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The genus-one global mirror theorem for the quintic 3-fold

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Abstract

We prove the genus-one restriction of the all-genus Landau–Ginzburg/Calabi–Yau conjecture of Chiodo and Ruan, stated in terms of the geometric quantization of an explicit symplectomorphism determined by genus-zero invariants. This gives the first evidence supporting the higher-genus Landau–Ginzburg/Calabi–Yau correspondence for the quintic 3-fold, and exhibits the first instance of the 'genus zero controls higher genus' principle, in the sense of Givental's quantization formalism, for non-semisimple cohomological field theories.

1. Introduction

Over the last twenty-five years, there have been a number of important developments that have advanced our understanding of Gromov–Witten (GW) theory. Among these results, the genus-zero mirror theorems have provided closed formulas for the genus-zero GW potentials of a large number of target geometries [Giv98, LLY97, Ber00, CK14], and Teleman's classification theorem for semisimple cohomological field theories [Tel12] has led to explicit formulas for all-genus partition functions in terms of Givental's quantization formula [Giv01a, Giv01b]. One of the most important remaining open problems is to understand the all-genus partition functions of non-semisimple cohomological field theories, of which the GW theory of the quintic 3-fold $X := V(W = x_0^5 + \cdots + x_4^5) \subseteq \mathbb{P}^4$ is the prototypical example.

The Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence, which first arose in the study of string theory [GVW89, VW89, Mar89], suggests an equivalence between the GW theory of a CY hypersurface and the LG model of the defining equation of the hypersurface. The latter model is now mathematically understood in terms of Fan–Jarvis–Ruan–Witten (FJRW) invariants. In the case of the quintic 3-fold, Chiodo and Ruan proved that the genus-zero GW theory of X can be identified with the genus-zero FJRW theory of the polynomial W after analytic continuation and an explicit linear symplectic transformation $\mathbb{U}$ [CR10].

Motivated by Givental's quantization formula, Chiodo and Ruan suggested that the geometric quantization $\widehat{\mathbb{U}}$, which is an explicit differential operator constructed from $\mathbb{U}$, should identify the higher-genus GW and FJRW partition functions after analytic continuation. The genus-zero restriction of their quantization conjecture follows from the fact that $\mathbb{U}$ identifies the genus-zero theories. The main result of this work is the genus-one verification of Chiodo and Ruan's all-genus LG/CY conjecture.

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Main Result Theorem 3.3. *The genus-one potential determined by the action of $\widehat{\mathbb{U}}$ on the Fan–Jarvis–Ruan–Witten partition function of $W = x_0^5 + \cdots + x_4^5$ is equal to the analytic continuation of the genus-one Gromov–Witten potential of the quintic 3-fold $X = V(W)$.*

The theorem is significant for several reasons. First of all, it provides the first evidence for the higher-genus LG/CY correspondence. It has been suggested that the LG model could be instrumental in computing higher-genus GW invariants of the quintic 3-fold, and this theorem lends validity to that approach. Secondly, the theorem gives evidence for a general 'genus zero controls higher genus' principle, in the sense of Givental, in which a correspondence between all-genus partition functions is determined by a genus-zero correspondence through an explicit quantization procedure. While such a principle has been studied extensively and proved in many cases for semisimple cohomological field theories, for example [Tel12, Giv01b, BCR13, Zon15, CI18, HLSW15, IMRS16], this is the first significant evidence for such a principle in the non-semisimple case.

1.1 Plan of the paper

We begin in § 2 by recalling the basic definitions in GW and FJRW theory. We recall some previously known results, including the genus-zero mirror theorems, the genus-zero LG/CY correspondence, and the genus-one mirror theorems. In § 3 we discuss the Birkhoff factorization of the symplectomorphism $\mathbb{U}$ and recall Givental's quantization formulas in order to make Theorem 3.3 precise. We also apply the string and dilaton equations to reduce the main theorem to the one-parameter 'small state space'. In § 4 we provide a proof of the genus-zero restriction of the quantization conjecture, mostly in order to set up notation for the genus-one correspondence. The proof of the genus-one correspondence occupies §§ 5, 6, and 7, where we carefully analyze the vertex- and loop-type graphs that appear in the quantization formula.

2. Recapitulation of global mirror symmetry for the quintic 3-fold

In this section we review the basic setup of GW and FJRW invariants, and we recall previously known mirror theorems concerning the genus-zero and genus-one invariants.

2.1 Review of Gromov–Witten theory

Let X denote the Fermat quintic 3-fold,

$$X := V(x_0^5 + \cdots + x_4^5) \subset \mathbb{P}^4,$$

and let $\overline{\mathcal{M}}_{g,n}(X,d)$ denote the moduli space of n-pointed, genus-g, degree-d stable maps to X. GW invariants of X encode virtual intersection numbers

$$\langle \alpha_1\psi^{k_1}\cdots\alpha_n\psi^{k_n}\rangle^{\mathrm{CY}}_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i)\psi_i^{k_i}, \tag{1}$$

where $\alpha_i \in H^*(X,\mathbb{C})$, $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,d) \to X$ is the ith evaluation map, ψ_i is the descendent cotangent-line class, and $[-]^{\mathrm{vir}}$ is the virtual fundamental class. The correlators defined in (1) are multilinear and symmetric. For the purposes of this paper, we focus on the *ambient sector* $H^{\mathrm{CY}} \subset H^*(X,\mathbb{C})$ of the state space, obtained by restricting the cohomological insertions to the image of the restriction map:

$$H^{\mathrm{CY}} := \mathrm{Im}(H^*(\mathbb{P}^4,\mathbb{C}) \to H^*(X,\mathbb{C})).$$

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A natural basis for H^{CY} is given by $\{\varphi_0, \ldots, \varphi_3\}$, where φ_m is the pullback of $c_1(\mathcal{O}(1))^m$ under the inclusion $X \hookrightarrow \mathbb{P}^4$. The genus-$g$ GW potential is defined by

$$F_g^{\mathrm{CY}}(\mathbf{t}) := \sum_{n,d} \frac{1}{n!} \langle \mathbf{t}(\psi)^n \rangle_{g,n,d}^{\mathrm{CY}},$$

where

$$\mathbf{t}(z) = \sum_{\substack{k \geqslant 0 \\ 0 \leqslant m \leqslant 3}} t_k^m \varphi_m z^k.$$

We view the set of variables $\mathbf{t} = \{t_k^m\}$ as formal parameters,[1] and we write $\mathbf{t}(z)$ when we want to emphasize the role of z. The sum is taken over all indices for which the underlying moduli space is non-empty. The GW partition function is defined by

$$\mathcal{D}^{\mathrm{CY}}(\mathbf{t}, \hbar) = \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} F_g^{\mathrm{CY}}(\mathbf{t}) \right).$$

Following Givental [Giv04], we define an infinite-dimensional vector space

$$\mathcal{H}^{\mathrm{CY}} := H^{\mathrm{CY}}((z^{-1}))$$

with symplectic form

$$\Omega^{\mathrm{CY}}(f(z), g(z)) = \mathrm{Res}_{z=0}(f(z), g(-z))^{\mathrm{CY}},$$

where $(-,-)^{\mathrm{CY}}$ denotes the Poincaré pairing on X. Let $(\mathbf{q}, \mathbf{p})$ be the Darboux coordinates on $\mathcal{H}^{\mathrm{CY}}$ with respect to the basis $\varphi_m z^k$, so that a general element of $\mathcal{H}^{\mathrm{CY}}$ can be written

$$\sum_{\substack{k \geqslant 0 \\ 0 \leqslant m \leqslant 3}} q_k^m \varphi_m z^k + \sum_{\substack{k \geqslant 0 \\ 0 \leqslant m \leqslant 3}} p^{m,k} \varphi^m (-z)^{-k-1},$$

where φ^m is Poincaré dual to φ_m. Viewing $F_0(\mathbf{t})$ as a formal function on $\mathcal{H}_+^{\mathrm{CY}} := H^{\mathrm{CY}}[z]$ via the dilaton shift

$$\mathbf{t}(z) = \mathbf{q}(z) + \varphi_0 z,$$

the genus-zero GW invariants are encoded in a Lagrangian subspace $\mathcal{L}^{\mathrm{CY}}$, defined as the graph of the differential of F_0^{CY}:

$$\mathcal{L}^{\mathrm{CY}} := \left\{ p^{m,k} = \frac{\partial F_0^{\mathrm{CY}}(\mathbf{t})}{\partial q_k^m} \right\} \subset \mathcal{H}^{\mathrm{CY}}.$$

A general point of $\mathcal{L}^{\mathrm{CY}}$ has the form

$$J^{\mathrm{CY}}(\mathbf{t}, -z) := -z\varphi_0 + \mathbf{t}(z) + \sum_{n,d,m} \frac{1}{n!} \left\langle \mathbf{t}(\psi)^n \frac{\varphi_m}{-z-\psi} \right\rangle_{0,n,d}^{\mathrm{CY}} \varphi^m.$$

[1] Typically, one introduces an additional Novikov parameter to keep track of the degree d. However, the divisor equation implies that the Novikov parameter and t_0^1 are redundant, allowing us to omit the Novikov parameter in our discussion.

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Givental proved that $\mathcal{L}^{\mathrm{CY}}$ is a cone centered at the origin, and that every tangent space T is tangent to $\mathcal{L}^{\mathrm{CY}}$ exactly along zT. In particular, $\mathcal{L}^{\mathrm{CY}}$ (and hence, the totality of genus-zero GW invariants) is determined by the finite-dimensional slice

$$J^{\mathrm{CY}}(t,-z) = -z\varphi_0 + t + \sum_{n,d,m} \frac{1}{n!}\Big\langle t^n \frac{\varphi_m}{-z-\psi}\Big\rangle^{\mathrm{CY}}_{0,n,d} \varphi^m,$$

where $t = \sum_{0\leqslant m\leqslant 3} t^m \varphi_m$. The properties of the cone imply that

$$\mathcal{L}^{\mathrm{CY}} = \bigg\{\sum_r c_r(t,z) S^{\mathrm{CY}}(t,z)^*(\varphi_r) : c_r(t,z) \in \mathcal{H}_+^{\mathrm{CY}}\bigg\},$$

where[2]

$$S^{\mathrm{CY}}(t,z)^*(\varphi_r) = \frac{\partial J^{\mathrm{CY}}(t,-z)}{\partial t^r} = \varphi_r + \sum_{n,d,m} \frac{1}{n!}\Big\langle \phi_r t^n \frac{\varphi_m}{-z-\psi}\Big\rangle^{\mathrm{CY}}_{0,n,d} \varphi^m.$$

In the particular case of the quintic 3-fold X, even more is true. It follows from dimension arguments along with the string and dilaton equations that $\mathcal{L}^{\mathrm{CY}}$ is, in fact, determined by the one-dimensional slice along the *small state space* $t = \tau\varphi_1$:

$$J^{\mathrm{CY}}(\tau\varphi_1,-z) = -z\varphi_0 + \tau\varphi_1 + \sum_{n,d,m} \frac{1}{n!}\Big\langle (\tau\varphi_1)^n \frac{\varphi_m}{-z-\psi}\Big\rangle^{\mathrm{CY}}_{0,n+1,d} \varphi^m.$$

By a slight abuse of notation, we often drop φ_1 in the notation when we restrict to the small state space: $J^{\mathrm{CY}}(\tau,-z) := J^{\mathrm{CY}}(\tau\varphi_1,-z)$.

2.2 Review of Fan–Jarvis–Ruan–Witten theory

Let $\overline{\mathcal{M}}^{1/5}_{g,\vec{m}}$ denote the moduli space of stable 5-spin curves with n orbifold marked points having multiplicities $\vec{m} = (m_1,\ldots,m_n)$. More precisely, a point in $\overline{\mathcal{M}}^{1/5}_{g,\vec{m}}$ parameterizes a tuple $(C, q_1,\ldots,q_n, L, \kappa)$ where

- $(C, q_1,\ldots,q_n)$ is a stable orbifold curve with μ_5 orbifold structure at all marks and nodes;
- L is an orbifold line bundle on C and the μ_5-representation $L|_{q_i}$ is multiplication by $\mathrm{e}^{2\pi \mathrm{i} m_i/5}$;
- κ is an isomorphism
$$\kappa : L^{\otimes 5} \cong \omega_{C,\log}.$$

The (narrow) FJRW invariants of the quintic 3-fold encode the intersection numbers

$$\langle \phi_{m_1}\psi^{k_1}\cdots\phi_{m_n}\psi^{k_n}\rangle^{\mathrm{LG}}_{g,n} := 5^{2-2g}\int_{[\overline{\mathcal{M}}^{1/5}_{g,\vec{m}+\vec{1}}]^{\mathrm{vir}}} \prod_{i=1}^n \psi_i^{k_i}, \tag{2}$$

where ψ_i is the ith cotangent line class on the coarse curve, and $[-]^{\mathrm{vir}}$ is the fifth power of the Witten class associated to the quintic 3-fold.[3] By convention, the correlators (2) vanish if $m_i = 4$ for any i. We let H^{LG} denote the *narrow* state space, which is the complex vector

[2] The asterisk in the notation refers to the fact that $S^{\mathrm{CY}}(t,z)^*$ is the adjoint of a fundamental solution of the Dubrovin connection.

[3] The sign convention we use for the Witten class agrees with the original construction of Fan, Jarvis, and Ruan [FJR13].

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space generated by the formal symbols $\phi_0, \ldots, \phi_3$ and with a non-degenerate pairing defined by $(\phi_i, \phi_j)^{\mathrm{LG}} = 5\delta_{i+j=3}$.[4]

Analogously to GW theory, we define formal generating series $F_g^{\mathrm{LG}}(\mathbf{t})$ and $\mathcal{D}^{\mathrm{LG}}(\mathbf{t})$, we define a vector space $\mathcal{H}^{\mathrm{LG}}$ with symplectic form Ω^{LG}, and we define a Lagrangian subspace $\mathcal{L}^{\mathrm{LG}} \subset \mathcal{H}^{\mathrm{LG}}$ which is determined by the slice $J^{\mathrm{LG}}(t, -z)$ via the derivatives $S^{\mathrm{LG}}(t, z)^*$. As in GW theory, the totality of genus-zero FJRW invariants are determined by the one-dimensional slice

$$J^{\mathrm{LG}}(\tau, -z) = -z\phi_0 + \tau\phi_1 + \sum_{n,d,m} \frac{1}{n!}\Big\langle (\tau\phi_1)^n \frac{\phi_m}{-z-\psi}\Big\rangle_{0,n,d}^{\mathrm{LG}} \phi^m.$$

2.3 Genus-zero mirror theorems and the Landau–Ginzburg/Calabi–Yau correspondence

Define I-functions $I^{\mathrm{CY}}(\mathfrak{q}, z) \in \mathcal{H}^{\mathrm{CY}}$ and $I^{\mathrm{LG}}(\mathfrak{t}, z) \in \mathcal{H}^{\mathrm{LG}}$ by

$$I^{\mathrm{CY}}(\mathfrak{q}, z) := z \sum_{d \geqslant 0} \mathfrak{q}^{\varphi_1/z+d} \frac{\prod_{k=1}^{5d}(5\varphi_1 + kz)}{\prod_{k=1}^{d}(\varphi_1 + kz)^5},$$

where $\varphi_1^k := \varphi_k$ and $\varphi_4 = 0$, and

$$I^{\mathrm{LG}}(\mathfrak{t}, z) := z \sum_{a \geqslant 0} \frac{\mathfrak{t}^a}{z^a a!} \prod_{\substack{0<k<(a+1)/5 \\ \langle k\rangle = \langle (a+1)/5\rangle}} (kz)^5 \phi_a,$$

where $\phi_4 = 0$.[5]

The leading z-coefficients of the I-functions are especially important:

$$I^{\mathrm{CY}}(\mathfrak{q}, z) =: I_0^{\mathrm{CY}}(\mathfrak{q})\varphi_0 z + I_1^{\mathrm{CY}}(\mathfrak{q})\varphi_1 + \mathcal{O}(z^{-1})$$

and

$$I^{\mathrm{LG}}(\mathfrak{t}, z) =: I_0^{\mathrm{LG}}(\mathfrak{t})\phi_0 z + I_1^{\mathrm{LG}}(\mathfrak{t})\phi_1 + \mathcal{O}(z^{-1}).$$

The genus-zero mirror theorems, conjectured by Candelas, de la Ossa, Green, and Parkes [CdlOGP91] in the GW setting and Huang, Klemm, and Quackenbush [HKQ08] in the FJRW setting, provide an explicit solution to genus-zero GW and FJRW invariants in terms of the respective I-functions.

THEOREM 2.1 (Givental [Giv98], Lian, Liu, and Yau [LLY97]). *Setting*

$$\tau^{\mathrm{CY}} = \frac{I_1^{\mathrm{CY}}(\mathfrak{q})}{I_0^{\mathrm{CY}}(\mathfrak{q})},$$

we have

$$J^{\mathrm{CY}}(\tau^{\mathrm{CY}}, z) = \frac{I^{\mathrm{CY}}(\mathfrak{q}, z)}{I_0^{\mathrm{CY}}(\mathfrak{q})}.$$

[4] This pairing is different from the standard pairing in FJRW theory that was defined in [FJR13], but it is consistent with our previous work [GR19] and matches better with the pairing in GW theory.

[5] We warn the reader that the LG I-function defined here differs from the I-function defined in [CR10] by a factor of $\mathfrak{t}$. This keeps the notation consistent with our previous work [GR19].

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Theorem 2.2 (Chiodo and Ruan [CR10]). *Setting*

$$\tau^{\mathrm{LG}} = \frac{I_1^{\mathrm{LG}}(\mathfrak{t})}{I_0^{\mathrm{LG}}(\mathfrak{t})},$$

we have

$$J^{\mathrm{LG}}(\tau^{\mathrm{LG}}, z) = \frac{I^{\mathrm{LG}}(\mathfrak{t}, z)}{I_0^{\mathrm{LG}}(\mathfrak{t})}.$$

Chiodo and Ruan also studied the relationship between the respective I-functions. They proved the following theorem, which verifies the genus-zero LG/CY correspondence for the quintic 3-fold.

Theorem 2.3 (Chiodo and Ruan [CR10]). *Define a linear transformation* $\mathbb{U}(-z) : \mathcal{H}^{\mathrm{LG}} \to \mathcal{H}^{\mathrm{CY}}$ *by*

$$\mathbb{U}(-z)(\phi_m) = \frac{\xi^{m+1}}{\mathrm{e}^{-2\pi\mathrm{i}\varphi_1/z} - \xi^{m+1}} \frac{-2\pi\mathrm{i}(-z)^m}{\Gamma(1+5\varphi_1/z)} \frac{\Gamma^5(1+\varphi_1/z)}{\Gamma^5(1-(m+1)/5)}.$$

Then $\mathbb{U}(z)$ *is symplectic and, upon identifying* $\mathfrak{q}^{-1} = \mathfrak{t}^5$, *there exists an analytic continuation of* $I^{\mathrm{CY}}(\mathfrak{q}, z)$ *such that*

$$\mathbb{U}(z)(\mathfrak{t} I^{\mathrm{LG}}(\mathfrak{t}, -z)) = 5\widetilde{I}^{\mathrm{CY}}(\mathfrak{t}, -z).$$

From the discussion above, it follows that Theorem 2.3 can be rephrased as the statement that the symplectomorphism $\mathbb{U}(z)$ identifies Givental's Lagrangian cones upon analytic continuation. Following ideas due to Givental, Chiodo and Ruan wrote in [CR10] that

> the quantization $\widehat{\mathbb{U}}$ is a differential operator which we expect to yield the full higher genus Gromov–Witten partition function when applied to the full higher genus Fan–Jarvis–Ruan–Witten partition function.

In other words, Chiodo and Ruan conjectured that the higher-genus LG/CY correspondence can be formulated as an explicit relationship, depending only on genus-zero data, between the GW and FJRW partition functions. In § 3 below, we make this conjecture more explicit, and we give a precise statement of our main result, which proves the genus-one part of their conjecture.

2.4 Genus-one mirror theorems

Our proof of the genus-one LG/CY correspondence relies on the genus-one mirror theorems. In GW theory, the genus-one mirror theorem was conjectured by Bershadsky, Cecotti, Ooguri, and Vafa [BCOV94] and originally proved by Zinger [Zin09] (by combining the results in Kim and Lho [KL18] and Ciocan, Fontanine, and Kim [CFK16], there is also a new proof using quasimap techniques).

Theorem 2.4 (Zinger [Zin09]). *Setting*

$$\tau^{\mathrm{CY}} = \frac{I_1^{\mathrm{CY}}(\mathfrak{q})}{I_0^{\mathrm{CY}}(\mathfrak{q})},$$

we have

$$F_1^{\mathrm{CY}}(\tau^{\mathrm{CY}}) = \log\left(I_0^{\mathrm{CY}}(\mathfrak{q})^{-31/3}\mathfrak{q}^{-25/12}(1-5^5\mathfrak{q})^{-1/12}\left(\mathfrak{q}\frac{d\tau^{\mathrm{CY}}}{d\mathfrak{q}}\right)^{-1/2}\right).$$

In FJRW theory, the genus-one mirror theorem was conjectured by Huang, Klemm, and Quackenbush [HKQ08] and proved by the authors [GR19].

THEOREM 2.5 (Guo and Ross [GR19]). *Setting*

$$\tau^{\mathrm{LG}} = \frac{I_1^{\mathrm{LG}}(\mathfrak{t})}{I_0^{\mathrm{LG}}(\mathfrak{t})},$$

we have

$$F_1^{\mathrm{LG}}(\tau^{\mathrm{LG}}) = \log\left(I_0^{\mathrm{LG}}(\mathfrak{t})^{-31/3}(1-(\mathfrak{t}/5)^5)^{-1/12}\left(\frac{d\tau^{\mathrm{LG}}}{d\mathfrak{t}}\right)^{-1/2}\right).$$

3. Birkhoff factorization and geometric quantization

In order to make the higher-genus LG/CY correspondence more explicit, we write the linear transformation $\mathbb{U}(z)$ as a matrix in the bases $\{\phi_m\}$ and $\{\varphi_m\}$. Following Coates and Ruan [CR13], we consider the Birkhoff factorization of the matrix $\mathbb{U}(z)$,

$$\mathbb{U}(z) = \mathbb{U}_-\mathbb{U}_0\mathbb{U}_+,$$

where $\mathbb{U}_- = 1 + \mathcal{O}(z^{-1})$ is upper triangular, $\mathbb{U}_+ = 1 + \mathcal{O}(z)$ is lower triangular, and $\mathbb{U}_0$ is a diagonal matrix that is constant in z. By analogy with Givental [Giv01a], we define

$$S^{-1}(z) = \mathbb{U}_-(z)$$

and

$$R(z) = \mathbb{U}_0\mathbb{U}_+(z)\mathbb{U}_0^{-1},$$

so that

$$\mathbb{U}(z) = S^{-1}(z)R(z)\mathbb{U}_0.$$

We view R and S as linear automorphisms of $\mathcal{H}^{\mathrm{CY}}$, and $\mathbb{U}_0$ as a linear identification of $\mathcal{H}^{\mathrm{LG}}$ and $\mathcal{H}^{\mathrm{CY}}$. Since $\mathbb{U}$ is symplectic (i.e. $\mathbb{U}(z)\mathbb{U}(-z)^* = 1$, where the asterisk denotes adjoint), it is not hard to see that S, R, and $\mathbb{U}_0$ are also symplectic:

$$S(z)S(-z)^* = R(z)R(-z)^* = \mathbb{U}_0\mathbb{U}_0^* = 1.$$

Consider the geometric quantizations $\widehat{R}$, $\widehat{S}^{-1}$, and $\widehat{\mathbb{U}}_0$, defined, for example, in [Giv01a]. These are differential operators, which can be computed explicitly by the following result.

THEOREM 3.1 (Givental [Giv01a]). *Let* $\mathbf{q}(z) = q_k^m\varphi_m z^k$ *be coordinates on* $\mathcal{H}_+^{\mathrm{CY}}$*. Given a partition function* $\mathcal{D}(\mathbf{q})$ *on* $\mathcal{H}_+^{\mathrm{CY}}$*, the quantized operators act as follows.*

(i) *The quantization of* $\mathbb{U}_0$ *acts by*

$$\widehat{\mathbb{U}}_0\,\mathcal{D}(\mathbf{q}) = \mathcal{D}(\mathbb{U}_0^{-1}\mathbf{q}).$$

(ii) *The quantization of* S^{-1} *acts by*

$$\widehat{S^{-1}}\,\mathcal{D}(\mathbf{q}) = e^{W(\mathbf{q},\mathbf{q})/2\hbar}\mathcal{D}([S\mathbf{q}]_+),$$

where $[S\mathbf{q}]_+$ *is the power series truncation of* $S(z)\mathbf{q}(z)$ *and the quadratic form* $W(\mathbf{q},\mathbf{q}) = \sum_{k,l}(W_{kl}q_k, q_l)^{\mathrm{CY}}$ *is defined by*

$$\sum_{k,l\geqslant 0}\frac{W_{kl}}{w^k z^l} := \frac{S(w)^*S(z)-1}{w^{-1}+z^{-1}}.$$

(iii) *The quantization of R acts by*

$$\widehat{R}\,\mathcal{D}(\mathbf{q}) = [e^{(\hbar/2)V(\partial/\partial_{\mathbf{q}},\partial/\partial_{\mathbf{q}})}\mathcal{D}](R^{-1}\mathbf{q}),$$

where $R^{-1}\mathbf{q}$ is the power series $R^{-1}(z)\mathbf{q}(z)$ and the quadratic form $V = \sum_{k,l}(p_k, V_{kl}p_l)^{\mathrm{CY}}$ is defined by

$$V(w,z) = \sum_{k,l\geqslant 0} V_{kl}w^k z^l = \frac{1 - R(-w)^* R(-z)}{w+z}.$$

When a partition function is written in the coordinates $\mathbf{t}(z)$, we apply the formulas in Theorem 3.1 by first identifying $\mathbf{t}(z)$ and $\mathbf{q}(z)$ via the *dilaton shift*:

$$\mathbf{q}(z) = \mathbf{t}(z) - \Phi_0 z,$$

where $\Phi_0 = \varphi_0$ or ϕ_0 depending on the context. To simplify notation, we introduce the convention

$$\overline{\mathcal{D}}(\mathbf{q}) = \mathcal{D}(\mathbf{t}),$$

where $\mathbf{q}$ and $\mathbf{t}$ are related by the dilaton shift. It is important to notice that, even though we might start with a partition function that is a formal series centered at $\mathbf{t}(z) = 0$, the outcome of acting by the quantized operator may be divergent at $\mathbf{t}(z) = 0$.

The Chiodo–Ruan conjecture can be stated more explicitly in the following form.

Conjecture 3.2 (Chiodo and Ruan [CR10]). There exists an analytic continuation of $\mathcal{D}^{\mathrm{CY}}$ such that

$$\widetilde{\mathcal{D}^{\mathrm{CY}}(\mathbf{t})} \propto \widehat{S^{-1}}\,\widehat{R}\,\widehat{\mathbb{U}}_0\,\mathcal{D}^{\mathrm{LG}}(\mathbf{t}),$$

where the symbol '$\propto$' denotes equivalence up to a scalar multiple.

The main result of this paper is the following partial verification of Conjecture 3.2.

Theorem 3.3. *Conjecture 3.2 holds for the genus-zero and genus-one potentials. In other words, there exist an analytic continuation and a constant c such that, for $g \leqslant 1$,*

$$[\hbar^{g-1}]\log(\widetilde{\mathcal{D}^{\mathrm{CY}}(\mathbf{t})}) = [\hbar^{g-1}]\log(\widehat{S^{-1}}\,\widehat{R}\,\widehat{\mathbb{U}}_0\,\mathcal{D}^{\mathrm{LG}}(\mathbf{t})) + \delta_{g,1}c.$$

Remark 3.4. In order to interpret the analytic continuation, we consider both sides as formal power series in the variables $\{t_k^m : (k,m) \neq (0,1)\}$ with coefficients that are analytic in t_0^1, and we analytically continue coefficient by coefficient. Implicit in Conjecture 3.2 is the claim that both sides are analytic in t_0^1. The question of whether genus-g potentials are analytic is open in general. We verify the necessary convergence of genus-zero and genus-one potentials throughout the course of our arguments.

3.1 Quantized operators, potential functions, and graph sums

In order to investigate Theorem 3.3, we consider intermediate partition functions

$$\overline{\mathcal{D}}^A(\mathbf{q}) := \widehat{\mathbb{U}}_0\,\overline{\mathcal{D}}^{\mathrm{LG}}(\mathbf{q}),$$
$$\overline{\mathcal{D}}^B(\mathbf{q}) := \widehat{R}\,\overline{\mathcal{D}}^A(\mathbf{q}),$$
$$\overline{\mathcal{D}}^C(\mathbf{q}) := \widehat{S^{-1}}\,\overline{\mathcal{D}}^B(\mathbf{q}).$$

Notice that $\overline{\mathcal{D}}^{\mathrm{LG}}(\mathbf{q})$ is centered at $\mathbf{q}(z)=-\phi_0 z$ while $\overline{\mathcal{D}}^{A}(\mathbf{q})$, $\overline{\mathcal{D}}^{B}(\mathbf{q})$, and $\overline{\mathcal{D}}^{C}(\mathbf{q})$ are centered at $\mathbf{q}(z)=-\mathbb{U}_0\phi_0 z$, $\mathbf{q}(z)=-R(z)\mathbb{U}_0\phi_0 z$, and $\mathbf{q}(z)=-[\mathbb{U}(z)\phi_0 z]_+$, respectively. For each partition function, we can write

$$\overline{\mathcal{D}}^{\bullet}(\mathbf{q}) =: \mathrm{e}^{\sum_{g\geqslant 0}\hbar^{g-1}\overline{F}_g^{\bullet}(\mathbf{q})}.$$

Theorem 3.1 implies that the $\mathbb{U}_0$-action is a change of variables:

$$\overline{\mathcal{D}}^{A}(\mathbf{q}) = \overline{\mathcal{D}}^{\mathrm{LG}}(\mathbb{U}_0^{-1}\mathbf{q}) \quad\Longrightarrow\quad \overline{F}_g^{A}(\mathbf{q}) = \overline{F}_g^{\mathrm{LG}}(\mathbb{U}_0^{-1}\mathbf{q}).$$

The R-action is more interesting. We have

$$\begin{aligned}\sum_{g\geqslant 0}\hbar^{g-1}\overline{F}_g^{B}(\mathbf{q}) &= \log(\overline{\mathcal{D}}^{B}(\mathbf{q}))\\ &= \log([e^{(\hbar/2)V(\partial/\partial_{\mathbf{q}},\partial/\partial_{\mathbf{q}})}\overline{\mathcal{D}}^{A}](R^{-1}\mathbf{q})).\end{aligned} \tag{3}$$

The action of the exponential of the quadratic differential operator in (3) has a Feynman graph expansion, and the logarithm outputs only the connected graphs. Let Γ denote a connected graph consisting of vertices V, edges E, and legs L, with each vertex v labeled by a genus g_v. For each v, let $\mathrm{val}(v)$ be the total number of legs and edges adjacent to v, define $g(\Gamma)=b_1(\Gamma)+\sum_v g_v$ where b_1 denotes the first Betti number of the graph, let $F=\{v,e\}$ denote the set of flags, and let F_v and L_v denote the flags and legs adjacent to a vertex v. We have

$$\overline{F}_g^{B}(\mathbf{q}) = \sum_{\Gamma:g(\Gamma)=g}\frac{1}{|\mathrm{Aut}(\Gamma)|}\underline{\mathrm{Contr}}(\Gamma),$$

where

$$\underline{\mathrm{Contr}}(\Gamma) = \mathrm{Res}_{z_f=0}\prod_v \underline{\mathrm{Contr}}(v)\prod_e \underline{\mathrm{Contr}}(e),$$

with vertices contributing

$$\underline{\mathrm{Contr}}(v) = \Bigg(\sum_{m_f,k_f}\Bigg(\prod_{f\in F_v}\frac{\varphi^{m_f}}{z^{k_f+1}}\otimes\frac{\partial}{\partial q_{k_f}^{m_f}}\Bigg)\overline{F}_g^{A}(\mathbf{q})\Bigg)_{\mathbf{q}(z)\to R^{-1}\mathbf{q}(z)}$$

contracted along the edges by pairing with the 2-tensor

$$\underline{\mathrm{Contr}}(e) = V(z_f,z_{f'}) = \sum_{m,m'}V(z_f,z_{f'})_{m,m'}\varphi_m\otimes\varphi^{m'}.$$

Including the S-action, we have

$$\overline{F}_g^{C}(\mathbf{q}) = \delta_{g,0}\hbar^{-1}W(\mathbf{q},\mathbf{q})/2 + \sum_{\Gamma:g(\Gamma)=g}\frac{1}{|\mathrm{Aut}(\Gamma)|}\mathrm{Res}_{z_f=0}\prod_v \mathrm{Contr}(v)\prod_e \mathrm{Contr}(e),$$

where $\mathrm{Contr}(e)=\underline{\mathrm{Contr}}(e)$, but we replace the vertex contributions with

$$\mathrm{Contr}(v) = \sum_{m_f}\Bigg\langle\prod_{l\in L_v}(\overline{\mathbf{q}}(\psi_l)+\varphi_0\psi_l)\prod_{f\in F_v}\frac{\mathbb{U}_0^{-1}\varphi_{m_f}}{z_f-\psi_f}\Bigg\rangle^{\mathrm{LG}}_{g_v,\mathrm{val}(v)}\bigotimes_f\varphi^{m_f},$$

where

$$\overline{\mathbf{q}}(z) := \mathbb{U}_0^{-1}R^{-1}(z)[S(z)\mathbf{q}(z)]_+.$$

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3.2 String and dilaton equations

In this section, we show that the dilaton and string equations commute with quantization, allowing us to reduce Conjecture 3.2 to the small state space.

The dilaton equation asserts that, for $\bullet = \mathrm{CY}$ or LG and $\Phi_m = \varphi_m$ or ϕ_m, we have

$$\langle \Phi_0\psi\Phi_{m_1}\psi^{k_1}\cdots\Phi_{m_n}\psi^{k_n}\rangle^{\bullet}_{g,n+1,(d)} = (2g-2+n)\langle \Phi_{m_1}\psi^{k_1}\cdots\Phi_{m_n}\psi^{k_n}\rangle^{\bullet}_{g,n,(d)},$$

whenever the moduli space on the right-hand side exists. The string equation asserts that

$$\langle \Phi_0\Phi_{m_1}\psi^{k_1}\cdots\Phi_{m_n}\psi^{k_n}\rangle^{\bullet}_{g,n+1,(d)} = \sum_{i=1}^{n}\langle \Phi_{m_1}\psi^{k_1}\cdots\Phi_{m_i}\psi^{k_i-1}\cdots\Phi_{m_n}\psi^{k_n}\rangle^{\bullet}_{g,n,(d)},$$

whenever the moduli space on the right-hand side exists. We interpret $\psi^{-1}=0$. In addition, by a virtual dimension count, the correlator $\langle \Phi_{m_1}\psi^{k_1}\cdots\Phi_{m_n}\psi^{k_n}\rangle^{\bullet}_{g,n,(d)}$ vanishes unless $\sum m_i + \sum k_i = n$. Using this vanishing, it is not hard to see that $\mathcal{D}^{\bullet}(\mathbf{t})$ can be reconstructed from its restriction to $\mathbf{t}(z) = t_0^1\Phi_1$ by the dilaton and string equations and the initial conditions

$$\langle \Phi_a\Phi_b\Phi_0\rangle^{\bullet}_{0,3,(0)} = 5\delta_{a+b,3} \quad \text{and} \quad \langle \Phi_0\psi\rangle_{1,1,(0)} = -\tfrac{25}{3}. \tag{4}$$

The first initial condition in (4) can be computed directly from the definitions, while the second can be computed using the CohFT axioms (see, for example, [GR19, § 7.6]).

It is useful to rephrase the string and dilaton equations as differential operators. In terms of total descendent potentials, the dilaton equation can be rewritten as

$$\bigg(\sum_{m,k} q_k^m \frac{\partial}{\partial q_k^m} + 2\hbar\frac{\partial}{\partial \hbar} - \frac{25}{3}\bigg)\mathcal{D}^{\bullet}(\mathbf{t}) = 0, \tag{5}$$

and it is well known (see, for example, [Coa03, Example 1.3.3.2]) that the string equation takes the form

$$\widehat{1/z}\,\mathcal{D}^{\bullet}(\mathbf{t}) = 0. \tag{6}$$

Moreover, the equations (5) and (6) take into account the initial conditions (4), and thus determine $\mathcal{D}^{\bullet}(\mathbf{t})$ uniquely from its restriction to $\mathbf{t}(z) = t_0^1\Phi_1$. The following compatibility is important in order to reduce Conjecture 3.2 to the small state space.

Lemma 3.5. *The formal series $\widehat{S^{-1}}\widehat{R}\,\widehat{\mathbb{U}}_0\,\overline{\mathcal{D}}^{\mathrm{LG}}(\mathbf{q})$ centered at $\mathbf{q}(z) = -[\mathbb{U}(z)\phi_0 z]_+$ satisfies the dilaton equation (5) and the string equation (6).*

Proof. We start with the dilaton equation. We must prove

$$\bigg(\sum_{m,k} q_k^m \frac{\partial}{\partial q_k^m} + 2g-2\bigg)\overline{F}^C_g(\mathbf{q}) = \delta_{g,1}\frac{25}{3}. \tag{7}$$

First of all, notice that the genus-zero shift $W(\mathbf{q},\mathbf{q})/2$ is annihilated by the operator in (7), simply because it is homogenous of degree 2 in $\mathbf{q}$. Next, notice that

$$\sum_{m,k} q_k^m \frac{\partial}{\partial q_k^m} = \sum_{m,k} \overline{q}_k^m \frac{\partial}{\partial \overline{q}_k^m}.$$

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Therefore, by applying the dilaton equation for FJRW invariants to each vertex in the graph sum expression of $\overline{F}_g^C(\mathbf{q})$, along with fact that Euler characteristics add

$$\sum_{v\in\Gamma}(2-2g_v-|F_v|)=2-2g_\Gamma,$$

we observe that

$$\sum_{m,k} q_k^m \frac{\partial}{\partial q_k^m}\overline{F}_g^C(\mathbf{q})=\sum_{m,k} \overline{q}_k^m \frac{\partial}{\partial \overline{q}_k^m}\overline{F}_g^C(\mathbf{q})=(2-2g)\overline{F}_g^C(\mathbf{q})+\delta_{g,1}\frac{25}{3}.$$

This proves (7).

We now verify the compatibility of the string equation. We must prove

$$\widehat{1/z}\widehat{S^{-1}}\widehat{R}\,\widehat{\mathbb{U}}_0\,\overline{\mathcal{D}}^{\mathrm{LG}}(\mathbf{q})=0. \tag{8}$$

By the string equation in FJRW theory, we know

$$\widehat{1/z}\,\mathcal{D}^{\mathrm{LG}}(\mathbf{t})=0.$$

Therefore, it suffices to check that $\widehat{1/z}$ commutes with $\widehat{S^{-1}}\widehat{R}\,\widehat{\mathbb{U}}_0$. Clearly, $1/z$ commutes with each of S^{-1}, R, and $\mathbb{U}_0$, but a little care must be taken because the quantization procedure is not an algebra homomorphism. However, by the formula for the cocycle given in [Coa03, § 1.3.4], we see immediately that the cocycle vanishes when we commute $\widehat{1/z}$ with $\widehat{S^{-1}}$ and $\widehat{\mathbb{U}}_0$. Upon noticing that the linear-in-z terms of R are strictly above the diagonal, we also see from [Coa03, Example 1.3.4.1] that the cocycle vanishes when we commute $\widehat{1/z}$ with $\widehat{R}$. This proves (8). □

Using the reconstruction by the dilaton and string equations, we can make the following reduction.

COROLLARY 3.6. *In order to prove Conjecture 3.2, it suffices to prove the restriction*

$$\widetilde{F}_g^{\mathrm{CY}}(t_0^1)=F_g^C(t_0^1)+\delta_{g,1}c. \tag{9}$$

The rest of this paper is devoted to proving (9) in the case of genus-zero and genus-one potential functions. The analytic continuation in (9) is described as follows. By the $g\leqslant 1$ mirror theorems for GW invariants, $F_g^{\mathrm{CY}}(\tau^{\mathrm{CY}})$ is an analytic function near $\mathfrak{q}=0$. In the course of our arguments below, we verify that $F_g^C(\widetilde{\tau}^{\mathrm{CY}})$ is also an analytic function at $\mathfrak{t}=0$. The analytic continuation of τ^{CY} in this expression can be computed explicitly by Theorem 2.3. Thus, the analytic continuation occurring in (9) occurs after substituting $t_0^1=\tau^{\mathrm{CY}}$ and takes $F_g^{\mathrm{CY}}(\tau^{\mathrm{CY}})$ from $\mathfrak{q}=0$ to $\mathfrak{t}=0$ along the same path that identifies I-functions in Theorem 2.3.

4. Genus-zero correspondence and tail series

Our goal in this section is to prove the genus-zero correspondence in Theorem 3.3 and to set up some notation for studying generating series of rational tails that appear in the Feynman graph expansions for F_g^C. We begin by recalling a few important points about genus-zero descendent invariants and semi-classical limits.

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If $M(z)$ is a symplectomorphism such that $\widehat{M}\mathcal{D}^{\bullet} = \mathcal{D}^{\star}$, then a careful study of the genus-zero Feynman graphs (see, for example, [CPS13, § 3.5]) implies that

$$M\mathcal{L}^{\bullet} = \mathcal{L}^{\star},$$

where, as in § 2, the Lagrangian cone $\mathcal{L}$ is the differential of the genus-zero potential. In particular, by identifying the parts that have non-negative powers of z, this implies that

$$\overline{J}^{\star}(\mathbf{q}, -z) = M(z)\overline{J}^{\bullet}(M \cdot \mathbf{q}, -z), \tag{10}$$

where

$$M \cdot \mathbf{q}(z) := [M(z)^{-1}\overline{J}^{\star}(\mathbf{q}, -z)]_{+}.$$

Keep in mind that the change of variables $M \cdot \mathbf{q}(z)$ shifts the center of the power series. The next result is a consequence of Theorem 2.3.

Proposition 4.1. *Setting* $\tau^{C} := \widetilde{\tau}^{\mathrm{CY}}$, *we have*

$$\widetilde{J}^{\mathrm{CY}}(\tau^{C}, z) = J^{C}(\tau^{C}, z).$$

Proof. By (10), we see that

$$\mathbb{U}(z)J^{\mathrm{LG}}(\tau^{\mathrm{LG}}, -z) = \overline{J}^{C}([\mathbb{U}(z)J^{\mathrm{LG}}(\tau^{\mathrm{LG}}, -z)]_{+}, -z).$$

By Theorem 2.3, the left-hand side can be rewritten as

$$\mathbb{U}(z)J^{\mathrm{LG}}(\tau^{\mathrm{LG}}, -z) = \frac{5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})}{tI_0^{\mathrm{LG}}(\mathfrak{t})}\widetilde{J}^{\mathrm{CY}}(\tau^{C}, -z).$$

On the other hand, we have

$$\begin{aligned}\overline{J}^{C}([\mathbb{U}(z)J^{\mathrm{LG}}(\tau^{\mathrm{LG}}, -z)]_{+}, -z) &= J^{C}\bigg(\frac{-5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})\varphi_0 z + 5\widetilde{I}_1^{\mathrm{CY}}(\mathfrak{t})\varphi_1}{tI_0^{\mathrm{LG}}(\mathfrak{t})} + \varphi_0 z, -z\bigg)\\ &=: J^{C}(T_0\varphi_0 z + T_1\varphi_1, -z),\end{aligned}$$

which is centered at $T_0 = T_1 = 0$. Expanding as a Taylor series, we have

$$J^{C}(T_0\varphi_0 z + T_1\varphi_1, -z) = \sum_{i,j}\frac{A_{i,j}}{i!j!}T_0^{i}T_1^{j},$$

and the dilaton equation (7) implies that

$$A_{i,j} = (i + j - 2)A_{i-1,j}.$$

Using the fact that, for $j \geqslant 2$,

$$\sum_{m}\binom{m + j - 2}{m}T_0^{m} = \frac{1}{(1 - T_0)^{j-1}}, \tag{11}$$

we see that

$$J^{C}(T_0\varphi_0 z + T_1\varphi_1, -z) = \frac{5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})}{tI_0^{\mathrm{LG}}(\mathfrak{t})}J^{C}(\tau^{C}, -z),$$

concluding the proof. □

Remark 4.2. The identity (11) allows us to write J^C, which is *a priori* centered at $T_0 = T_1 = 0$, as an analytic function at $\mathfrak{t} = 0$.

COROLLARY 4.3. *We have the following genus-zero correspondence:*

$$\widetilde{F}_0^{\mathrm{CY}}(\tau^C) = F^C(\tau^C).$$

Proof. By Proposition 4.1

$$\left(\frac{\partial F_0^C(\mathbf{q})}{\partial q_0^1}\right)_{\mathbf{q}(z)=\tau^C\varphi_1} = \left(\frac{\partial \widetilde{F_0^{\mathrm{CY}}(\mathbf{q})}}{\partial q_0^1}\right)_{\mathbf{q}(z)=\tau^{\mathrm{CY}}\varphi_1} =: J_2(\tau^C),$$

$$\left(\frac{\partial F_0^C(\mathbf{q})}{\partial q_1^0}\right)_{\mathbf{q}(z)=\tau^C\varphi_1} = \left(\frac{\partial \widetilde{F_0^{\mathrm{CY}}(\mathbf{q})}}{\partial q_1^0}\right)_{\mathbf{q}(z)=\tau^{\mathrm{CY}}\varphi_1} =: J_3(\tau^C),$$

and all other partial derivatives vanish at $\mathbf{q}(z) = \tau^C\varphi_1$. Thus, applying the dilaton equation, we have

$$2F_0^C(\tau^C) = 2\widetilde{F}_0^{\mathrm{CY}}(\tau^C) = \tau^C J_2(\tau^C) - J_3(\tau^C). \qquad \square$$

By Corollary 3.6, this completes the proof of Theorem 3.3 for $g = 0$.

4.1 Tail series

A significant portion of our analysis of the action of $\widehat{\mathbb{U}}$ on $\mathcal{D}^{\mathrm{LG}}(\mathbf{t})$ concerns packaging genus-zero tails in the Feynman graph expansions introduced in § 3.1. More specifically, define

$$T(\mathbf{q}, z) = \overline{\mathbf{q}}(z) + \left(\mathbb{U}_0^{-1} \operatorname*{Res}_{z_f=0} V(z, z_f) \sum_{k,m} \frac{\varphi^m}{z_f^{k+1}} \frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial (R^{-1}\mathbf{q})_k^m}\right)_{\mathbf{q}(z)\to[S(z)\mathbf{q}(z)]_+}.$$

Before continuing, let us briefly parse the definition of $T(\mathbf{q}, z)$. First, since

$$\overline{F}_0^B(\mathbf{q}) = \sum_{\Gamma : g(\Gamma)=0} \underline{\mathrm{Contr}}(\Gamma),$$

we see that the partial derivatives

$$\sum_{k,m} \frac{\varphi^m}{z_f^{k+1}} \frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial (R^{-1}\mathbf{q})_k^m}$$

specify in each graph contribution a leg with a particular insertion on it. Contracting with $V(z, z_f)$ and taking the residue turns the specified leg into a specified edge. Finally, applying $\mathbb{U}_0^{-1}$ and specializing the variables $\mathbf{q}(z) \to [S(z)\mathbf{q}(z)]_+$, we see that $T(\mathbf{q}, z)$ is the contribution of all possible genus-zero trees attaching to a specified vertex in the graph contribution for $\overline{F}^C(\mathbf{q})$. Adding $\overline{\mathbf{q}}(z)$ simply corresponds to the contribution of the degenerate tree. We call $T(\mathbf{q}, z)$ the *tail series*. The next lemma describes $T(\mathbf{q}, z)$ explicitly.

LEMMA 4.4. *With notation as above, we have*

$$T(\mathbf{q}, z) = \mathbb{U}(z) \cdot \mathbf{q}(z).$$

Proof. We compute directly:

$$\begin{aligned}\operatorname*{Res}_{z_f=0} V(z,z_f)\sum_{k,m}\frac{\varphi^m}{z_f^{k+1}}\frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial (R^{-1}\mathbf{q})_k^m} &= \operatorname*{Res}_{z_f=0} V(z,z_f)R(z_f)^*\sum_{k,m}\frac{\varphi^m}{z_f^{k+1}}\frac{\partial}{\partial q_k^m}F_0^B(\mathbf{t})\\ &= \operatorname*{Res}_{z_f=0}\frac{R(z_f)^*-R(-z)^*}{z+z_f}\sum_{k,m}\frac{\varphi^m}{z_f^{k+1}}\frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial q_k^m}\\ &= \operatorname*{Res}_{z_f=0}\frac{R(z_f)^*}{z+z_f}\sum_{k,m}\frac{\varphi^m}{z_f^{k+1}}\frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial q_k^m}\\ &= \operatorname*{Res}_{z_f=0}\frac{R(z_f)^*}{z+z_f}(\overline{J}^B(\mathbf{q},z_f)-\mathbf{q}(-z_f)).\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname*{Res}_{z_f=0} V(z,z_f)\sum_{k,m}\frac{\varphi^m}{z_f^{k+1}}\frac{\partial \overline{F}_0^B(\mathbf{q})}{\partial (R^{-1}\mathbf{q})_k^m} &= \operatorname{Res}_{z_f=0}\frac{R^{-1}(z_f)}{z_f-z}(\overline{J}^B(\mathbf{q},-z_f)-\mathbf{q}(z_f))\\ &= [R^{-1}(z)\overline{J}^B(\mathbf{q},-z)]_+ - R^{-1}(z)\mathbf{q}(z).\end{aligned}$$

To obtain $T(\mathbf{q},z)$ from this, we multiply both sides by $\mathbb{U}_0^{-1}$, substitute $\mathbf{q}(z)\to[S(z)\mathbf{q}(z)]_+$, and add $\overline{\mathbf{q}}(z)$, obtaining

$$\begin{aligned}T(\mathbf{q},z) &= \mathbb{U}_0^{-1}[R^{-1}(z)\overline{J}^B([S(z)\mathbf{q}(z)]_+,-z)]_+\\ &= \mathbb{U}_0^{-1}[R^{-1}(z)S(z)\overline{J}^C(\mathbf{q}(z),-z)]_+\\ &= \mathbb{U}(z)\cdot\mathbf{q}(z).\end{aligned}$$

□

5. Genus-one correspondence

In regard to the genus-one potential, there are two types of graphs which appear: the *vertex-type graphs* consist of trees with a unique genus-one vertex, and the *loop-type graphs* consist of graphs Γ with $b_1(\Gamma)=1$ and with $g_v=0$ for all $v\in V$. We separate the contributions from the two types of graphs, and we write

$$F_1^C(\mathbf{t}) = F_1^C(\mathbf{t})_V + F_1^C(\mathbf{t})_L.$$

We now analyze these contributions.

5.1 Vertex-type graphs

By definition of the tail series, the contribution from the vertex-type graphs to $\overline{F}_1^C$ is equal to

$$\overline{F}_1^C(\mathbf{q})_V = \overline{F}_1^{\mathrm{LG}}(T(\mathbf{q},z)).$$

Restricting to the small state space, we obtain the following result.

Proposition 5.1. *We have*

$$F_1^C(\tau^C)_V = F_1^{\mathrm{LG}}(\tau^{\mathrm{LG}}) + \tfrac{25}{3}(\log(\mathfrak{t}I_0^{\mathrm{LG}}(\mathfrak{t})) - \log(5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})))$$

where the variables are related by $\tau^C := \widetilde{\tau}^{\mathrm{CY}} = \widetilde{I}_1^{\mathrm{CY}}(\mathfrak{t})/\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})$, *and* $\tau^{\mathrm{LG}} = I_1^{\mathrm{LG}}(\mathfrak{t})/I_0^{\mathrm{LG}}(\mathfrak{t})$.

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Proof. By Lemma 4.4, we have

$$\begin{aligned}\overline{F}_1^C(\mathbf{q})_V &= \overline{F}_1^{\mathrm{LG}}(T(\mathbf{q},z))\\ &= \overline{F}_1^{\mathrm{LG}}(\mathbb{U}(z)\cdot\mathbf{q}(z))\\ &= \overline{F}_1^{\mathrm{LG}}([\mathbb{U}(z)^{-1}\overline{J}^C(\mathbf{q},-z)]_+).\end{aligned}$$

Specializing $\mathbf{t}=\widetilde{\tau}^{\mathrm{CY}}$, the GW mirror theorem (Theorem 2.1) and the genus-zero LG/CY correspondence (Proposition 4.1) imply that

$$\begin{aligned}F_1^C(\tau^C)_V &= F_1^{\mathrm{LG}}([\mathbb{U}(z)^{-1}\widetilde{J}^{\mathrm{CY}}(\tau^C,-z)]_+ + z\phi_0)\\ &= F_1^{\mathrm{LG}}\bigg(\frac{-\mathfrak{t}I_0^{\mathrm{LG}}(\mathfrak{t})z\phi_0+\mathfrak{t}I_1^{\mathrm{LG}}(\mathfrak{t})\phi_1}{5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})}+z\phi_0\bigg)\\ &= F_1^{\mathrm{LG}}\bigg(\frac{I_1^{\mathrm{LG}}(\mathfrak{t})}{I_0^{\mathrm{LG}}(\mathfrak{t})}\bigg)-\log\bigg(\frac{\mathfrak{t}I_0^{\mathrm{LG}}(\mathfrak{t})}{5\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})}\bigg)\langle\psi_1\phi_0\rangle_{1,1}^{\mathrm{LG}},\end{aligned}$$

where the final equality follows from the dilaton equation. □

5.2 Loop-type graphs

In order to study the loop-type graph contributions to F_1^C, we consider the 1-form $d\overline{F}_1^C(\mathbf{q})_L$, which packages loop-type graph contributions with one specified leg. We break the loop at the vertex where the tree supporting the specified leg attaches and analyze the resulting genus-zero graph contributions. Define the 2-tensors

$$\overline{V}^\bullet(\mathbf{q},w,z) := \sum_m \frac{\varphi_m\otimes\varphi^m}{w+z} + \sum_{m,m',k,k'}\frac{\varphi^m\otimes\varphi^{m'}}{w^{k+1}z^{k'+1}}\frac{\partial^2\overline{F}^\bullet(\mathbf{q})}{\partial q_k^m\partial q_{k'}^{m'}}.$$

The next lemma determines $d\overline{F}_1^C(\mathbf{t})_L$ in terms of $\overline{V}^\bullet(\mathbf{q},w,z)$.

LEMMA 5.2. *We have*

$$d\overline{F}_1^C(\mathbf{q})_L = \tfrac{1}{2}\operatorname{Res}_{\substack{w=0\\z=0}}(d\overline{V}^{\mathrm{LG}}(\mathbb{U}\cdot\mathbf{q},w,z),\mathbb{U}^{-1}(w)\otimes\mathbb{U}^{-1}(z)\overline{V}^C(\mathbf{q},-w-z))^{\mathrm{LG}},$$

where the pairing contracts along each factor of the 2-tensors.

Proof. By arguing as in the proof of Lemma 4.4, we have

$$d\overline{F}_1^B(\mathbf{q})_L = \tfrac{1}{2}\operatorname{Res}_{\substack{w=0\\z=0}}(d\overline{V}^A(T(\mathbf{q},z),w,z),R^{-1}(w)\otimes R^{-1}(z)\overline{V}^B(\mathbf{q},-w,-z)). \tag{12}$$

Therefore, to obtain $d\overline{F}_1^C(\mathbf{q})$, we must replace $\mathbf{q}$ in (12) with $S^{-1}\cdot\mathbf{q}$. Using the facts that $\overline{J}^B(S^{-1}\cdot\mathbf{q},-z)=S(z)\overline{J}^C(\mathbf{q},-z)$ and that $\overline{V}^\bullet(\mathbf{q},-w,-z)$ is obtained from $\overline{J}^\bullet(\mathbf{q},-z)$ by applying the operator

$$\sum_{k,m}\frac{\varphi^m}{(-w)^{k+1}}\otimes\frac{\partial}{\partial q_k^m} = S^{-1}(w)\sum_{k,m}\frac{\varphi^m}{(-w)^{k+1}}\otimes\frac{\partial}{\partial(S^{-1}\cdot\mathbf{q})_k^m},$$

we have

$$\overline{V}^B(S^{-1}\cdot\mathbf{q},-w,-z) = S(w)\otimes S(z)\,\overline{V}^C(\mathbf{q},-w-z).$$

Therefore, the second term in the pairing in (12) becomes

$$R(w)^{-1}S(w)\otimes R(z)^{-1}S(z)\cdot\overline{V}^C(\mathbf{q},-w-z).$$

Similarly, the first term becomes

$$d\overline{V}^A(R(z)\cdot(S^{-1}\cdot\mathbf{q}),w,z) = \mathbb{U}_0\otimes\mathbb{U}_0\, d\overline{V}^{\mathrm{LG}}(\mathbb{U}\cdot\mathbf{q},w,z).$$

The lemma then follows from the fact that $\mathbb{U}_0^* = \mathbb{U}_0^{-1}$. □

If we turn off the descendent parameters by setting $\mathbf{t} = t$, then the string and Witten–Dijkgraaf–Verlinde–Verlinde equations (see, for example, [Coa03, Proposition 1.4.1]) imply that

$$V^{\mathrm{CY}}(t,z,w) = \frac{\sum_m S^{\mathrm{CY}}(t,w)^*(\varphi_m)\otimes S^{\mathrm{CY}}(t,z)^*(\varphi^m)}{w+z}$$

and

$$V^{\mathrm{LG}}(t,z,w) = \frac{\sum_m S^{\mathrm{LG}}(t,w)^*(\phi_m)\otimes S^{\mathrm{LG}}(t,z)^*(\phi^m)}{w+z}.$$

Therefore, by further specializing $t = \tau^C$ $(= \widetilde{\tau}^{\mathrm{CY}})$, using the genus-zero correspondence of Theorem 3.3, and applying the dilaton equation as in the proof of Proposition 5.1, the residue in Lemma 5.2 simplifies as follows.

Lemma 5.3. *We have*

$$dF_1^C(\tau^C)_L = \frac{1}{2}\operatorname{Res}_{\substack{w=0\\z=0}}\left(d\frac{\sum_m \mathbb{U}(-w)S^{\mathrm{LG}}(\tau^{\mathrm{LG}},w)^*(\phi_m)\otimes\mathbb{U}(-z)S^{\mathrm{LG}}(\tau^{\mathrm{LG}},z)^*(\phi^m)}{w+z}, \frac{\sum_m \widetilde{S}^{\mathrm{CY}}(\tau^C,-w)^*(\varphi_m)\otimes(z)\widetilde{S}^{\mathrm{CY}}(\tau^C,-z)^*(\varphi^m)}{-w-z}\right)^{\mathrm{CY}}. \quad (13)$$

In order to further study the residue (13), it will be useful to work in canonical bases for quantum products. Although the GW and FJRW invariants associated to the quintic 3-fold do not yield semisimple Frobenius manifolds, they both admit twisted extensions in genus zero that do admit semisimple Frobenius manifolds. In the next section, we recall and study the twisted extensions.

6. Interlude on twisted invariants

In this section we describe semisimple twisted theories that extend the genus-zero GW and FJRW invariants.

6.1 Twisted GW and 5-spin invariants

Twisted GW invariants associated to the quintic 3-fold take inputs from the extended state space $\overline{H}^{\mathrm{CY}}$ with basis $\varphi_0,\ldots,\varphi_4$ where $\varphi_i = c_1(\mathcal{O}(1))^m \in H^*(\mathbb{P}^4,\mathbb{C})$. To define them, we consider the natural $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$:

$$(\alpha_1,\ldots,\alpha_5)\cdot(z_1,\ldots,z_5) := (\alpha_1 z_1,\ldots,\alpha_5 z_5).$$

There is an induced $(\mathbb{C}^*)^5$-action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4,d)$ and a natural lift to $R\pi_*\mathcal{L}^{\otimes 5}$. Lifting the φ_i to equivariant cohomology where $\prod(\varphi_i-\lambda_i)=0$, the twisted GW invariants are defined by

$$\langle\varphi_{m_1}\psi^{a_1}\cdots\varphi_{m_n}\psi^{a_n}\rangle_{g,n,d}^{\mathrm{CY},\lambda} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4,d)]^{\mathrm{vir}}}\left(\prod_{i=1}^n \mathrm{ev}_i^*(\varphi_{m_i})\psi_i^{a_i}\right)e_{(\mathbb{C}^*)^5}(R\pi_*\mathcal{L}^{\otimes 5}),$$

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where $e_{(\mathbb{C}^*)^5}(-)$ is the equivariant Euler class. These invariants take values in localized equivariant cohomology

$$H^*_{\mathrm{loc}}(\mathcal{B}(\mathbb{C}^*)^5, \mathbb{C}) = \mathbb{C}[\lambda_1^{\pm 1}, \dots, \lambda_5^{\pm 1}].$$

We recover the genus-zero GW invariants of the quintic by restricting the genus-zero twisted invariants to the ambient state space $H^{\mathrm{CY}} \subset \overline{H}^{\mathrm{CY}}$ and taking the non-equivariant limit $\lambda_i = 0$. We define the *shifted twisted GW invariants* by

$$\langle\langle \varphi_{m_1}\psi^{a_1} \cdots \varphi_{m_n}\psi^{a_n} \rangle\rangle^{\mathrm{CY},\lambda}_{g,n}(\tau) := \sum_d \sum_{k \geqslant 0} \frac{\tau^k}{k!} \langle \varphi_{m_1}\psi^{a_1} \cdots \varphi_{m_n}\psi^{a_n} \ \varphi_1 \cdots \varphi_1 \rangle^{\mathrm{CY},\lambda}_{g,n+k,d}.$$

We are primarily interested in the specialization $\lambda_i = \xi^i \lambda$ where $\xi = \exp(2\pi \mathrm{i}/5)$. Since the unspecialized correlators are symmetric in $\{\lambda_i\}$, the specialized correlators are Laurent polynomials in λ^5. The CY I-function can be extended to the (specialized) twisted setting:

$$I^{\mathrm{CY},\lambda}(\mathfrak{q}, z) := z\varphi_0 \sum_{d \geqslant 0} \mathfrak{q}^{\varphi_1/z + d} \frac{\prod_{k=1}^{5d}(5\varphi_1 + kz)}{\prod_{k=1}^{d}((\varphi_1 + kz)^5 - \lambda^5)},$$

where $\varphi_1^a := \lambda^{5\lfloor a/5 \rfloor}\varphi_a$.

Analogously, twisted 5-spin invariants take inputs from the extended state space $\overline{H}^{\mathrm{LG}}$ with basis $\phi_0, \dots, \phi_4$. To define them, we consider the natural $(\mathbb{C}^*)^5$-action on $L^{\oplus 5}$. This induces an action on $R\pi_*\mathcal{L}(-\Sigma_5)^{\oplus 5}$, where Σ_5 is the universal divisor of untwisted points. The twisted 5-spin invariants are defined by

$$\langle \phi_{m_1}\psi^{a_1} \cdots \phi_{m_n}\psi^{a_n} \rangle^{\mathrm{LG},\lambda}_{g,n} := 5^{2-2g} \int_{[\overline{\mathcal{M}}^{1/5}_{g,\vec{m}+\vec{1}}]} \bigg(\prod_{i=1}^{n} \psi_i^{a_i} \bigg) e_{(\mathbb{C}^*)^5}((-R\pi_*\mathcal{L}(-\Sigma_5)^{\oplus 5})^\vee),$$

taking values in

$$H^*_{\mathrm{loc}}(\mathcal{B}(\mathbb{C}^*)^5, \mathbb{C}) = \mathbb{C}[\lambda_1^{\pm 1}, \dots, \lambda_5^{\pm 1}].$$

We recover the genus-zero FJRW invariants associated to the quintic by restricting the genus-zero twisted invariants to the narrow state space $H^{\mathrm{LG}} \subset \overline{H}^{\mathrm{LG}}$ and taking the non-equivariant limit $\lambda_i = 0$. We define the *shifted twisted* 5*-spin invariants* by

$$\langle\langle \varphi_{m_1}\psi^{a_1} \cdots \varphi_{m_n}\psi^{a_n} \rangle\rangle^{\mathrm{LG},\lambda}_{g,n}(\tau) := \sum_{k \geqslant 0} \frac{\tau^k}{k!} \langle \phi_{m_1}\psi^{a_1} \cdots \phi_{m_n}\psi^{a_n} \ \phi_1 \cdots \phi_1 \rangle^{\mathrm{LG},\lambda}_{g,n+k}.$$

As in the CY case, we are primarily interested in the specialization $\lambda_i = \xi^i \lambda$. The LG I-function can be extended to the (specialized) twisted setting:

$$I^{\mathrm{LG},\lambda}(\mathfrak{t}, z) = z \sum_{a \geqslant 0} \frac{\mathfrak{t}^a}{z^a a!} \prod_{\substack{0 < k < (a+1)/5 \\ \langle k \rangle = \langle (a+1)/5 \rangle}} ((kz)^5 + \lambda^5)\phi_a.$$

Notice that $I^{\mathrm{CY},\lambda}$ is annihilated by the Picard–Fuchs operator

$$-\left(\mathfrak{q}\frac{d}{d\mathfrak{q}}\right)^5 + \left(\frac{\lambda}{z}\right)^5 + \mathfrak{q} \prod_{i=1}^{5} \left(5\mathfrak{q}\frac{d}{d\mathfrak{q}} + i\right), \tag{14}$$

while $\mathfrak{t}I^{\mathrm{LG},\lambda}$ is annihilated by the Picard–Fuchs operator

$$\left(\frac{1}{5}\mathfrak{t}\frac{d}{d\mathfrak{t}}\right)^5 + \left(\frac{\lambda}{z}\right)^5 - \mathfrak{t}^{-5} \prod_{i=1}^{5} \left(\mathfrak{t}\frac{d}{d\mathfrak{t}} - 1\right). \tag{15}$$

Moreover, the differential operators (14) and (15) agree upon setting $\mathfrak{q}^{-1} = \mathfrak{t}^5$.

6.2 Genus-zero computations

In what follows, we use Φ_m to denote φ_m or ϕ_m, depending on the context, and we use x to denote $\mathfrak{q}$ or $\mathfrak{t}$. For $\bullet = \mathrm{CY}$ or LG, we study the semisimple Frobenius manifold on $\overline{H}^{\bullet} \otimes \mathbb{C}[\lambda^{\pm 5}]$ where the pairing is defined by

$$(\Phi_a, \Phi_b)^{\bullet,\lambda} := \langle\langle \Phi_a\ \Phi_b\ \Phi_0 \rangle\rangle_{0,3}^{\bullet,\lambda}$$

and the quantum product is defined by

$$\Phi_a \star_{\tau}^{\bullet} \Phi_b := \sum_m \langle\langle \Phi_a \Phi_b \Phi_m \rangle\rangle_{0,3}^{\bullet,\lambda} \Phi^m, \tag{16}$$

where Φ^m is dual to Φ_m under the pairing.

For any $F(x,z) \in \mathbb{C}[[x, z^{-1}]]$, define

$$D^{\bullet} = \begin{cases} \mathfrak{q}\dfrac{d}{d\mathfrak{q}}, & \bullet = \mathrm{CY}, \\ \dfrac{d}{d\mathfrak{t}}, & \bullet = \mathrm{LG}, \end{cases}$$

and define the Birkhoff factorization operator

$$\mathbf{M}^{\bullet} F(x,z) := z D^{\bullet} \frac{F(x,z)}{F(x,\infty)},$$

where, in the presence of state-space insertions, we set $\Phi_i = 1$ in the denominator.

We inductively define series $I_{p,q}^{\bullet}(x)$ by

$$I_{0,q}^{\bullet}(x) := I_q^{\bullet,\lambda}(x) \quad \text{and} \quad I_{p,q}^{\bullet,\lambda}(x) := D^{\bullet}\left(\frac{I_{p-1,q}^{\bullet}(x)}{I_{p-1,p-1}^{\bullet}(x)}\right) \quad \text{for } q \geqslant p > 0, \tag{17}$$

so that

$$(\mathbf{M}^{\bullet})^p (I^{\bullet,\lambda}(x,z)/z) = \sum_{q \geqslant 0} I_{p,p+q}^{\bullet}(x) z^{-q} \Phi_{p+q}$$

for $p \geqslant 0$.

We have the following expression of twisted S-operators in terms of I-functions.

Proposition 6.1. *Define the twisted S-operators by*

$$S^{\bullet,\lambda}(x,z)^*(\Phi) := \Phi + \sum_m \left\langle\left\langle \Phi \frac{\Phi_m}{z-\psi} \right\rangle\right\rangle_{0,2}^{\bullet,\lambda}(x) \Phi^m.$$

Then

$$S^{\bullet,\lambda}(x,z)^*(\Phi_m) := \frac{(\mathbf{M}^{\bullet})^m (I^{\bullet,\lambda}(x,z)/z)}{I_{m,m}^{\bullet}(x)}.$$

Proof. This follows from standard properties of Givental's Lagrangian cone. See, for example, [GR19, Lemma 7.4] for the proof in the LG setting. □

We have the following important properties of $I_{p,p}^{\bullet}$, which were proved in [ZZ08] for the CY case and in [GR19] for the LG case.

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PROPOSITION 6.2 ([ZZ08, Theorem 2], [GR19, Lemma 7.6]). *Define*

$$L^{\bullet} = \begin{cases} (1-\mathfrak{q}5^5)^{-1/5}, & \bullet = \mathrm{CY}, \\ (1-(\mathfrak{t}/5)^5)^{-1/5}, & \bullet = \mathrm{LG}. \end{cases}$$

Then the following properties hold:

(i) $I^{\bullet}_{0,0} \cdots I^{\bullet}_{4,4} = (L^{\bullet})^5$;

(ii) $I^{\bullet}_{5+p,5+p} = \lambda^5 I^{\bullet}_{p,p}$;

(iii) *for* $0 \leqslant p \leqslant 4$, $I^{\bullet}_{p,p} = I^{\bullet}_{4-p,4-p}$.

Following the arguments of [GR19], we see that the quantum product (16) is semisimple. In particular, define

$$E_\alpha = \begin{Bmatrix} \epsilon_\alpha, & \bullet = \mathrm{CY} \\ e_\alpha, & \bullet = \mathrm{LG} \end{Bmatrix} := \frac{1}{5}\sum_i \tilde{\Phi}_i \xi^{-i\alpha}, \quad \alpha = 0,1,2,3,4,$$

where

$$\tilde{\Phi}_0 = \Phi_0, \quad \tilde{\Phi}_1 = \frac{g^{-2/5} f^{-1/5}}{\lambda} \cdot \Phi_1, \quad \tilde{\Phi}_2 = \frac{g^{-4/5} f^{3/5}}{\lambda^2} \cdot \Phi_2,$$
$$\tilde{\Phi}_3 = \frac{g^{-6/5} f^{2/5}}{\lambda^3} \cdot \Phi_3, \quad \tilde{\Phi}_4 = \frac{g^{-3/5} f^{1/5}}{\lambda^4} \cdot \Phi_4,$$

with $f := I^{\bullet}_{2,2}/I^{\bullet}_{1,1}$ and $g = I^{\bullet}_{0,0}/I^{\bullet}_{1,1}$. Then

$$E_\alpha \star^{\bullet}_{\tau} E_\beta = \delta_{\alpha,\beta} E_\alpha.$$

Let $\{u^{\bullet,\alpha}\}$ be canonical coordinates, determined up to a constant by

$$\sum_\alpha E_\alpha \, du^{\bullet,\alpha} = \Phi_1 \, d\tau^{\bullet}.$$

The next result computes the canonical coordinates explicitly in terms of a global 1-form.

PROPOSITION 6.3. *We have*

$$du^{\bullet,\alpha} = \xi^\alpha \lambda \cdot du,$$

where du *is the global one-form*

$$du = L^{\mathrm{CY}} \frac{d\mathfrak{q}}{\mathfrak{q}} = L^{\mathrm{LG}} \, d\mathfrak{t}.$$

Proof. The LG case is proved in [GR19, Lemma 7.8], and the CY case follows from the same arguments. □

We fix the constants of integration by declaring

$$u^{\mathrm{CY},\alpha} = \xi^\alpha \lambda \log(\mathfrak{q}) + \mathcal{O}(q)$$

and

$$u^{\mathrm{LG},\alpha} = \mathcal{O}(\mathfrak{t}).$$

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The normalized canonical coordinates are defined by

$$\tilde{E}_\alpha := (\Delta^\bullet_\alpha)^{1/2} E_\alpha$$

where

$$\Delta^\bullet_\alpha = \frac{1}{\eta(E_\alpha, E_\alpha)^{\bullet,\lambda}}.$$

We compute the pairing on the canonical coordinates explicitly.

Proposition 6.4. *We have*

$$\Delta^\bullet_\alpha = (\xi^\alpha \lambda)^3 \frac{(I^\bullet_{0,0})^2}{(L^\bullet)^2}.$$

Proof. The LG case is proved in [GR19, Lemma 7.9], and the CY case follows from the same arguments. □

The change of basis matrix between flat and normalized canonical coordinates is denoted by

$$\Psi^\bullet_{\alpha m} := (\widetilde{E}_\alpha, \Phi_m)^{\bullet,\lambda}.$$

From the above definitions, we can compute the change of basis explicitly.

Proposition 6.5. *We have*

$$\Psi^\bullet_{\alpha m} = \xi^{\alpha(m-3/2)} c^\bullet_{3-m},$$

where $c^\bullet_i$ satisfy

$$c^\bullet_{-1} := \lambda^{5/2}, \quad c^\bullet_0 := \lambda^{3/2} \frac{I^\bullet_{0,0}}{L^\bullet}, \quad c^\bullet_1 := \lambda^{1/2} \frac{I^\bullet_{0,0} I^\bullet_{1,1}}{(L^\bullet)^2}$$
$$c^\bullet_2 := (c^\bullet_1)^{-1} = \lambda^{-1/2} \frac{I^\bullet_{0,0} I^\bullet_{1,1} I^\bullet_{2,2}}{(L^\bullet)^3}, \quad c^\bullet_3 := (c^\bullet_0)^{-1} = \lambda^{-3/2} \frac{I^\bullet_{0,0} I^\bullet_{1,1} I^\bullet_{2,2} I^\bullet_{3,3}}{(L^\bullet)^4}.$$

For convenience, we also define $c^\bullet_4 := \lambda^{-5/2}$, so that $c^\bullet_m = (c^\bullet_{3-m})^{-1}$ for $m = 0, \ldots, 4$. The inverse matrix of $\Psi^\bullet$ is given by

$$(\Psi^\bullet)^{-1}_{m\alpha} = \frac{\xi^{\alpha(3/2-m)}}{5} c^\bullet_m.$$

Since the quantum product is semisimple for both types of twisted invariant, there is a canonical R-matrix that yields the higher-genus twisted invariants via Teleman's reconstruction theorem [Tel12]. The diagonal entries of the linear term of the R-matrix can be computed explicitly.

Proposition 6.6. *Define*

$$(R^{\mathrm{CY}}_1)_{\alpha\alpha} = \frac{1}{5}\frac{d}{du^\alpha}\left(\frac{5}{4}\log(L^{\mathrm{CY}}) - 4\log(I^{\mathrm{CY}}_0) - \log(I^{\mathrm{CY}}_{1,1}) - \frac{3}{4}\log(\mathfrak{q})\right)$$

and

$$(R^{\mathrm{LG}}_1)_{\alpha\alpha} = \frac{1}{5}\frac{d}{du^\alpha}\left(\frac{5}{4}\log(L^{\mathrm{LG}}) - 4\log(I^{\mathrm{LG}}_0) - \log(I^{\mathrm{LG}}_{1,1})\right).$$

Then, up to constant terms, these matrices are equal to the linear terms of the canonical R-matrices associated to the respective semisimple Frobenius manifolds.

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Proof. In the LG case, this is [GR19, Proposition 7.10]. In the CY case, the proof in [GR19] can be mimicked up to the point where

$$(dR_1^{\mathrm{CY}})_{\alpha\alpha} = \frac{1}{5\lambda\xi^\alpha}\left(\left(\frac{d\log c_2^{\mathrm{CY}}}{du}\right)^2 + \left(\frac{d\log c_3^{\mathrm{CY}}}{du}\right)^2\right) du.$$

Letting $(-)'$ denote $\mathfrak{q}(d/d\mathfrak{q})$, we then apply [ZZ08, Lemma 3] at the second equality below to rewrite the right-hand side as

$$\begin{aligned}\left(\frac{d\log c_2^{\mathrm{CY}}}{du}\right)^2 + \left(\frac{d\log c_3^{\mathrm{CY}}}{du}\right)^2 &= \frac{1}{(L^{\mathrm{CY}})^2}\left(\frac{(c_2^{\mathrm{CY}})'}{c_2^{\mathrm{CY}}} + \frac{(c_3^{\mathrm{CY}})'}{c_3^{\mathrm{CY}}}\right)\\ &= \frac{1}{L^{\mathrm{CY}}}\left(-\frac{3}{4}(L^{\mathrm{CY}})^4 - \frac{1}{L^{\mathrm{CY}}}\left(-5\frac{(L^{\mathrm{CY}})'}{L^{\mathrm{CY}}} + 4\frac{(I_0^{\mathrm{CY}})'}{I_0^{\mathrm{CY}}} + \frac{(I_1^{\mathrm{CY}})'}{I_1^{\mathrm{CY}}}\right)\right)'\\ &= \frac{d}{du}\left(\frac{1}{L^{\mathrm{CY}}}\left(\frac{1}{4}(L^{\mathrm{CY}})^5 - 1 - 4\frac{(I_0^{\mathrm{CY}})'}{I_0^{\mathrm{CY}}} - \frac{(I_1^{\mathrm{CY}})'}{I_1^{\mathrm{CY}}}\right)\right)\\ &= \frac{d^2}{du^2}\left(\frac{5}{4}\frac{(L^{\mathrm{CY}})'}{L^{\mathrm{CY}}} - 4\frac{(I_0^{\mathrm{CY}})'}{I_0^{\mathrm{CY}}} - \frac{(I_1^{\mathrm{CY}})'}{I_1^{\mathrm{CY}}} - \frac{3}{4}\log(\mathfrak{q})\right).\end{aligned}$$

In the last equality, we have used the fact that $(L^{\mathrm{CY}})'/L^{\mathrm{CY}} = \frac{1}{5}((L^{\mathrm{CY}})^5 - 1)$. □

Notice that the genus-one formulas can be obtained from the above R-matrices by the formulas

$$dF_1^{\mathrm{CY}}(\tau^{\mathrm{CY}}) = -\frac{200}{24}d\log(q^{1/5}I_0^{\mathrm{CY}}(\mathfrak{q})) - \frac{5}{24}d\log(\mathfrak{q}^{1/5}L^{\mathrm{CY}}(\mathfrak{q})) + \frac{1}{2}\sum_\alpha (R_1^{\mathrm{CY}})_{\alpha\alpha}\,du^\alpha \tag{18}$$

and

$$dF_1^{\mathrm{LG}}(\tau^{\mathrm{LG}}) = -\frac{200}{24}d\log(I_0^{\mathrm{LG}}(\mathfrak{t})) - \frac{5}{24}d\log(L^{\mathrm{LG}}(\mathfrak{t})) + \frac{1}{2}\sum_\alpha (R_1^{\mathrm{LG}})_{\alpha\alpha}\,du^\alpha. \tag{19}$$

6.3 The twisted genus-zero correspondence

We can extend Theorem 2.3 to the twisted setting.

THEOREM 6.7. *Define the linear transformation $\mathbb{U}^\lambda(-z) : \overline{\mathcal{H}}^{\mathrm{LG}} \to \overline{\mathcal{H}}^{\mathrm{CY}}$ by*

$$\mathbb{U}^\lambda(-z)(\phi_m) = \frac{\xi^{m+1}}{\mathrm{e}^{-2\pi\mathrm{i}\varphi_1/z} - \xi^{m+1}}\frac{-2\pi\mathrm{i}(-z)^m}{\Gamma(1+5\varphi_1/z)}\prod_{i=0}^4\frac{\Gamma(1+\varphi_1/z - \xi^i\lambda/z)}{\Gamma(1-(m+1)/5 - \xi^i\lambda/z)},$$

where $\varphi_1^a := \lambda^{5\lfloor a/5\rfloor}\varphi_a$. Then $\mathbb{U}^\lambda(-z)$ is a symplectic transformation and, upon identifying $\mathfrak{q}^{-1} = \mathfrak{t}^5$, there exists an analytic continuation of $I^{\mathrm{CY},\lambda}(\mathfrak{q}, z)$ such that

$$\mathbb{U}^\lambda(-z)(\mathfrak{t}I^{\mathrm{LG},\lambda}(\mathfrak{t}, z)) = 5\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t}, z).$$

Proof. The Mellin–Barnes method employed in [CR10] to prove Theorem 2.3 (Corollary 4.2.4 in their paper) easily generalizes. □

Using Proposition 6.1 and Theorem 6.7, we can study the action of the symplectic transformation $\mathbb{U}^\lambda$ on the S-operators.

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Proposition 6.8. *We have*

$$\mathbb{U}^\lambda(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^* = \widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*M(\mathfrak{t},z), \tag{20}$$

where $M(\mathfrak{t},z)$ has only non-negative powers of z. Moreover, $M(\mathfrak{t},0)$ is diagonal with

$$M(\mathfrak{t},0) = \left(\delta_{m,m'}(-1)^m \frac{5^{m+1}}{\mathfrak{t}^{m+1}} \frac{\widetilde{I}^{\mathrm{CY}}_{0,0}\cdots \widetilde{I}^{\mathrm{CY}}_{m,m}}{I^{\mathrm{LG}}_{0,0}\cdots I^{\mathrm{LG}}_{m,m}}\right)_{mm'} = -(\widetilde{\Psi}^{\mathrm{CY}})^{-1}\Psi^{\mathrm{LG}} \tag{21}$$

and

$$M(0,z) = R^\lambda(-z)\mathbb{U}^\lambda_0. \tag{22}$$

Proof. Statement (20) follows from Theorem 6.7 and general properties of Lagrangian cones. However, in order to prove (21), we first provide a more constructive proof of (20). From Proposition 6.1 and Theorem 6.7, we compute

$$\begin{aligned}
\widetilde{S}^{\mathrm{CY}}(\tau^C,z)^*(\varphi_0) &= \frac{\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)}{\widetilde{I}^{\mathrm{CY}}_{0,0}} = \frac{\mathfrak{t}}{5}\frac{I^{\mathrm{LG}}_{0,0}}{\widetilde{I}^{\mathrm{CY}}_{0,0}}\mathbb{U}^\lambda(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi_0),\\
\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*(\varphi_1) &= \frac{z}{\widetilde{I}^{\mathrm{CY}}_{1,1}}\left(-\frac{\mathfrak{t}}{5}\frac{d}{d\mathfrak{t}}\right)\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*(\varphi_0)\\
&= \mathcal{O}(z)\mathbb{U}^\lambda(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi_0)\\
&\quad - \frac{\mathfrak{t}^2}{5^2}\frac{I^{\mathrm{LG}}_{0,0}I^{\mathrm{LG}}_{1,1}}{\widetilde{I}^{\mathrm{CY}}_{0,0}\widetilde{I}^{\mathrm{CY}}_{1,1}}\mathbb{U}^\lambda(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi_1),\\
&\ \ \vdots\\
\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*(\varphi_4) &= \mathcal{O}(z)\sum_{m=0}^{3}\mathbb{U}^\lambda(-z)\mathfrak{t}^{m+1}S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi_m)\\
&\quad + \frac{\mathfrak{t}^5}{5^5}\frac{I^{\mathrm{LG}}_{0,0}I^{\mathrm{LG}}_{1,1}I^{\mathrm{LG}}_{2,2}I^{\mathrm{LG}}_{3,3}I^{\mathrm{LG}}_{4,4}}{\widetilde{I}^{\mathrm{CY}}_{0,0}\widetilde{I}^{\mathrm{CY}}_{1,1}\widetilde{I}^{\mathrm{CY}}_{2,2}\widetilde{I}^{\mathrm{CY}}_{3,3}\widetilde{I}^{\mathrm{CY}}_{4,4}}\mathbb{U}^\lambda(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi_4).
\end{aligned}$$

The explicit formula for $M(\mathfrak{t},0)$ in (21) then follows from this computation and Proposition 6.5.

To prove (22), multiply both sides of (20) by $S^\lambda(-z)$ to obtain

$$R^\lambda(-z)\mathbb{U}^\lambda_0 S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^* = S^\lambda(-z)\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*M(\mathfrak{t},z).$$

Since $S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*|_{\mathfrak{t}=0} = 1$, it suffices to prove that

$$S^\lambda(-z)\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*|_{\mathfrak{t}=0} = 1. \tag{23}$$

By Proposition 6.1 and the definition of the operator $\mathbf{M}^{\mathrm{CY}}$, we compute, using the fact that $S^\lambda(-z) = 1+\mathcal{O}(z^{-1})$, that

$$\begin{aligned}
S^\lambda(-z)\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,z)^*(\varphi_k) &= S^\lambda(-z)\frac{(\mathbf{M}^{\mathrm{CY}})^k(\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)/z)}{[z^0\varphi_k](\mathbf{M}^{\mathrm{CY}})^k(\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)/z)}\\
&= \frac{(\mathbf{M}^{\mathrm{CY}})^k(S^\lambda(-z)\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)/z)}{[z^0\varphi_k](\mathbf{M}^{\mathrm{CY}})^k(S^\lambda(-z)\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)/z)}.
\end{aligned} \tag{24}$$

Since $\mathbb{U}^\lambda_0$ is diagonal and $R^\lambda(-z) = 1+\mathcal{O}(z)$, we see that

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$$
\begin{aligned}
S^{\lambda}(-z)\widetilde{I}^{\mathrm{CY},\lambda}(\mathfrak{t},z)/z &= \frac{\mathfrak{t}}{\sqrt{5}}R^{\lambda}(-z)\mathbb{U}_0^{\lambda}I^{\mathrm{LG},\lambda}(\mathfrak{t},z)/z \\
&= \frac{\mathfrak{t}}{\sqrt{5}}\sum_{i=0}^{4}\bigg(\frac{(\mathbb{U}_0^{\lambda})_{ii}\varphi_i}{i!}t^i + \mathcal{O}(t^{i+1})\bigg)z^{-i} + \mathcal{O}(t^6z^{-5}). \qquad (25)
\end{aligned}
$$

Reinserting (25) into (24), one verifies (23) from the definition of $\mathbf{M}^{\mathrm{CY}}$. □

7. Loop-type graphs revisited

We now return to the task of computing the residue in (13). We begin by reinterpreting the residue in terms of a non-equivariant limit of twisted invariants.

LEMMA 7.1. *The loop-type contributions can be expressed as a non-equivariant limit:*

$$
\begin{aligned}
dF_1^C(\tau^C)_L = \lim_{\lambda\to 0}\frac{1}{2}\operatorname{Res}_{\substack{w=0\\z=0}}\Bigg(&d\frac{\sum_m \mathbb{U}^{\lambda}(-w)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},w)^*(\phi_m)\otimes \mathbb{U}^{\lambda}(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi^m)}{w+z}, \\
&\frac{\sum_m \widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,-w)^*(\varphi_m)\otimes (z)\widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,-z)^*(\varphi^m)}{-w-z}\Bigg)^{\mathrm{CY}}. \qquad (26)
\end{aligned}
$$

Proof. Recall that

$$
S^{\bullet,\lambda}(\tau,z)^*(\Phi) = \sum_m \bigg\langle\!\bigg\langle \Phi\frac{\Phi_m}{z-\psi}\bigg\rangle\!\bigg\rangle_{0,2}^{\bullet,\lambda}\Phi^m.
$$

Since

$$
\Phi^4 = \frac{1}{5\lambda^5}\Phi_4,
$$

some care is required in taking the non-equivariant limit. It is not difficult to check that the LG correlators have a zero of order 5 × (number of ϕ_4 insertions) at $\lambda = 0$. Along with the fact that $\mathbb{U}^{\lambda}(z)(\phi_4)$ has a zero of order 5 at $\lambda = 0$, we see that

$$
\lim_{\lambda\to 0}\sum_m \mathbb{U}^{\lambda}(-w)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},w)^*(\phi_m)\otimes \mathbb{U}^{\lambda}(-z)S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)^*(\phi^m)
$$

always exists, and it vanishes whenever there is a ϕ_4 or ϕ^4 insertion in either of the correlators. Similarly, it is not hard to see that the CY correlators have a zero of order 5 at $\lambda = 0$ whenever there is a φ_4 insertion, and therefore

$$
\lim_{\lambda\to 0}\Bigg(\bigg(\sum_m \widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,-w)^*(\varphi_m)\otimes \widetilde{S}^{\mathrm{CY},\lambda}(\tau^C,-z)^*(\varphi^m)\bigg)[\varphi^i\otimes\varphi^j]\Bigg)
$$

always exists, and it vanishes whenever there is a φ_4 or φ^4 insertion in the correlators. Therefore, the limit of the pairing exists and vanishes whenever there is a Φ_4 or Φ^4 insertion in any of the correlators. □

Now that we understand the residue in terms of twisted S-matrices, we can rewrite it in terms of R-matrices, where the residue will become easy to compute. To do this, let p_α denote the equivariant cohomology class of the αth $(\mathbb{C}^*)^5$-fixed point of $\mathbb{P}^4$, and set

$$
\tilde{p}_\alpha := \frac{p_\alpha}{\sqrt{(p_\alpha,p_\alpha)^{\mathrm{CY}}}}.
$$

The following result allows us to rewrite S-matrices in terms of R-matrices.

Proposition 7.2. *As a linear map from the basis $\{p_\alpha\}$ to the basis $\{\varphi_i\}$, the matrix series of the twisted CY fundamental solution*

$$S^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}},z)(\tilde{p}_\alpha)=\tilde{p}_\alpha+\sum_i\left\langle\!\!\left\langle\varphi_i\ \frac{\tilde{p}_\alpha}{z-\psi}\right\rangle\!\!\right\rangle_{0,2}^{\mathrm{CY},\lambda}\varphi^i$$

factors canonically as

$$(\Psi^{\mathrm{CY}})^{-1}\underline{R}^{\mathrm{CY}}(\mathfrak{q},z)e^{U^{\mathrm{CY}}/z},$$

where $U^{\mathrm{CY}}:=\mathrm{diag}(u^{\mathrm{CY},1},\ldots,u^{\mathrm{CY},5})$ and $\underline{R}^{\mathrm{CY}}(\mathfrak{q},z)$ is an R-matrix of the Frobenius manifold associated to twisted GW invariants. In addition, the matrix series of the twisted 5-spin fundamental solution

$$S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)\mathbb{U}^\lambda(-z)^*(\tilde{p}_\alpha)=\mathbb{U}^\lambda(-z)^*(\tilde{p}_\alpha)+\sum_i\left\langle\!\!\left\langle\phi_i\frac{\mathbb{U}^\lambda(-z)^*(\tilde{p}_\alpha)}{z-\psi}\right\rangle\!\!\right\rangle_{0,2}^{\mathrm{LG},\lambda}\phi^i$$

factors canonically as

$$-(\Psi^{\mathrm{LG}})^{-1}\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)e^{U^{\mathrm{CY}}/z},$$

where $\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)$ is an R-matrix of the Frobenius manifold associated to the twisted 5-spin invariants.

Remark 7.3. The reader should note that it does *not* follow from Proposition 7.2 that $\underline{R}^\bullet=R^\bullet$. However, since they are both R-matrices of the same semisimple Frobenius manifold, they differ, at most, by right multiplication by a matrix of the form $\mathrm{diag}(\sum_{k\geqslant 0}a_{2i+1}z^{2k+1})$.

Proof. The factorization of $S^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}},z)$ was proved by Givental [Giv01b] using materialization. A detailed proof can be found in [LP04, ch. 7]. To prove the factorization of $S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)\mathbb{U}^\lambda(-z)^*$, we apply Proposition 6.8 to see that

$$\begin{aligned}S^{\mathrm{LG},\lambda}(\tau^{\mathrm{LG}},z)\mathbb{U}^\lambda(-z)^*(\tilde{p}_\alpha)&=M(\mathfrak{t},z)^*\widetilde{S}^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}},z)(\tilde{p}_\alpha)\\&=M(\mathfrak{t},z)^*(\widetilde{\Psi}^{\mathrm{CY}})^{-1}\widetilde{\underline{R}}^{\mathrm{CY}}(\mathfrak{t},z)e^{U^{\mathrm{CY}}/z}(\tilde{p}_\alpha)\\&=-(\Psi^{\mathrm{LG}})^{-1}\underline{R}^{\mathrm{CY}}(\mathfrak{t},z)e^{U^{\mathrm{LG}}/z}(\tilde{p}_\alpha),\end{aligned}$$

where

$$\underline{R}^{\mathrm{LG}}(\mathfrak{t},z):=-(\Psi^{\mathrm{LG}})M(\mathfrak{t},z)^*(\widetilde{\Psi}^{\mathrm{CY}})^{-1}\widetilde{\underline{R}}^{\mathrm{CY}}(\mathfrak{t},z).$$

To verify that $\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)$ is an R-calibration of the Frobenius manifold, the following properties must be checked:

(i) U^{CY} is a diagonal matrix of canonical coordinates for the twisted 5-spin Frobenius manifold;
(ii) $\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)=1+\mathcal{O}(z)$; and
(iii) $\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)\underline{R}^{\mathrm{LG}}(\mathfrak{t},-z)^*=1$.

The first property follows from Proposition 6.3. The second follows from the fact that $\underline{R}^{\mathrm{CY}}(\mathfrak{q},z)=1+\mathcal{O}(z)$, along with the second part of Proposition 6.8 and the observation that $(\Psi^\bullet)(\Psi^\bullet)^*=1$. The third property follows from the observations $(\Psi^\bullet)(\Psi^\bullet)^*=1$ and $M(\mathfrak{t},z)M(\mathfrak{t},-z)^*=1$. □

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We now compute the residue.

PROPOSITION 7.4. *We have*

$$dF_1^C(\tau^C)_L = \lim_{\lambda\to 0}\frac{1}{2}\sum_\alpha(\underline{\widetilde{R}}_1^{\mathrm{CY}}(\mathfrak{t})^*_{\alpha\alpha} - \underline{R}_1^{\mathrm{LG}}(\mathfrak{t})^*_{\alpha\alpha})\, du^\alpha,$$

where $\underline{R}_1^\bullet(\mathfrak{t})_{\alpha\alpha}$ *denotes the diagonal entries of the linear-in-*z *part of* $\underline{R}^\bullet$.

Proof. Inserting the factorizations of Proposition 7.2 into the residue in Lemma 7.1, we obtain

$$\begin{aligned} dF_1^C(\tau^C)_L = \frac{1}{2}\lim_{\lambda\to 0}\mathrm{Res}_{\substack{w=0\\z=0}}\bigg(& d\frac{\sum_\alpha \underline{R}^{\mathrm{LG}}(\mathfrak{t},w)^*(e_\alpha)\otimes \underline{R}^{\mathrm{LG}}(\mathfrak{t},z)^*(e_\alpha)}{w+z}\\ &+\frac{\sum_\alpha dU^{\mathrm{CY}}\underline{R}^{\mathrm{LG}}(\mathfrak{t},w)^*(e_\alpha)\otimes \underline{R}^{\mathrm{LG}}(\mathfrak{t},z)^*(e_\alpha)}{w(w+z)}\\ &+\frac{\sum_\alpha \underline{R}^{\mathrm{LG}}(\mathfrak{t},w)^*(e_\alpha)\otimes dU^{\mathrm{CY}}\underline{R}^{\mathrm{LG}}(\mathfrak{t},z)^*(e_\alpha)}{(w+z)z},\\ &\frac{\sum_\alpha \underline{\widetilde{R}}^{\mathrm{CY}}(\mathfrak{t},-w)^*(\epsilon_\alpha)\otimes \underline{\widetilde{R}}^{\mathrm{CY}}(\mathfrak{t},-z)^*(\epsilon_\alpha)}{-w-z}\bigg)^{\mathrm{CY}}. \end{aligned}$$

Expanding the denominators as Taylor series in either z/w or w/z, the residue is easily computed to yield

$$dF_1^C(\tau^C)_L = \frac{1}{2}\lim_{\lambda\to 0}\sum_\alpha(\underline{\widetilde{R}}_1^{\mathrm{CY}}(\mathfrak{t})^*_{\alpha\alpha} - \underline{R}_1^{\mathrm{LG}}(\mathfrak{t})^*_{\alpha\alpha})\, du^\alpha.$$

The proposition follows from the fact that $\underline{R}_1^\bullet(x)^* = \underline{R}_1^\bullet(x)$, which follows easily from the properties that $R^\bullet(x,z) = 1+\mathcal{O}(z)$ and $\underline{R}^\bullet(x,z)\underline{R}^\bullet(x,-z)^* = 1$. □

7.1 Comparing constants

Since both $R_1^\bullet$ and $\underline{R}_1^\bullet$ are linear terms of R-matrices, they differ by at most an additive constant. The purpose of this subsection is to compare the constants in order to obtain the following improvement of Proposition 7.4.

PROPOSITION 7.5. *We have*

$$dF_1^C(\tau^C)_L = \lim_{\lambda\to 0}\frac{1}{2}\sum_\alpha(\widetilde{R}_1^{\mathrm{CY}}(\mathfrak{t})_{\alpha\alpha} - R_1^{\mathrm{LG}}(\mathfrak{t})_{\alpha\alpha})\, du^\alpha.$$

Proof. We prove Proposition 7.5 via a sequence of lemmas.

LEMMA 7.6. *We have*

$$\underline{R}^{\mathrm{CY}}(\mathfrak{q}=0,z) = 1$$

and

$$\sum_\alpha \underline{R}_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha}\, du^\alpha = \sum_\alpha R_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha}\, du^\alpha + \frac{3}{4}\, du.$$

Proof of Lemma 7.6. By a standard application of the divisor equation and the localization isomorphism, we compute

$$S^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}},z)(\tilde{p}_\alpha) = \mathrm{e}^{\xi^\alpha\lambda\tau^{\mathrm{CY}}/z}\tilde{p}_\alpha + \sum_{\substack{i\\d>0}}\mathrm{e}^{\tau^{\mathrm{CY}}d}\bigg\langle\varphi_i\ \frac{\mathrm{e}^{\xi^\alpha\lambda\tau^{\mathrm{CY}}/z}\tilde{p}_\alpha}{z-\psi}\bigg\rangle_{0,2,d}\varphi^i.$$

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Multiplying on the right by $\mathrm{e}^{-U^{\mathrm{CY}}/z}$ and using the facts that $\tau^{\mathrm{CY}} = \log(\mathfrak{q}) + \mathcal{O}(\mathfrak{q})$ and $u^{\mathrm{CY},\alpha} = \xi^\alpha \lambda \log(q) + \mathcal{O}(q)$, we see that

$$(S^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}}, z)\mathrm{e}^{-U^{\mathrm{CY}}/z})|_{\mathfrak{q}=0}(\tilde{p}_\alpha) = \tilde{p}_\alpha.$$

Moreover, since $(\Psi^{\mathrm{CY}})|_{\mathfrak{q}=0}$ is simply the change of basis from the $\{\varphi_i\}$ to $\{\tilde{p}_\alpha\}$, we see that, as a matrix,

$$\underline{R}^{\mathrm{CY}}(\mathfrak{q}=0, z) = (\Psi^{\mathrm{CY}} S^{\mathrm{CY},\lambda}(\tau^{\mathrm{CY}}, z)\mathrm{e}^{-U^{\mathrm{CY}}/z})|_{\mathfrak{q}=0} = 1.$$

To prove the second statement, we note that $R_1^{\mathrm{CY}}(\mathfrak{q})$ and $\underline{R}_1^{\mathrm{CY}}(\mathfrak{q})$ are both linear terms of R-calibrations, so they only differ by an additive constant. Observing that

$$R_1^{\mathrm{CY}}(\mathfrak{q}=0)_{\alpha\alpha} = -\frac{3}{20\lambda\xi^\alpha},$$

the computation above shows that

$$\begin{aligned}\sum_\alpha \underline{R}_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha}\, du^\alpha &= \sum_\alpha \bigg(R_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha} + \frac{3}{20\lambda\xi^\alpha} + 1\bigg)\, du^\alpha \\ &= \bigg(\sum_\alpha \bigg(R_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha} + \frac{3}{20\lambda\xi^\alpha} + 1\bigg)\lambda\xi^\alpha\bigg)\, du \\ &= \sum_\alpha R_1^{\mathrm{CY}}(\mathfrak{q})_{\alpha\alpha}\, du^\alpha + \frac{3}{4}\, du. \end{aligned}$$

□

Lemma 7.7. *We have*

$$\widetilde{R}_1^{\mathrm{CY}}(\mathfrak{t}=0)_{\alpha\alpha} = \frac{1}{5\lambda\xi^\alpha}\bigg(2\frac{\xi}{1+\xi}\frac{\Gamma^5(4/5)}{\Gamma^5(3/5)} + \frac{\xi(1+\xi)^3}{(1+\xi+\xi^2)^2}\frac{\Gamma^5(3/5)}{\Gamma^5(2/5)}\bigg).$$

Proof of Lemma 7.7. We know that

$$\widetilde{R}_1^{\mathrm{CY}}(\mathfrak{t})_{\alpha\alpha} = \frac{1}{5\lambda\xi^\alpha}\frac{1}{L^{\mathrm{LG}}}\frac{d}{d\mathfrak{t}}\bigg(-\frac{1}{4}\log(1-\mathfrak{t}^{-5}5^5) - 4\log(\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})) - \log\bigg(-\frac{t}{5}\frac{d}{dt}\frac{\widetilde{I}_1^{\mathrm{CY}}(\mathfrak{t})}{\widetilde{I}_0^{\mathrm{CY}}(\mathfrak{t})}\bigg) + \frac{15}{4}\log(\mathfrak{t})\bigg).$$

Write

$$\widetilde{I}_i^{\mathrm{CY}}(\mathfrak{t}) = \frac{t}{5}\sum_{j=0}^{3} b_{ij} I_j^{\mathrm{LG}}(\mathfrak{t}) = \frac{t}{5}\bigg(b_{i0} + b_{i1}\mathfrak{t} + b_{i2}\frac{\mathfrak{t}^2}{2} + \mathcal{O}(t^3)\bigg),$$

where the coefficients b_{ij} can be computed explicitly from the formula for $\mathbb{U}$ (see below, for example). Notice that the $\log(\mathfrak{t})$ terms cancel and, disregarding constant terms in the derivative, we are left with

$$\begin{aligned}\widetilde{R}_1^{\mathrm{CY}}(\mathfrak{t})_{\alpha\alpha} = \frac{1}{5\lambda\xi^\alpha}\frac{1}{L^{\mathrm{LG}}}\frac{d}{d\mathfrak{t}}\bigg(&-\frac{1}{4}\log(\mathfrak{t}^5 - 5^5) - 4\log(b_{00} + b_{01}t + \cdots) \\ &- \log(b_{00}(b_{00}b_{11} - b_{01}b_{10}) + (b_{00}^2 b_{12} - 2b_{00}b_{01}b_{11} - b_{00}b_{02}b_{10} + 2b_{01}^2 b_{10})\mathfrak{t} + \cdots)\bigg),\end{aligned}$$

where $+\cdots$ denotes higher-order terms in $\mathfrak{t}$. Computing the derivative and setting $\mathfrak{t}=0$, we obtain

$$\widetilde{R}_1^{\mathrm{CY}}(\mathfrak{t}=0)_{\alpha\alpha} = \frac{1}{5\lambda\xi^\alpha}\bigg(\frac{b_{00}^2 b_{12} + 2b_{00}b_{01}b_{11} - b_{00}b_{02}b_{10} - 2b_{01}^2 b_{10}}{b_{00}(b_{01}b_{10} - b_{00}b_{11})}\bigg).$$

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Using the fact that, for $i = 0, 1$,

$$b_{ij} = \frac{(-1)^{j+1}(2\pi \mathrm{i})^{i+1}}{\Gamma^5(1-(j+1)/5)} \frac{\xi^{j+1}}{(1-\xi^{j+1})^{i+1}},$$

the lemma follows by direct computation. □

LEMMA 7.8. *We have*

$$\lim_{\lambda\to 0} \sum_\alpha \lambda \xi^\alpha \underline{R}_1^{\mathrm{LG}}(\mathfrak{t}=0)_{\alpha\alpha} = \frac{3}{4}$$

and

$$\lim_{\lambda\to 0} \sum_\alpha \underline{R}_1^{\mathrm{LG}}(\mathfrak{t})_{\alpha\alpha}\, du^\alpha = \lim_{\lambda\to 0} \sum_\alpha R_1^{\mathrm{LG}}(\mathfrak{t})_{\alpha\alpha}\, du^\alpha + \frac{3}{4}\, du.$$

Proof of Lemma 7.8. Recall that

$$\underline{R}^{\mathrm{LG}}(\mathfrak{t}, z) = -(\Psi^{\mathrm{LG}}) M(\mathfrak{t}, z)^* (\widetilde{\Psi}^{\mathrm{CY}})^{-1} \underline{\widetilde{R}}^{\mathrm{CY}}(\mathfrak{t}, z),$$

so that

$$\underline{R}_1^{\mathrm{LG}}(\mathfrak{t})_{\alpha\alpha} = -((\Psi^{\mathrm{LG}}) M_1(\mathfrak{t})^* (\widetilde{\Psi}^{\mathrm{CY}})^{-1})_{\alpha\alpha} + \underline{\widetilde{R}}_1^{\mathrm{CY}}(\mathfrak{t})_{\alpha\alpha}.$$

By the previous two lemmas, it suffices to prove that

$$\lim_{\lambda\to 0} \sum_\alpha \lambda\xi^\alpha ((\Psi^{\mathrm{LG}}) M_1(\mathfrak{t})^* (\widetilde{\Psi}^{\mathrm{CY}})^{-1})_{\alpha\alpha}\Big|_{\mathfrak{t}=0} = 2\frac{\xi}{1+\xi}\frac{\Gamma^5(4/5)}{\Gamma^5(3/5)} + \frac{\xi(1+\xi)^3}{(1+\xi+\xi^2)^2}\frac{\Gamma^5(3/5)}{\Gamma^5(2/5)}.$$

Let Ψ_0^{LG} and $\widetilde{\Psi}_0^{\mathrm{CY}}$ denote the specializations at $\mathfrak{t} = 0$. Then by (21) and (22), we have

$$\begin{aligned}(\Psi_0^{\mathrm{LG}}) M_1(0)^* (\Psi^{\mathrm{CY}})^{-1} &= (\Psi_0^{\mathrm{LG}})(R^\lambda(-z)\mathbb{U}_0^\lambda)_1^*(-\mathbb{U}_0^\lambda(\Psi_0^{\mathrm{LG}})^{-1}) \\ &= (\Psi_0^{\mathrm{LG}})(\mathbb{U}_0^\lambda)^{-1} R_1^\lambda (\mathbb{U}_0^\lambda)(\Psi_0^{\mathrm{LG}})^{-1} \\ &= (\Psi_0^{\mathrm{LG}})(\mathbb{U}_+^\lambda)_1(\Psi_0^{\mathrm{LG}})^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_\alpha \lambda\xi^\alpha((\Psi_0^{\mathrm{LG}}) M_1(0)^* (\widetilde{\Psi}_0^{\mathrm{CY}})^{-1})_{\alpha\alpha} &= \mathrm{tr}(\mathrm{diag}(\lambda\xi^\alpha)(\Psi_0^{\mathrm{LG}})(\mathbb{U}_+^\lambda)_1(\Psi_0^{\mathrm{LG}})^{-1}) \\ &= \mathrm{tr}(\Lambda(\mathbb{U}_+^\lambda)_1),\end{aligned}$$

where

$$\Lambda := (\Psi_0^{\mathrm{LG}})^{-1}\,\mathrm{diag}(\lambda\xi^\alpha)(\Psi_0^{\mathrm{LG}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda^5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the non-equivariant limit can be computed explicitly in terms of coefficients of $\mathbb{U}_+$:

$$\lim_{\lambda\to 0} \sum_\alpha \lambda\xi^\alpha((\Psi_0^{\mathrm{LG}}) M_1(0)^* (\widetilde{\Psi}_0^{\mathrm{CY}})^{-1})_{\alpha\alpha} = (\mathbb{U}_+)_{01} + (\mathbb{U}_+)_{12} + (\mathbb{U}_+)_{23}.$$

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By expanding the gamma functions in terms of constants $D := 5/12$ and $E := -40\zeta(3)/(2\pi\mathrm{i})^3$, Chiodo and Ruan [CR10] computed

$$\mathbb{U}(-z) = \begin{pmatrix} \dfrac{(-1)^{k+1}(2\pi\mathrm{i})}{\Gamma^5(1-(k+1)/5)}\dfrac{\xi^{k+1}}{1-\xi^{k+1}}z^k & \\ \dfrac{(-1)^{k+1}(2\pi\mathrm{i})^2}{\Gamma^5(1-(k+1)/5)}\dfrac{\xi^{k+1}}{(1-\xi^{k+1})^2}z^{k-1}, & k=0,1,2,3 \\ \dfrac{(-1)^{k+1}(2\pi\mathrm{i})^3}{\Gamma^5(1-(k+1)/5)}\left(\dfrac{\xi^{k+1}(1+\xi^{k+1})}{2(1-\xi^{k+1})^3}+D\dfrac{\xi^{k+1}}{1-\xi^{k+1}}\right)z^{k-2} & \\ \dfrac{(-1)^{k+1}(2\pi i)^4}{\Gamma^5(1-(k+1)/5)}\left(\dfrac{\xi^{k+1}(1+4\xi^{k+1}+\xi^{2k+2})}{6(1-\xi^{k+1})^4}+D\dfrac{\xi^{k+1}}{(1-\xi^{k+1})^2}-E\dfrac{\xi^{k+1}}{1-\xi^{k+1}}\right)z^{k-3} & \end{pmatrix}. \tag{27}$$

By explicitly constructing $S(-z)$ using elementary row operations, we have

$$S(-z)\mathbb{U}(-z) = \begin{pmatrix} \dfrac{-2\pi\mathrm{i}}{\Gamma^5(4/5)}\dfrac{\xi}{1-\xi}, & \dfrac{2\pi\mathrm{i}}{\Gamma^5(3/5)}\dfrac{\xi^2}{1-\xi^2}z, & \dfrac{-2\pi\mathrm{i}}{\Gamma^5(2/5)}\dfrac{\xi^3}{1-\xi^3}z^2, & \dfrac{2\pi\mathrm{i}}{\Gamma^5(1/5)}\dfrac{\xi^4}{1-\xi^4}z^3 \\ 0 & \dfrac{-(2\pi\mathrm{i})^2}{\Gamma^5(3/5)}\dfrac{\xi^3}{(1-\xi^2)^2}, & \dfrac{(2\pi\mathrm{i})^2}{\Gamma^5(2/5)}\dfrac{\xi^4(1+\xi)}{(1-\xi^3)^2}z, & \dfrac{-(2\pi\mathrm{i})^2}{\Gamma^5(1/5)}\dfrac{\xi^5(1+\xi+\xi^2)}{(1-\xi^4)^2}z^2 \\ 0 & 0 & \dfrac{-(2\pi\mathrm{i})^3}{\Gamma^5(2/5)}\dfrac{\xi^6}{(1-\xi^3)^3}, & \dfrac{(2\pi\mathrm{i})^3}{\Gamma^5(1/5)}\dfrac{\xi^7(1+\xi+\xi^2)}{(1-\xi^4)^3}z \\ 0 & 0 & 0 & \dfrac{-(2\pi\mathrm{i})^4}{\Gamma^5(1/5)}\dfrac{\xi^{10}}{(1-\xi^4)^4} \end{pmatrix},$$

where, by definition, the right-hand side is $\mathbb{U}_0\mathbb{U}_+(-z)$. Therefore, we can compute

$$(\mathbb{U}_+)_{01} = \frac{\xi}{1+\xi}\frac{\Gamma^5(4/5)}{\Gamma^5(3/5)},$$

$$(\mathbb{U}_+)_{12} = \frac{\xi(1+\xi)^3}{(1+\xi+\xi^2)^2}\frac{\Gamma^5(3/5)}{\Gamma^5(2/5)},$$

and

$$(\mathbb{U}_+)_{23} = \frac{\xi(1+\xi+\xi^2)(1-\xi^3)^3}{(1-\xi^4)^3}\frac{\Gamma^5(2/5)}{\Gamma^5(1/5)}.$$

By applying the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ and simplifying, it is straightforward to show that $(\mathbb{U}_+)_{23} = (\mathbb{U}_+)_{01}$, finishing the proof of the lemma. □

Proposition 7.5 now follows easily from Proposition 7.4 and Lemmas 7.6 and 7.8. □

Combining Propositions 5.1 and 7.5 with equations (18) and (19), we conclude that

$$dF_1^C(\tau^C) = d\widetilde{F}_1^{\mathrm{CY}}(\tau^C),$$

which, by Corollary 3.6, finishes the proof of Theorem 3.3.

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