = \{t_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}^r} \in \Sigma$ such that, if $\mathbf{q} = \max_{1\le j\le 3}\{\mathbf{q}_j\}$, then for $\mathbf{i} \le \mathbf{q}$ and $\mathbf{i} = \mathbf{q}_j^{(r)}$, $t_{\mathbf{i}} = s_{\mathbf{q}_j^{(r)}}$. It follows that $\|Q_j\| = \|Q_j\|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma} = \theta_j$ and $\{\| \ \|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma}\}_{\sigma\in\Sigma}$ verifies the conditions of Lemma 2.

Lastly, take $R \in K[\mathbf{X}]$. Since $\|R\| \le \|R\|_{\mathcal{F}_\sigma,\mathcal{N}_\sigma}$, it can be proved in the same manner that the norm is equal to the norm defined by (24). □

Remark 3. On $K[\mathbf{X}]$ there exist non-Archimedean norms which are not Gauss norms and extend $| \ |$. Even in the case of multiplicative norms and $r = 1$ such examples can be found. For instance, one may take $K = \mathbb{Q}$, p a prime number and $x \in \mathbb{Q}_p$ a transcendental element over $\mathbb{Q}$. Then we consider on $\mathbb{Q}[x]$ the absolute value induced by the p-adic absolute value $|\ |_p$ defined on $\mathbb{Q}_p$. If $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers which tends to x in $\mathbb{Q}_p$, the polynomials $P_n(X) = X - a_n \in \mathbb{Q}[X]$ define a sequence such that $|P_n(x)|_p$ tends to zero. Hence, by Proposition 4, it follows that one obtains a norm as in the second case of Theorem 1.

4. Completions of $K[\mathbf{X}]$ with respect to non-Archimedean norms. We now proceed to study the completion of $K[\mathbf{X}]$ with respect to a Gauss norm $\|\ \|_{\mathcal{F},\mathcal{N}}$. We denote by $\tilde{K}$ the completion of K with respect to $|\ |$, and consider the set of formal sums

$$\widetilde{K[\mathbf{X}]} = \left\{ f = \sum_{\mathbf{i}\in\mathbb{N}^r} a_{\mathbf{i}} P_{\mathbf{i}};\ a_{\mathbf{i}} \in \tilde{K},\ \lim_{N(\mathbf{i})\to\infty} |a_{\mathbf{i}}|\,\delta_{\mathbf{i}} = 0 \right\}. \tag{30}$$

If $f \in \widetilde{K[\mathbf{X}]}$, define

$$\|f\|_{\mathcal{F},\mathcal{N}} = \sup_{\mathbf{i}\in\mathbb{N}^r} \{|a_{\mathbf{i}}|\,\delta_{\mathbf{i}}\}. \tag{31}$$

THEOREM 2. *Suppose that* $(K, |\ |)$ *is a valued field and* $\|\ \|_{\mathcal{F},\mathcal{N}}$ *is a Gauss norm of* K*-algebra on* $K[\mathbf{X}]$. *Then* $\widetilde{K[\mathbf{X}]}$ *is a* K*-algebra which contains* $K[\mathbf{X}]$. *Furthermore the map given by* (31) *is a* K*-algebra norm and* $\widetilde{K[\mathbf{X}]}$ *is the completion of* $K[\mathbf{X}]$ *with respect to the Gauss norm.*

Proof. If $f, g = \sum_{\mathbf{j}\in\mathbb{N}^r} b_{\mathbf{j}} P_{\mathbf{j}} \in \widetilde{K[\mathbf{X}]}$, then

$$fg = \sum_{\mathbf{u}\in\mathbb{N}^r} c_{\mathbf{u}} P_{\mathbf{u}},$$

with

$$c_{\mathbf{u}} = \sum_{\mathbf{u}\leq\mathbf{v}} \tau_{\mathbf{v}}^{(\mathbf{u})},\ \ \tau_{\mathbf{v}}^{(\mathbf{u})} = \sum_{\mathbf{w}\leq\mathbf{v}} a_{\mathbf{w}} b_{\mathbf{v}-\mathbf{w}} \gamma_{\mathbf{u}}(\mathbf{w}, \mathbf{v}-\mathbf{w}). \tag{32}$$

Since, for $\mathbf{v} \geq \mathbf{u}$,

$$\left|\tau_{\mathbf{v}}^{(\mathbf{u})}\right| \leq \max_{\mathbf{w}\leq\mathbf{v}} \{|a_{\mathbf{w}} b_{\mathbf{v}-\mathbf{w}} \gamma_{\mathbf{u}}(\mathbf{w}, \mathbf{v}-\mathbf{w})|\} \leq \max_{\mathbf{w}\leq\mathbf{v}} \left\{ |a_{\mathbf{w}}|\,|b_{\mathbf{v}-\mathbf{w}}|\, \frac{\delta_{\mathbf{w}}\delta_{\mathbf{v}-\mathbf{w}}}{\delta_{\mathbf{u}}} \right\}, \tag{33}$$

it follows that $\lim_{N(\mathbf{v})\to\infty} \tau_{\mathbf{v}}^{(\mathbf{u})} = 0$ and $c_{\mathbf{u}} \in \tilde{K}$. Moreover, $\lim_{N(\mathbf{u})\to\infty} |c_{\mathbf{u}}|\,\delta_{\mathbf{u}} = 0$ and $fg \in \widetilde{K[\mathbf{X}]}$. Then it follows easily that $\widetilde{K[\mathbf{X}]}$ is a K-algebra which contains $K[\mathbf{X}]$. Since

$$\|fg\|_{\mathcal{F},\mathcal{N}} = \sup_{\mathbf{u}\in\mathbb{N}^r} \{|c_{\mathbf{u}}|\,\delta_{\mathbf{u}}\}$$

by (32) and (33) we obtain that the map given by (31) is a K-algebra norm on $\widetilde{K[\mathbf{X}]}$.

We need to show that $(\widetilde{K[\mathbf{X}]}, \|\ \|_{\mathcal{F},\mathcal{N}})$ is complete. Let $f^{[t]} = \sum_{i\in\mathbb{N}^r} a_{\mathbf{i},t} P_{\mathbf{i}} \in \widetilde{K[\mathbf{X}]}$, $t \geq 1$ a Cauchy sequence. Since, for a fixed $\mathbf{i}$,

$$|a_{\mathbf{i},t+1} - a_{\mathbf{i},t}| \leq \frac{\|f^{[t+1]} - f^{[t]}\|_{\mathcal{F},\mathcal{N}}}{\delta_{\mathbf{i}}}, \tag{34}$$

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it follows that each sequence $a_{\mathbf{i},t},\ t = 1, 2, \dots$ is a Cauchy sequence in $\tilde{K}$. For $\mathbf{i} \in \mathbb{N}^r$, let $a_{\mathbf{i}} \in \tilde{K}$ be the limit of this sequence and $f = \sum_{\mathbf{i}\in\mathbb{N}^r} a_{\mathbf{i}} P_{\mathbf{i}}$. We have to prove that $f \in \widetilde{K[\mathbf{X}]}$ and $\lim_{t\to\infty} \|f - f^{[t]}\|_{\mathcal{F},\mathcal{N}} = 0$. By restricting to a subsequence we may assume that

$$\|f^{[s]} - f^{[t]}\|_{\mathcal{F},\mathcal{N}} \leq \frac{1}{t} \tag{35}$$

for all $s \geq t,\ t = 1, 2, \dots$. By (34) and (35) we obtain $|a_{\mathbf{i},s} - a_{\mathbf{i},t}| \leq \frac{1}{t\delta_{\mathbf{i}}},\ s = t, t+1, \dots$ and hence $|a_{\mathbf{i}} - a_{\mathbf{i},t}| \leq \frac{1}{t\delta_{\mathbf{i}}}$, for any $\mathbf{i} \in \mathbb{N}^r$, $t \geq 1$. Since $f^{[t]} \in \widetilde{K[\mathbf{X}]}$, we obtain $\lim_{N(\mathbf{i})\to\infty} |a_{\mathbf{i},t}|\delta_{\mathbf{i}} = 0$. But, for every t, $|a_{\mathbf{i}}|\delta_{\mathbf{i}} \leq \max\{|a_{\mathbf{i},t}|\delta_{\mathbf{i}}, \frac{1}{t}\}$. Hence $\lim_{N(\mathbf{i})\to\infty} |a_{\mathbf{i}}|\delta_{\mathbf{i}} = 0$ and $f \in \widetilde{K[\mathbf{X}]}$. Then $\|f - f^{[t]}\|_{\mathcal{F},\mathcal{N}} = \sup_{\mathbf{i}\in\mathbb{N}^r} \{|a_{\mathbf{i}} - a_{\mathbf{i},t}|\, \delta_{\mathbf{i}}\} \leq \frac{1}{t}$ and $\lim_{t\to\infty} \|f - f^{[t]}\|_{\mathcal{F},\mathcal{N}} = 0$. This proves the theorem. □

5. Non-Archimedean absolute values on $K(\mathbf{X})$. In the following we deal with non-Archimedean absolute values (multiplicative norms) on $K[\mathbf{X}]$ which extend an absolute value of K.

Let $(K, |\ |)$ be a valued field and $|\ |_L$ an absolute value on $L = K(\mathbf{X})$ which extend $|\ |$. We call $|\ |_L$ a *residual transcendental* (r.t.) extension of $|\ |$ if the residue field $L_{|\ |_L}$ is a transcendental extension of $K_{|\ |}$ of transcendence degree r. We call $|\ |_L$ a *Gauss absolute value* if its restriction to $K[\mathbf{X}]$ is a non-Archimedean Gauss norm. If a Gauss absolute value $|\ |_L$ is defined by $\mathcal{F} = \{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^r}$ and $\mathcal{N} = \{|P_{\mathbf{j}}|_L\}_{\mathbf{j}\in\mathbb{N}^r}$, where $P_{\mathbf{j}} = P_{\mathbf{e}_1}^{j_1} \dots P_{\mathbf{e}_r}^{j_r}$, $P_{\mathbf{e}_i} = X_i - \alpha_i$ and $\alpha_i \in K$, then it is called a *canonical Gauss absolute value*. In this case we denote $|\ |_L = |\ |_{(\alpha_1,\delta_1),\dots,(\alpha_r,\delta_r)}$, where $\delta_i = |X_i - \alpha_i|_L$. For $r = 1$ and $\bar{K}$ a fixed algebraic closure of K, we denote by $|\ |_{\bar{K}}$ a fixed extension of $|\ |$ to $\bar{K}$. If $|\ |_{K(X)}$ is an extension of $|\ |$ to $L = K(X)$, then there exists an extension $|\ |_{\bar{K}(X)}$ of $|\ |_{K(X)}$ to $\bar{K}(X)$ which is also an extension of $|\ |_{\bar{K}}$. Moreover, if $|\ |_{K(X)}$ is an r.t. extension of $|\ |$, then $|\ |_{\bar{K}(X)}$ is an r.t. extension of $|\ |_{\bar{K}}$ and there exist $\alpha \in \bar{K}$ and $\delta \in |\bar{K}^\times|$ such that $|\ |_{\bar{K}(X)} = |\ |_{(\alpha,\delta)}$ is a canonical Gauss absolute value. The pair (α, δ) is called a *pair of definition* for $|\ |_{\bar{K}(X)}$. It is known that two pairs (α_1, δ_1) and (α_2, δ_2) define the same valuation $|\ |_{\bar{K}(X)}$ if and only if

$$\delta_1 = \delta_2 \ \text{ and } \ |\alpha_1 - \alpha_2|_{\bar{K}} \leq \delta_1. \tag{36}$$

By a *minimal pair* (*of definition*) (see [1–3]) for $|\ |_{\bar{K}(X)}$ we mean a pair of definition (α, δ) such that $[K(\alpha) : K]$ is minimal.

PROPOSITION 5. *Let $|\ |_L$ be a residual transcendental extension of $|\ |$. Then there exist polynomials $f_1, \dots, f_r$, with $f_i \in K[X_1, \dots, X_i]$, which are algebraically independent over K, such that the restriction $|\ |_A$ of $|\ |_L$ to $A = K[f_1, \dots, f_r]$ is a Gauss absolute value with $P_{\mathbf{e}_i} = f_i$, for $i = 1, 2, \dots, r$ and $\delta_{\mathbf{j}} = 1$ for every $\mathbf{j}$. Moreover, if $K_1 = K(f_1, \dots, f_r)$ and $|\ |_{K_1}$ is the canonical extension of $|\ |_A$ to K_1, then L is an algebraic extension of K_1 and $|\ |_L$ is an extension of $|\ |_{K_1}$.*

Proof. Consider a transcendence basis $\bar{F}_1, \dots, \bar{F}_r$ of $L_{|\ |_L}$ over $K_{|\ |}$. Since $|\ |_L$ is a residual transcendental extension of $|\ |$ and $K \subset K(X_1) \subset K(X_1, X_2) \subset \dots \subset K(X_1, \dots, X_r)$, we can choose $F_i \in K(X_1, \dots, X_i)$. Then for every i $|F_i|_L = 1$, and if $P = \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{F}^{\mathbf{j}} \in K[\mathbf{F}]$, there exists $a \in K$ such that $aP \in \bar{B}_L(0, 1)$. Hence we may suppose that $P \in \bar{B}_L(0, 1)$ and at least a coefficient of P has absolute value equal to 1. Since

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$\bar{F}_1, \ldots, \bar{F}_r$ are algebraically independent over $K_{|\,|}$ it follows that $|P|_L = \max_{\mathbf{j}} \{|a_{\mathbf{j}}|\}$. Thus $|K_1^\times|_{K_1} = |K^\times|$ and the index e of the subgroup $|K^\times|$ in $|L^\times|_L$ is finite (see **[4]**, Ch.VI, §8, Sec. 1, Lemma 2). Since $F_i = \frac{g_i}{h_i}$ with $g_i, h_i \in K[X_1, \ldots, X_i]$ and $|F_i|_L = 1$ it follows that $|g_i|_L = |h_i|_L$ and $|g_i|_L^e \in |K^\times|$. But $\bar{F}_1^e, \ldots, \bar{F}_r^e$ are algebraically independent over $|\;|_K$. Hence we may suppose $|g_i|_L \in |K^\times|_K$ and there exist elements $b_i \in K^\times$ such that $|g_i|_L = |b_i|$. Thus we can consider $|g_i|_L = |h_i|_L = 1$.

Now we prove that one can replace $F_1, \ldots, F_r$ by polynomials. Since F_1 is transcendental over $K(F_2, \ldots, F_r)$ at least one of g_1 and h_1 is transcendental over $K(F_2, \ldots, F_r)$. Thus we can replace F_1 by a polynomial $f_1 \in K[X_1]$. Since F_2 is transcendental over $K(f_1, F_3, \ldots, F_r)$ we can replace F_2 by a polynomial $f_2 \in K[X_1, X_2]$ and the proposition follows by induction on i. □

Corollary 2. *If, in Proposition 5, K is an algebraically closed valued field, then $|K^\times| = |L^\times|_L$ and we can choose the polynomials f_i to be irreducible for every $i = 1, 2, \ldots, r$.*

Proof. Since, in this case, the group $|K^\times|$ is divisible, it follows that $|K^\times| = |L^\times|_L$. By Proposition 5, $|\;|_L$ is a canonical Gauss absolute value with $\delta_{\mathbf{j}} = 1$. If $f_1 = \prod_{j=1}^{n_1} f_{1,j}$, where $f_{1,j}$ are irreducible polynomials, there exists j_0 such that f_{1,j_0} is transcendental over $K(f_2, \ldots, f_r)$. Hence by multiplying by suitable elements from K, the corollary follows by induction. □

Remark 4. Let $(K, |\;|)$ be an algebraically closed valued field and $r = 1$. If $|\;|_L$ is a non-Archimedean absolute value on $L = K(X)$ which extends $|\;|_K$ and there exists $P_1 \in M^{(1)}$ such that $|P_1|_L = \inf M^{(1)}_{|\;|_L}$, then for all positive j there exists $Q_j \in M^{(j)}$ such that $|Q_j|_L = \inf M^{(j)}_{|\;|_L}, = |P_1|_L^j$ and $|\;|_L$ is a canonical Gauss absolute value defined by $P_j = P_1^j$ and $\delta_j = |P_j|_L$. To prove this statement it is enough to take $Q \in M^{(j)}$. Then $Q = (X - \alpha_1)...(X - \alpha_j)$ with $\alpha_i \in K$. Hence $|Q|_L \geq (\min_{1 \leq i \leq j} |X - \alpha_i|_L)^j \geq |P_1|_L^j$. Since $|P_1^j|_L = |P_1|_L^j$, the remark follows.

Now let $(K, |\;|)$ be a (not necessarily algebraically closed) valued field. We consider a r.t. extension $|\;|_L$ of $|\;|$ to $L = K(\mathbf{X})$ and $|\;|_{L_i}$ the restriction of $|\;|_L$ to $L_i = K(X_1, \ldots, X_i)$, $i = 0, 1, 2, \ldots, r$, with $L_0 = K$ and $L_r = L$. Then $|\;|_{L_{i+1}}$ is a r.t. extension of $|\;|_{L_i}$. Let us denote by $\bar{L}_i$ a fixed algebraic closure of L_i such that

$$\bar{K} \subset \bar{L}_1 \subset \ldots \subset \bar{L},$$

and by $|\;|_{\bar{L}_i}$ a fixed extension of $|\;|_{L_i}$ to $\bar{L}_i$, $i = 0, 1, \ldots, r$.

Theorem 3. *Let $(K, |\;|)$ be a valued field and $|\;|_L$ a r.t. extension of $|\;|$ to $L = K(\mathbf{X})$. Then there exist pairs (α_i, δ_i) with $\alpha_i \in \bar{L}_{i-1}$, $\delta_i \in |\bar{L}_{i-1}^\times|_{\bar{L}_{i-1}}$, $i = 1, 2, \ldots, r$, such that $|\;|_L$ is defined by $|\;|_K$ in the following manner. If $P \in K[\mathbf{X}]$ and $P_j = (X_r - \alpha_r)^j$, then*

$$P = \sum_{j \leq d(P)} b_j (X_r - \alpha_r)^j, b_j \in \bar{L}_{r-1},$$

and

$$|P|_L = \max_{j \leq d(P)} \left\{ |b_j|_{\bar{L}_{r-1}} \delta_r^j \right\}. \tag{37}$$

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Then by using, for each j the minimal polynomial of b_j over L_{r-1} one can compute by (3) *its absolute value* $| \ |_{\bar{L}_{r-1}}$ *by means of* $| \ |_{\bar{L}_{r-2}}, \alpha_{r-1}, \delta_{r-1}$, *and so on.*

Proof. Since $| \ |_L$ is an absolute value it is enough to define it on $K[X_1, \dots, X_r]$. By [**1**], Proposition 1.1 it follows that $| \ |_{\bar{L}_{i+1}}$ is a r.t. extension of $| \ |_{\bar{L}_i}$ and $|\bar{L}_{i+1}^{\times}|_{\bar{L}_{i+1}} = |\bar{L}_i^{\times}|_{\bar{L}_i}$. From Corollary 2, for $i = r$, it follows that $| \ |_{L_{i+1}}$ is defined in (37) by means of $| \ |_{L_i}$ and a pair (α_i, δ_i), where $\alpha_i \in \bar{L}_i$ is the root of an irreducible polynomial P_i of degree 1 and $\delta_i = |P_i|_{\bar{L}_i}$. Now the theorem follows by induction on i. □

COROLLARY 3. *With the hypotheses and notations of Theorem* 3 *there exist* $\beta_{i,j}, \gamma_i \in L_{| \ |_L}$, $i = 1, 2, \dots, r$, $j = 1, 2, \dots, n_i$ *such that the following conditions are satisfied:*

(*a*) $L_{| \ |_L} = K_{| \ |}(\beta_{1,1} \dots, \beta_{1,n_1}, \gamma_1, \beta_{2,1}, \dots, \beta_{2,n_2}, \gamma_2, \dots, \beta_{r,1}, \dots, \beta_{r,n_r}, \gamma_r)$.

(*b*) $\gamma_1, \gamma_2, \dots, \gamma_r$ *are algebraically independent over* $K_{| \ |}$.

(*c*) *For every* i, j, $\beta_{i,j}$ *is an algebraic element over* $K_{| \ |}(\beta_{1,1}, \dots, \beta_{1,n_1}, \gamma_1, \dots, \beta_{i-1,1}, \dots, \beta_{i-1,n_{i-1}}, \gamma_{i-1})$.

(*d*) *The algebraic closure of* $K_{| \ |}$ *in* $L_{| \ |_L}$ *is a finite dimensional extension of* $K_{| \ |}$.

Proof. Since $| \ |_{\bar{L}_{i+1}}$ is a r.t. extension of $| \ |_{\bar{L}_i}$, by [**1**] Corollary 2.3 there exist $\beta_{i+1,1}, \dots, \beta_{i+1,n_{i+1}}, \gamma_{i+1} \in L_{i+1 | \ |_{L_{i+1}}}$ such that $L_{i+1 | \ |_{L_{i+1}}} = L_{i | \ |_{L_i}}(\beta_{i+1,1}, \dots, \beta_{i+1,n_{i+1}}, \gamma_{i+1})$ and γ_{i+1} is transcendental over $L_{i | \ |_{L_i}}$. Now the statements (*a*)–(*c*) follow by induction, and (*d*) holds because (*c*) implies that the algebraic closure of $K_{| \ |}$ in $L_{| \ |_L}$ is a finitely generated extension of $K_{| \ |}$. □

Next, we consider the problem when $L_{| \ |_L}$ is a transcendental extension of a finite algebraic extension of $K_{| \ |}$ (Nagata's problem) in the case $r \geq 2$. We need the following three lemmas.

LEMMA 3. *Let* $(K, | \ |)$ *be a valued field,* $L = K(\mathbf{X})$ *and* $| \ |_L$ *the absolute value defined on* $K[\mathbf{X}]$ *by*

$$\left| \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{X}^{\mathbf{j}} \right|_L = \max_{\mathbf{j}} |a_{\mathbf{j}}|. \tag{38}$$

If X_i^*, $i = 1, 2, \dots, r$ *is the image of* X_i *in* $L_{| \ |_L}$, *then* $X_1^*, \dots, X_r^*$ *are algebraically independent over* $K_{| \ |}$.

Proof. If

$$\sum_{\mathbf{j}} b_{\mathbf{j}}^* \mathbf{X}^{*\mathbf{j}} = 0,$$

where $b_{\mathbf{j}} \in \bar{B}_K(0, 1)$, then

$$\left| \sum_{\mathbf{j}} b_{\mathbf{j}} \mathbf{X}^{\mathbf{j}} \right|_L < 1.$$

By (38), it follows that all $b_{\mathbf{j}} \in B_K(0, 1)$. Hence $b_{\mathbf{j}}^* = 0$ and $X_1^*, \dots, X_r^*$ are algebraically independent over $K_{| \ |}$. □

LEMMA 4. *Let* $(K, | \ |)$ *be a valued field,* $L = K(\mathbf{X})$. *Then there exists a uniquely defined absolute value* $| \ |_L$ *on* $K(\mathbf{X})$ *which extends* $| \ |$ *such that for every i,* $|X_i|_L = 1$ *and*

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$X_1^*, \ldots, X_r^*$ are algebraically independent over $K_{|\,|}$. Moreover

$$|K^\times| = |L^\times|_{|\,|_L} \text{ and } L_{|\,|_L} = K_{|\,|}(\mathbf{X}^*). \tag{39}$$

Proof. The proof is similar to the proof of Proposition 2, Ch.VI, §10 of [**4**]. To show the uniqueness it is enough to show that if $|\ |_L$ is an absolute value on $K[\mathbf{X}]$ which extends $|\ |$ such that for every i, $|X_i|_L = 1$ and $X_1^*, \ldots, X_r^*$ are algebraically independent over $K_{|\,|}$, then it is defined by (38).

Without loss of generality we can consider $P \in K[\mathbf{X}]$ given by (1) such that all $a_{\mathbf{j}} \in \bar{B}_K(0,1)$ and at least one of the coefficients has the absolute value equal to one. Since for every i, $|X_i|_L = 1$, it follows that

$$P^* = \sum_{\mathbf{j}} a_{\mathbf{j}}^* \mathbf{X}^{*\mathbf{j}}.$$

By using the fact that $X_1^*, \ldots, X_r^*$ are algebraically independent over $K_{|\,|}$, we obtain that $P^* \neq 0$ and $|P|_L = 1 = \max_{\mathbf{j}} |a_{\mathbf{j}}|$.

Now we prove the existence of the absolute value $|\ |_L$. It is easy to see that the absolute value defined by (38) extends $|\ |$, for every i, $|X_i|_L = 1$ and $|K^\times| = |L^\times|_{|\,|_L}$. From Lemma 3 it follows that $X_1^*, \ldots, X_r^*$ are algebraically independent over $K_{|\,|}$. To prove that $L_{|\,|_L} = K_{|\,|}(\mathbf{X}^*)$ we consider $R \in L$. Then we can write

$$R = \frac{c \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}}{\sum_{\mathbf{i}} b_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}}, \tag{40}$$

where $c, a_{\mathbf{i}}, b_{\mathbf{i}} \in \bar{B}_K(0,1)$ and at least one of the coefficients $a_{\mathbf{i}}$ and $b_{\mathbf{i}}$ has the absolute value equal to one. Thus $|R|_L = 1$ if and only if $|c| = 1$. In this case

$$R^* = \frac{c^* \sum_{\mathbf{i}} a_{\mathbf{i}}^* \mathbf{X}^{*\mathbf{i}}}{\sum_{\mathbf{i}} b_{\mathbf{i}}^* \mathbf{X}^{*\mathbf{i}}}. \tag{41}$$

and this completes the proof of the lemma. □

LEMMA 5. *Let $(K, |\ |)$ be a valued field. If $|\ |_L = |\ |_{(\alpha_1,\delta_1),\ldots,(\alpha_r,\delta_r)}$, with $\delta_i \in |K^\times|$ is a canonical Gauss absolute value defined on $L = K(\mathbf{X})$, then $K_{|\,|}$ is algebraically closed in $L_{|\,|_L}$.*

Proof. We take $\tau_i \in K^\times$ such that for every i, $|\tau_i| = \delta_i$. Then $|\frac{X_i - \alpha_i}{\tau_i}| = 1$ and every polynomial $P \in K[\mathbf{X}]$ can be written in the form

$$P = \sum_{\mathbf{i}} a_{\mathbf{i}} (\mathbf{X} - \alpha)^{\mathbf{i}} = \sum_{\mathbf{i}} b_{\mathbf{i}} \left(\frac{\mathbf{X} - \alpha}{\tau} \right)^{\mathbf{i}}, \tag{42}$$

where $\left(\frac{\mathbf{X}-\alpha}{\tau}\right)^{\mathbf{i}} = \left(\frac{X_1-\alpha_1}{\tau_1}\right)^{i_1} \ldots \left(\frac{X_r-\alpha_r}{\tau_r}\right)^{i_r}$, $b_{\mathbf{i}} = a_{\mathbf{i}} \tau^{\mathbf{i}}$ and

$$|P|_L = \max_{\mathbf{i}} |b_{\mathbf{i}}|. \tag{43}$$

By Lemma 3 it follows that $(\frac{X_1-\alpha_1}{\tau_1})^* \ldots (\frac{X_r-\alpha_r}{\tau_r})^*$ are algebraically independent over $K_{|\,|}$ and from Lemma 2 we obtain that $L_{|\,|_L} = K_{|\,|}(\frac{\mathbf{X}-\alpha}{\tau})^*$. Hence $K_{|\,|}$ is algebraically closed in $L_{|\,|_L}$. □

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Now we consider a valued field $(K, |\ |)$, $|\ |_{\bar{K}}$ an extension of $|\ |$ to $\bar{K}$ and $|\ |_{(\alpha_1,\delta_1),\ldots,(\alpha_r,\delta_r)}$, with $\alpha_i \in \bar{K}$, a canonical Gauss absolute value on $\bar{K}(\mathbf{X})$. Then the pair (α_1, δ_1) defines a canonical Gauss absolute value on $\bar{K}(X_1) \subset \overline{K(X_1)}$ such that

$$\left|\sum_i b_i(X_1-\alpha_1)^i\right|_{(\alpha_1,\delta_1)} = \max_i \left\{|b_i|_{\bar{K}}\delta_1^i,\ b_i \in \bar{K}\right\}. \tag{44}$$

Similarly, (α_2, δ_2) defines a canonical Gauss absolute value on $\bar{K}(X_1, X_2) = \bar{K}(X_1)(X_2)$ which is an extension of $|\ |_{(\alpha_1,\delta_1)}$ such that

$$\left|\sum_j c_j(X_2-\alpha_2)^j\right|_{(\alpha_2,\delta_2)} = \max_j \left\{|c_j|_{(\alpha_1,\delta_1)}\delta_2^j\right\},\ c_j \in \bar{K}(X_1). \tag{45}$$

Hence

$$\left|\sum_{i,j} a_{ij}(X_1-\alpha_1)^i(X_2-\alpha_2)^j\right|_{(\alpha_1,\delta_1),(\alpha_2,\delta_2)} = \max_j \left\{\left|\sum_i a_{ij}(X_1-\alpha_1)^i\right|_{(\alpha_1,\delta_1)}\delta_2^j\right\}$$

and for every $P \in \bar{K}[X_1, X_2] = \bar{K}[X_1][X_2]$,

$$|P|_{(\alpha_1,\delta_1),(\alpha_2,\delta_2)} = |P|_{(\alpha_2,\delta_2)}. \tag{46}$$

By induction it follows that for every $P \in \bar{K}[X_1, \ldots, X_i] = \bar{K}[X_1, \ldots, X_{i-1}]\,[X_i]$, and for every i,

$$|P|_{(\alpha_1,\delta_1),\ldots,(\alpha_i,\delta_i)} = |P|_{(\alpha_i,\delta_i)}. \tag{47}$$

The following result shows that Nagata's conjectures holds for $r \geq 1$, if $|\ |$ is a canonical Gauss absolute value.

THEOREM 4. *Suppose that $(K, |\ |)$ is a valued field, $L = K(\mathbf{X})$, $|\ |_L$ an absolute value which is the restriction of a canonical Gauss absolute value $|\ |_{(\alpha_1,\delta_1),\ldots,(\alpha_r,\delta_r)}$ on $\bar{K}(\mathbf{X})$ such that:*

(a)$[|L^\times|_L : |K^\times|_K] < \infty$.

(b) For every i, (α_i, δ_i) is a minimal pair of definition for the absolute value $|\ |_{(\alpha_i,\delta_i)}$ defined on $\overline{K(X_1, \ldots, X_{i-1})}(X_i)$.

Then $|\ |_{(\alpha_1,\delta_1),\ldots,(\alpha_r,\delta_r)}$ is a r.t. absolute value on $L(\mathbf{X})$ and there exists a finite algebraic extension K_1 of K such that $K_{1|\ |_{\bar{K}}} \subset L_{|\ |}$ and

$$L_{|\ |_L} = K_{1|\ |_{\bar{K}}}(\mathbf{Y}^*), \tag{48}$$

with $Y_1^*, \ldots, Y_r^* \in L_{|\ |_L}$ algebraically independent over $K_{1|\ |_{\bar{K}}}$.

Proof. We denote $K_1 = K(\alpha_1, \ldots, \alpha_r)$, $n_i = [K(\alpha_1, \ldots, \alpha_i) : K(\alpha_1, \ldots, \alpha_{i-1})]$ and we prove that $K_{1|\ |_{\bar{K}}} \subset L_{|\ |_L}$. If $P = \sum_{\mathbf{i}} a_{\mathbf{i}}(\mathbf{X}-\alpha)^{\mathbf{i}} \in L$ and for every i the degree d_i of $|P|$ with respect to X_i is less than n_i, then by (47) and Theorem 2.1 from **[1]** it follows that

$$|P(\mathbf{X})|_L = |P(X_1, \ldots, X_{r-1}, \alpha_r)|_{(\alpha_{r-1},\delta_{r-1})} = \ldots = |P(\alpha_1, \alpha_2, \ldots \alpha_r)|_{\bar{K}}. \tag{49}$$

Now, if $\gamma \in K_1$ there exists $P \in L$ with $d_i < n_i$ such that $\gamma = P(\alpha)$. Then by (49) it follows that

$$|\gamma|_{\bar{K}} = |P(\alpha)|_{\bar{K}} = |P|_L$$

and $K_{1| |_{\bar{K}}} \subset L_{| |}$.

We show that $K_{1| |_{\bar{K}}}$ is the algebraic closure of $K_{| |}$ in $L_{| |_L}$. We choose q_1 the smallest natural number such that $\delta_1^{q_1} = |\theta_1|_{\bar{K}}$, where $\theta_1 \in K_1$ and we take β_1 a root of the polynomial $Z_1^{q_1} - \theta_1$. Since

$$q_1 \le e(K_1(\beta_1)/K_1) \le [K_1(\beta_1) : K_1] \le q_1,$$

it follows that $f(K_1(\beta_1)/K_1) = 1$. Hence $K_1(\beta_1)_{| |_{\bar{K}}} = K_{1| |_{\bar{K}}}$. Similarly, we choose q_2 the smallest natural number such that $\delta_2^{q_2} = |\theta_2|_{\bar{K}}$, where $\theta_2 \in K_1(\beta_1)$ and we take β_2 a root of the polynomial $Z_2^{q_2} - \theta_2$. Then we obtain $K_1(\beta_1, \beta_2)_{| |_{\bar{K}}} = K_1(\beta_1)_{| |_{\bar{K}}}$ and by induction, for every i,

$$K_1(\beta_1, \ldots, \beta_i)_{| |_{\bar{K}}} = K_1(\beta_1, \ldots, \beta_{i-1})_{| |_{\bar{K}}}. \tag{50}$$

Now, by (50) and Lemma 5 for $M = K_1(\beta_1, \ldots, \beta_r)(\mathbf{X})$, it follows that $K_{1| |_{\bar{K}}} = K_1(\beta_1, \ldots, \beta_r)_{| |_{\bar{K}}}$ is algebraically closed in $M_{| |_{(\alpha_1,\delta_1),\ldots,(\alpha_r,\delta_r)}}$. Then the canonically defined commutative diagram

$$\begin{array}{ccc} K_{| |} & \longrightarrow & L_{| |_L} \\ \downarrow & & \downarrow \\ K_{1| |_{\bar{K}}} & \longrightarrow & M_{| |_M} \end{array}$$

implies that the algebraic closure of $K_{| |}$ in $L_{| |_L}$ is included in $K_{1| |_{\bar{K}}}$. Since $K_{1| |_{\bar{K}}}$ is a finite extension of $K_{| |}$, it follows that $K_{1| |_{\bar{K}}}$ is the algebraic closure of $K_{| |}$ in $L_{| |_L}$.

Finally, we prove (48). Since the multiplicative group G/H, where $G = |L^\times|_L$, $H = |K^\times|$, is generated by the images $\bar{\delta}_1, \ldots, \bar{\delta}_r$ of $\delta_1, \ldots, \delta_r$, from (a) it follows that G/H is a finite commutative group. Hence it is a direct product of cyclic groups:

$$G/H = < g_1 > \times < g_2 > \times \ldots \times < g_r >, \tag{51}$$

where it is possible that some of $g_i = 1$. We denote by o_i the order of g_i. If $P \in K[\mathbf{X}]$ is given by (42), then

$$P = \sum_{\mathbf{i}} b_{\mathbf{i}} \left(\frac{\mathbf{X} - \alpha}{\beta} \right)^{\mathbf{i}}, \tag{52}$$

where $(\frac{\mathbf{X}-\alpha}{\beta})^{\mathbf{i}} = (\frac{X_1-\alpha_1}{\beta_1})^{i_1} \ldots (\frac{X_r-\alpha_r}{\beta_r})^{i_r}$, $b_{\mathbf{i}} = a_{\mathbf{i}}\beta^{\mathbf{i}}$. Since $|\frac{X_i-\alpha_i}{\beta_i}|_{\bar{K}(\mathbf{X})} = 1$ it follows that $|P|_L = 1$ if and only if

$$\max_{\mathbf{i}} \{|a_{\mathbf{i}}\beta^{\mathbf{i}}|_{\bar{K}}\} = \max_{\mathbf{i}} \{|a_{\mathbf{i}}|_{\bar{K}}\delta^{\mathbf{i}}\} = 1. \tag{53}$$

Because, in G/H, $g_i = \overline{\delta}_1^{m(i,1)} \dots \overline{\delta}_r^{m(i,r)}$, then (53) holds if and only if $\overline{\delta}_1^{i_1} \dots \overline{\delta}_r^{i_r} = g_1^{o_1 s_1} \dots g_r^{o_r s_r}$, for each $\mathbf{i}$ such that $|a_{\mathbf{i}}|_{\bar{K}} \delta^{\mathbf{i}} = 1$. If we put $Y_i^* = (\frac{X_1 - \alpha_1}{\beta_1})^{*m(i,1)o_1} \dots (\frac{X_r - \alpha_r}{\beta_r})^{*m(i,r)o_r}$, it follows that for $P \in \bar{B}_L(0,1)$ we have

$$P^* = \sum_{\mathbf{s}} b_{\mathbf{s}}^* \mathbf{Y}^{*\mathbf{s}}, \tag{54}$$

which implies (48). □

REMARK 5. In order to prove that Nagata's conjecture does not hold generally we can take, for an odd prime p, $K = \mathbb{Q}_p$, $|\ | = |\ |_p$ the p-adic absolute value, $L = K(X_1, X_2)$, $|\ |_L$ an absolute value which is the restriction of a Gauss absolute value $|\ |_{(0,1),(\alpha_2,\delta_2)}$ on $\overline{K(X_1)}(X_2)$ such that: $X_1^q + \alpha_2^q = 1$, with q an odd prime different from p, $\delta_2 \notin |K^\times|$ and its order in the group $|L^\times|_L/|K^\times|$ is finite. Then $|\alpha_2|_{\overline{K(X_1)}} = 1$ and by using the notations from the proof of Theorem 4 we find $K_{|\ |} = \mathbb{F}_p$ (the field with p elements) and $L_{|\ |_L} = \mathbb{F}_p(X_1^*, \alpha_2^*, (\frac{X_2 - \alpha_2}{\beta_2})^*)$. Hence it is easy to see that $L_{|\ |_L}$ is not a transcendental extension of a finite extension of $K_{|\ |}$.

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