WEAK AMENABILITY FOR DUAL BANACH ALGEBRAS

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Abstract A suitable notion of weak amenability for dual Banach algebras, which we call weak Connes amenability, is defined and studied. Among other things, it is proved that the measure algebra $M(G)$ of a locally compact group G is always weakly Connes amenable. It can be a complement to Johnson's theorem that $L^1(G)$ is always weakly amenable [10].

1. Introduction

Amenable Banach algebras were introduced by Johnson in [12]. There are several variants of amenability, two of the most notable are weak amenability and Connes amenability. The concept of weak amenability for Banach algebras was introduced by Johnson in [11], and it generalizes that introduced by Bade, Curtis and Dales for commutative Banach algebras in [1]. The notion of Connes amenability was systematically introduced by Runde in [15]; however, it had been considered previously under different names. We recall the definitions in Definitions 2.1 and 2.2 below.

The purpose of this paper is to study a new notion of weak amenability for dual Banach algebras, which will play the role that weak amenability does for usual Banach algebras. Similar to Connes amenability, this new variant of amenability takes the dual space structure of a dual Banach algebra into account.

One of the most important dual Banach algebras is the measure algebra $M(G)$ of a locally compact group G. Weak amenability in the sense of [11] is too strong to deal with $M(G)$ in a satisfactory manner. In [3], Dales, Ghahramani and Helemskii prove that $M(G)$ is weakly amenable if and only if G is discrete. In contrast, weak Connes amenability of $M(G)$, in the sense of Definition 3.2 below, is not a restrictive demand.

The organization of the paper is as follows. In § 2, we recall some background notations and definitions.

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In §3, weak Connes amenability for dual Banach algebras is introduced. It is shown that the corresponding class of such algebras includes all Connes amenable dual Banach algebras (Theorem 3.5), as well as all weakly amenable dual Banach algebras (Theorem 3.6). For a locally compact group G, the measure algebra $M(G)$ is weakly Connes amenable (Theorem 3.7).

In §4, some basic and hereditary properties are given. We study weak Connes amenability of direct sums of dual Banach algebras (Theorem 4.3). Weak Connes amenability of the enveloping dual Banach algebras is also discussed (Corollary 4.5).

In §5, we verify this new notion for some certain algebras. Examples are given to distinguish between the new notion and the classical concepts of amenability. In particular, we present a class of weakly Connes amenable dual Banach algebras which are neither Connes amenable nor weakly amenable (Theorem 5.5).

2. Preliminaries

Suppose that $\mathfrak{A}$ is a Banach algebra. It is known that the projective tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in the canonical way. There is a continuous linear $\mathfrak{A}$-bimodule homomorphism $\pi : \mathfrak{A}\hat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ such that $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$. If E is a Banach $\mathfrak{A}$-bimodule, then so is the dual space E^*. A continuous linear map $D : \mathfrak{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathfrak{A}$. We call D *inner* if there is $x \in E$ such that $D(a) = ad_x(a) := a \cdot x - x \cdot a$ for every $a \in \mathfrak{A}$.

Definition 2.1. *A Banach algebra $\mathfrak{A}$ is weakly amenable if every derivation $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ is inner.*

Let X be a Banach space. We simply denote by wk and w^*, the $\sigma(X, X^*)$-topology and the $\sigma(X^*, X)$-topology on X and X^*, respectively.

Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A Banach algebra $\mathfrak{A} = (\mathfrak{A}_*)^*$ is *dual* if it is dual as a Banach $\mathfrak{A}$-bimodule. Equivalently, a Banach algebra $\mathfrak{A}$ is dual if it is a dual Banach space such that its multiplication is separately continuous in the w^*-topology.

Let $\mathfrak{A}$ be a dual Banach algebra, and let E be a dual Banach $\mathfrak{A}$-bimodule. Then, we say that E is *normal* if the module actions of $\mathfrak{A}$ on E are w^*–w^* continuous.

Definition 2.2. *A dual Banach algebra $\mathfrak{A}$ is Connes amenable if for every normal, dual Banach $\mathfrak{A}$-bimodule E, every w^*–w^* continuous derivation $D : \mathfrak{A} \longrightarrow E$ is inner.*

Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathfrak{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathfrak{A} \longrightarrow E \quad , \quad a \longmapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases} \quad ,$$

are w^*–wk continuous. It is well known that $\sigma wc(E)$ is a closed submodule of E, and $\sigma wc(E)^*$ is always normal. It is shown in [16, Corollary 4.6] that $\pi^*(\mathfrak{A}_*) \subseteq \sigma wc(\mathfrak{A}\hat{\otimes}\mathfrak{A})^*$.

Taking adjoints, we can extend π to an $\mathfrak{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^*$ to $\mathfrak{A}$. A *σwc-virtual diagonal* for a dual Banach algebra $\mathfrak{A}$ is an element $M \in \sigma wc((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^*$ such that $a \cdot M = M \cdot a$ and $a\pi_{\sigma wc}(M) = a$ for $a \in \mathfrak{A}$. It is known that Connes amenability of $\mathfrak{A}$ is equivalent to existence of a σwc-virtual diagonal for $\mathfrak{A}$ [16].

Our comprehensive references on amenability of Banach algebras are [2, 17].

3. Weak Connes amenability

Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. We denote by $j_E : E^* \longrightarrow \sigma wc(E)^*$ the adjoint of the inclusion map $\sigma wc(E) \hookrightarrow E$. It is clear that j_E is an $\mathfrak{A}$-bimodule homomorphism and $\langle x, j_E(f) \rangle = f(x)$ for all $x \in \sigma wc(E)$ and $f \in E^*$.

We start with the following observation.

Lemma 3.1. *Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra, let E be a Banach $\mathfrak{A}$-bimodule, and let $D : \mathfrak{A} \longrightarrow E^*$ be a derivation. Then:*
(i) *The map $j_E \circ D : \mathfrak{A} \longrightarrow \sigma wc(E)^*$ is a derivation;*
(ii) *If D is inner, then so is $j_E \circ D$. Furthermore, $j_E \circ ad_f = ad_{j_E(f)}$ for each $f \in E^*$;*
(iii) *For every $f \in E^*$, $j_E \circ ad_f : \mathfrak{A} \longrightarrow \sigma wc(E)^*$ is w^*-w^* continuous.*

Proof. (i) For every $a, b \in \mathfrak{A}$ and $x \in \sigma wc(E)$ we have

$$\langle x, a \cdot (j_E \circ D)(b) + (j_E \circ D)(a) \cdot b \rangle = \langle x, j_E(a \cdot D(b)) + j_E(D(a) \cdot b) \rangle$$
$$= \langle x, a \cdot D(b) + D(a) \cdot b \rangle$$
$$= \langle x, D(ab) \rangle = \langle x, (j_E \circ D)(ab) \rangle ,$$

and hence $j_E \circ D$ is a derivation.

(ii) This is a simple calculation.
(iii) It follows from (ii) and normality of $\sigma wc(E)^*$. $\qquad\square$

Definition 3.2. *A dual Banach algebra $\mathfrak{A} = (\mathfrak{A}_*)^*$ is weakly Connes amenable if for every derivation $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ such that $j_\mathfrak{A} \circ D : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{A})^*$ is w^*-w^* continuous, derivation $j_\mathfrak{A} \circ D$ is inner.*

Remark 3.3. For a dual Banach algebra $\mathfrak{A}$, $\sigma wc(\mathfrak{A})$ is a closed (two-sided) ideal of $\mathfrak{A}$. To see this, take $a \in \sigma wc(\mathfrak{A})$, $b \in \mathfrak{A}$, and let $c_\alpha \xrightarrow{w^*} c$ in $\mathfrak{A}$. Because $c_\alpha a \xrightarrow{wk} ca$, for every $\varphi \in \mathfrak{A}^*$, we have

$$lim_\alpha \langle \varphi, c_\alpha(ab) \rangle = lim_\alpha \langle b \cdot \varphi, c_\alpha a \rangle = \langle b \cdot \varphi, ca \rangle = \langle \varphi, c(ab) \rangle$$

so that $ab \in \sigma wc(\mathfrak{A})$. Next, by w^*-continuity of the multiplication, $c_\alpha b \xrightarrow{w^*} cb$. Then, $c_\alpha(ba) = (c_\alpha b)a \xrightarrow{wk} (cb)a = c(ba)$, which means $ba \in \sigma wc(\mathfrak{A})$.

For a given dual Banach algebra $\mathfrak{A}$, it would be interesting to determine the set $\sigma wc(\mathfrak{A})$. However, special care should be taken with the trivial cases $\sigma wc(\mathfrak{A}) = \{0\}$ and $\sigma wc(\mathfrak{A}) = \mathfrak{A}$, as follows.

Remark 3.4. Let $\mathfrak{A}$ be a dual Banach algebra. Then:

(i) If $\sigma wc(\mathfrak{A}) = \{0\}$, then $j_{\mathfrak{A}}$ is the zero map. Therefore, for every derivation $D : \mathfrak{A} \longrightarrow$ $\mathfrak{A}^*$, $j_{\mathfrak{A}} \circ D = 0$. So, $\mathfrak{A}$ is weakly Connes amenable.

(ii) If $\sigma wc(\mathfrak{A}) = \mathfrak{A}$ (or equivalently $\mathfrak{A}^*$ is normal by [16, Proposition 4.4]), then $j_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}^*$ and then $j_{\mathfrak{A}} \circ D = D$. In particular, Definition 3.2 becomes: $\mathfrak{A}$ is weakly Connes amenable if and only if every w^*-w^* continuous derivation $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ is inner.

As the name suggests, weak Connes amenability is weaker than Connes amenability.

Theorem 3.5. *Every Connes amenable dual Banach algebra is weakly Connes amenable.*

Proof. Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a Connes amenable dual Banach algebra, and let $D : \mathfrak{A} \longrightarrow$ $\mathfrak{A}^*$ be a derivation such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{A})^*$ is w^*-w^* continuous. By the assumption, $j_{\mathfrak{A}} \circ D$ is inner, as required. $\qquad\square$

For dual Banach algebras, weak amenability implies weak Connes amenability as follows.

Theorem 3.6. *Every weakly amenable dual Banach algebra is weakly Connes amenable.*

Proof. Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a weakly amenable dual Banach algebra, and let $D : \mathfrak{A} \longrightarrow$ $\mathfrak{A}^*$ be a derivation such that $j_{\mathfrak{A}} \circ D : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{A})^*$ is w^*-w^* continuous. By the assumption, D is inner. Then, by Lemma 3.1(ii), $j_{\mathfrak{A}} \circ D$ is inner. $\qquad\square$

It is well known that the measure algebra $M(G)$ of a locally compact group G is a dual Banach algebra with predual $C_0(G)$.

Theorem 3.7. *Let G be a locally compact group. Then, $M(G)$ is weakly Connes amenable.*

Proof. Let $D : M(G) \longrightarrow M(G)^*$ be a inner derivation such that $j_{M(G)} \circ D :$ $M(G) \longrightarrow \sigma wc(M(G))^*$ is w^*-w^* continuous. By the remark on the final page of [7], there exists $\varphi \in M(G)^*$ for which $D(\delta_g) = ad_{\varphi}(\delta_g)$ for every $g \in G$, where δ_g is the point mass at g. Then, $j_{M(G)} \circ D(\delta_g) = ad_{j(\varphi)}(\delta_g)$ for each $g \in G$, by Lemma 3.1 (ii). As $j_{M(G)} \circ D$ is w^*-w^* continuous and point masses are w^*-dense in $M(G)$, we conclude that $j_{M(G)} \circ D = ad_{j(\varphi)}$. $\qquad\square$

We shall require the following, pointed out by the referee.

Remark 3.8. Let $\mathfrak{A}$ be a finite-dimensional Banach algebra. It is known that $\sigma wc(\mathfrak{A}) = \mathfrak{A}$, and all norm-continuous derivations are automatically w^*–w^* continuous. Hence, for finite-dimensional algebras, weak Connes amenability is equivalent to weak amenability.

To end the current section, we wish to include an example of a dual Banach algebra which is not weakly Connes amenable. Indeed, Remark 3.8 is one way to find such examples: It suffices to give an example of a finite-dimensional Banach algebra, which is not weakly amenable.

Example 3.9. (Suggested by the referee) Let $\mathfrak{A}$ be the $\mathbb{C}$-algebra generated by an identity element e and a non-zero, square-zero element a. Define a linear map $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ by

$$\langle e, D(e) \rangle = \langle a, D(a) \rangle = \langle a, D(e) \rangle = 0, \text{ and } \langle e, D(a) \rangle = 1.$$

Then, a direct calculation shows that D is a non-zero derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. Since $\mathfrak{A}$ is commutative, D is a non-inner derivation. Thus, $\mathfrak{A}$ is not weakly Connes amenable by Remark 3.8.

4. Hereditary properties and relations

Suppose that $\mathfrak{A}$ is a Banach algebra and that E is a Banach $\mathfrak{A}$-bimodule. A derivation $D : \mathfrak{A} \longrightarrow E^*$ is w^*-*approximately inner* if there exists a net $(f_\alpha)_\alpha \subseteq E^*$ such that $D(a) = w^* - \lim_\alpha ad_{f_\alpha}(a)$ for all $a \in \mathfrak{A}$ [8]. The concept of w^*-approximate Connes amenability was introduced in [14]. A dual Banach algebra $\mathfrak{A}$ is w^*-*approximately Connes amenable* if for every normal, dual Banach $\mathfrak{A}$-bimodule E, every w^*–w^* continuous derivation $D : \mathfrak{A} \longrightarrow E$ is w^*-approximately inner.

Proposition 4.1. *Every commutative, w^*-approximately Connes amenable dual Banach algebra is weakly Connes amenable.*

Proof. Let $\mathfrak{A}$ be a commutative, w^*-approximately Connes amenable dual Banach algebra, and let $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ be a derivation such that $j_\mathfrak{A} \circ D : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{A})^*$ is w^*–w^* continuous. It follows from w^*-approximate Connes amenability of $\mathfrak{A}$ that $j_\mathfrak{A} \circ D = w^* - \lim_\alpha ad_{f_\alpha}$ for some net $(f_\alpha)_\alpha \subseteq \sigma wc(\mathfrak{A})^*$. As $\mathfrak{A}$ is commutative and $\sigma wc(\mathfrak{A})^*$ is a symmetric bimodule, $j_\mathfrak{A} \circ D$ must be zero and hence inner. $\square$

Suppose that $\mathfrak{A} = (\mathfrak{A}_*)^*$ and $\mathfrak{B} = (\mathfrak{B}_*)^*$ are dual Banach algebras. We consider the ℓ^1-direct sum $\mathfrak{A} \oplus^1 \mathfrak{B}$ with norm $||(a, b)|| = ||a|| + ||b||$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. This is a dual Banach algebra under pointwise-defined operations and with predual the ℓ^∞-direct sum $\mathfrak{A}_* \oplus^\infty \mathfrak{B}_*$, where the norm $||.||_\infty$ is defined through $||(\phi, \psi)||_\infty = \max(||\phi||, ||\psi||)$ for $\phi \in \mathfrak{A}_*$ and $\psi \in \mathfrak{B}_*$. The duality is given by

$$\langle (a, b), (\phi, \psi) \rangle = \langle a, \phi \rangle + \langle b, \psi \rangle \quad (a \in \mathfrak{A}, \ b \in \mathfrak{B}, \ \phi \in \mathfrak{A}_*, \ \psi \in \mathfrak{B}_*).$$

We write $\imath_\mathfrak{A} : \mathfrak{A} \longrightarrow \mathfrak{A} \oplus^1 \mathfrak{B}$ for the natural injective homomorphism. It is obvious that $\imath_\mathfrak{A}^*(\varphi, \psi) = \varphi$ for $(\varphi, \psi) \in \mathfrak{A}^* \oplus^\infty \mathfrak{B}^* = (\mathfrak{A} \oplus^1 \mathfrak{B})^*$. It follows from [14, Lemma 2.6] that $\sigma wc(\mathfrak{A} \oplus \mathfrak{B}) = \sigma wc(\mathfrak{A}) \oplus \sigma wc(\mathfrak{B})$. We also denote by $\nu_\mathfrak{A} : \sigma wc(\mathfrak{A} \oplus^1 \mathfrak{B})^* = \sigma wc(\mathfrak{A})^* \oplus^\infty \sigma wc(\mathfrak{B})^* \longrightarrow \sigma wc(\mathfrak{A})^*$ the adjoint of the natural embedding $\sigma wc(\mathfrak{A}) \hookrightarrow \sigma wc(\mathfrak{A}) \oplus^1 \sigma wc(\mathfrak{B}) = \sigma wc(\mathfrak{A} \oplus^1 \mathfrak{B})$, so that $\nu_\mathfrak{A}(\varphi, \psi) = \varphi$ for $(\varphi, \psi) \in \sigma wc(\mathfrak{A})^* \oplus^\infty \sigma wc(\mathfrak{B})^*$. Similarly, we define $\imath_\mathfrak{B}$ and $\nu_\mathfrak{B}$.

Lemma 4.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be dual Banach algebras, and let $D : \mathfrak{A} \oplus^1 \mathfrak{B} \longrightarrow (\mathfrak{A} \oplus^1 \mathfrak{B})^* = \mathfrak{A}^* \oplus^\infty \mathfrak{B}^*$ be a derivation. Then:*

(i) $j_{\mathfrak{A}} \circ (\imath_{\mathfrak{A}}^* \circ D) = \nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)$;
(ii) $j_{\mathfrak{B}} \circ (\imath_{\mathfrak{B}}^* \circ D) = \nu_{\mathfrak{B}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)$.

Proof. We only prove (i). For every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, and $c \in \sigma wc(\mathfrak{A})$, we have

$$
\begin{aligned}
\langle c, j_{\mathfrak{A}} \circ (\imath_{\mathfrak{A}}^* \circ D)(a,b) \rangle &= \langle c, (\imath_{\mathfrak{A}}^* \circ D)(a,b) \rangle = \langle (c,0), D(a,b) \rangle \\
&= \langle (c,0), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a,b) \rangle = \langle c, \nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a,b) \rangle \ ,
\end{aligned}
$$

as required. □

Theorem 4.3. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are weakly Connes amenable dual Banach algebras such that $\mathfrak{A}^2$ and $\mathfrak{B}^2$ are w^*-dense in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then, $\mathfrak{A} \oplus^1 \mathfrak{B}$ is weakly Connes amenable.*

Proof. Take a derivation $D : \mathfrak{A} \oplus^1 \mathfrak{B} \longrightarrow (\mathfrak{A} \oplus^1 \mathfrak{B})^* = \mathfrak{A}^* \oplus^\infty \mathfrak{B}^*$ for which $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D : \mathfrak{A} \oplus^1 \mathfrak{B} \longrightarrow \sigma wc(\mathfrak{A} \oplus^1 \mathfrak{B})^* = \sigma wc(\mathfrak{A})^* \oplus^\infty \sigma wc(\mathfrak{B})^*$ is w^*-w^* continuous. Then, $\imath_{\mathfrak{A}}^* \circ D \circ \imath_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ is a derivation. An easy calculation shows that $j_{\mathfrak{A}} \circ (\imath_{\mathfrak{A}}^* \circ D \circ \imath_{\mathfrak{A}}) : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{A})^*$ is w^*-w^* continuous. Since $\mathfrak{A}$ is weakly Connes amenable, there exists $\varphi \in \sigma wc(\mathfrak{A})^*$ such that $j_{\mathfrak{A}} \circ (\imath_{\mathfrak{A}}^* \circ D)(a,0) = ad_\varphi(a)$ and then, by Lemma 4.2(i), $\nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a,0) = ad_\varphi(a)$ for every $a \in \mathfrak{A}$. We regard $\varphi = (\varphi, 0)$ as an element of $\sigma wc(\mathfrak{A})^* \oplus^\infty \sigma wc(\mathfrak{B})^*$. Then it is easily checked that $\nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ (D - ad_\varphi))(a,0) = 0$. So, by replacing D by $D - ad_\varphi$, we may suppose that $\nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a,0) = 0$ for every $a \in \mathfrak{A}$. For each $a_1, a_2 \in \mathfrak{A}$, $a \in \sigma wc(\mathfrak{A})$ and $b \in \sigma wc(\mathfrak{B})$, we see that

$$
\begin{aligned}
\langle (a,b), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1 a_2, 0) \rangle &= \langle (a,b), j_{\mathfrak{A} \oplus^1 \mathfrak{B}}((a_1, 0) \cdot D(a_2, 0) + D(a_1, 0) \cdot (a_2, 0)) \rangle \\
&= \langle (a,b), (a_1, 0) \cdot (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) \\
&\qquad + (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \cdot (a_2, 0) \rangle \\
&= \langle (a a_1, 0), \\
&\qquad (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) + (a_2 a, 0), (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \rangle \\
&= \langle a a_1, \nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_2, 0) \rangle \\
&\qquad + \langle a_2 a, \nu_{\mathfrak{A}} \circ (j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1, 0) \rangle \\
&= 0 + 0 = 0, \quad (a a_1, a_2 a \in \sigma wc(\mathfrak{A}) \text{ by Remark 3.3})
\end{aligned}
$$

and whence $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a_1 a_2, 0) = 0$ for all $a_1, a_2 \in \mathfrak{A}$. Now, it follows from w^*-density of $\mathfrak{A}^2$ and w^*-continuity of $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D$ that $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(a,0) = 0$ for every $a \in \mathfrak{A}$.

A similar argument, with $\mathfrak{B}$ in place of $\mathfrak{A}$, shows that $(j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D)(0,b) = 0$ for every $b \in \mathfrak{B}$. It follows that $j_{\mathfrak{A} \oplus^1 \mathfrak{B}} \circ D = 0$, and hence $\mathfrak{A} \oplus^1 \mathfrak{B}$ is weakly Connes amenable. □

Proposition 4.4. *Let $\mathfrak{A}$ be a commutative Banach algebra, let $\mathfrak{B}$ be a dual Banach algebra, and let $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a (continuous) homomorphism with w^*-dense range. If $\mathfrak{A}$ is weakly amenable, then $\mathfrak{B}$ is weakly Connes amenable.*

Proof. Take a derivation $D : \mathfrak{B} \longrightarrow \mathfrak{B}^*$ such that $j_{\mathfrak{B}} \circ D : \mathfrak{B} \longrightarrow \sigma wc(\mathfrak{B})^*$ is w^*-w^* continuous. Then, $(j_{\mathfrak{B}} \circ D) \circ \theta : \mathfrak{A} \longrightarrow \sigma wc(\mathfrak{B})^*$ is a derivation. By [2, Theorem 2.8.63 (iii)], $(j_{\mathfrak{B}} \circ D) \circ \theta = 0$. From w^*-continuity of $j_{\mathfrak{B}} \circ D$ and w^*-density of the range of θ, we conclude that $j_{\mathfrak{B}} \circ D = 0$, as required. $\square$

Let $\mathfrak{A}$ be a Banach algebra. We write $WAP(\mathfrak{A}^*)$ for the space of all *weakly almost periodic functionals* on $\mathfrak{A}$. It is noted by Runde that $WAP(\mathfrak{A}^*)^*$ is a dual Banach algebra with a universal property [16, Theorem 4.10]. Later, Daws called $WAP(\mathfrak{A}^*)^*$ the *dual Banach algebra enveloping algebra of* $\mathfrak{A}$ [6, Definition 2.10]. There is a (continuous) homomorphism $\kappa : \mathfrak{A} \longrightarrow WAP(\mathfrak{A}^*)^*$ whose range is w^*-dense. Indeed, the map κ was obtained by composing the canonical inclusion $\mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ with the adjoint of the inclusion map $WAP(\mathfrak{A}^*) \hookrightarrow \mathfrak{A}^*$ [16].

Corollary 4.5. *Let $\mathfrak{A}$ be a commutative, weakly amenable Banach algebra. Then, $WAP(\mathfrak{A}^*)^*$ is weakly Connes amenable.*

Proof. We apply Proposition 4.4 for the canonical homomorphism $\kappa : \mathfrak{A} \longrightarrow WAP(\mathfrak{A}^*)^*$. $\square$

5. More examples

In this section, we give some examples to compare weak Connes amenability and some older notions such as Connes amenability, weak amenability and pseudo (Connes) amenability.

Example 5.1. It is well known that $\ell^1 = \ell^1(\mathbb{N})$ with pointwise multiplication is a commutative, weakly amenable dual Banach algebra. It follows from Proposition 3.6 that ℓ^1 is weakly Connes amenable. Further, $WAP((\ell^1)^*)^* = WAP(\ell^\infty)^*$ is also weakly Connes amenable, by Corollary 4.5. However, they fail to be Connes amenable because of the lack of an identity.

Example 5.2. Consider the semigroup $\mathbb{N}_\vee$ which is $\mathbb{N}$ with the operation $m \vee n := \max\{m, n\}$, $(m, n \in \mathbb{N})$. It is known that $\ell^1(\mathbb{N}_\vee)$ is a commutative, weakly amenable dual Banach algebra. An argument similar to Example 5.1 shows that both $\ell^1(\mathbb{N}_\vee)$ and $WAP(\ell^1(\mathbb{N}_\vee)^*)^*$ are weakly Connes amenable. Notice that none of them are Connes amenable by [5, Theorem 5.13] and [6, p. 262], respectively.

Example 5.3. We denote by $\mathbb{N}_\wedge$ the semigroup $\mathbb{N}$ with the operation $m \wedge n := \min\{m, n\}$, $(m, n \in \mathbb{N})$. It is known that $\ell^1(\mathbb{N}_\wedge)$ is a commutative, weakly amenable Banach algebra. By Corollary 4.5, $WAP(\ell^1(\mathbb{N}_\wedge)^*)^*$ is weakly Connes amenable. It was shown in [6, Theorem 7.6] that $WAP(\ell^1(\mathbb{N}_\wedge)^*)^*$ is not Connes amenable.

Let $\mathfrak{A}$ be a dual Banach algebra. Composing the canonical inclusion $\mathfrak{A} \hat{\otimes} \mathfrak{A} \longrightarrow (\mathfrak{A} \hat{\otimes} \mathfrak{A})^{**}$ with the quotient map $(\mathfrak{A} \hat{\otimes} \mathfrak{A})^{**} \longrightarrow \sigma wc((\mathfrak{A} \hat{\otimes} \mathfrak{A})^*)^*$, we obtain an $\mathfrak{A}$-bimodule homomorphism $\zeta : \mathfrak{A} \hat{\otimes} \mathfrak{A} \longrightarrow \sigma wc((\mathfrak{A} \hat{\otimes} \mathfrak{A})^*)^*$ with w^*-dense range.

Definition 5.4. ([14]) *A dual Banach algebra* $\mathfrak{A}$ *is pseudo Connes amenable if there exists a net* $(m_\alpha)_\alpha$ *in* $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, *called an approximate* σwc-*diagonal for* $\mathfrak{A}$, *such that* $a \cdot \zeta(m_\alpha) - \zeta(m_\alpha) \cdot a \xrightarrow{w^*} 0$, *and* $a\pi_{\sigma wc}(m_\alpha) \xrightarrow{w^*} a$ *for every* $a \in \mathfrak{A}$.

We write $\mathfrak{A}^\sharp$ for the unitization of an algebra $\mathfrak{A}$.

Theorem 5.5. *Let* G *be a non-discrete locally compact group, and set* $\mathfrak{A} := M(G) \oplus^1 (\ell^1)^\sharp$. *Then:*

(i) $\mathfrak{A}$ *is not weakly amenable;*
(ii) $\mathfrak{A}$ *is not pseudo amenable;*
(iii) $\mathfrak{A}$ *is not Connes amenable;*
(iv) $\mathfrak{A}$ *is not pseudo Connes amenable;*
(v) $\mathfrak{A}$ *is weakly Connes amenable.*

Proof.

(i) If $\mathfrak{A}$ were weakly amenable, then so is its image under the natural projection $\mathcal{P}_{M(G)} : \mathfrak{A} \longrightarrow M(G)$ by [13, Lemma 2.3]. However, this is not possible for non-discrete groups [17, Theorem 4.2.13].
(ii) By [9, Theorem 3.1] and [4, Theorem 4.1], $(\ell^1)^\sharp$ is not pseudo amenable. As $(\ell^1)^\sharp$ is the image of $\mathfrak{A}$ under the projection $\mathcal{P}_{(\ell^1)^\sharp} : \mathfrak{A} \longrightarrow (\ell^1)^\sharp$, $\mathfrak{A}$ cannot be pseudo amenable by [9, Proposition 2.2].
(iii) If $\mathfrak{A}$ were Connes amenable, then so is $(\ell^1)^\sharp$, its image under the natural projection $\mathcal{P}_{(\ell^1)^\sharp} : \mathfrak{A} \longrightarrow (\ell^1)^\sharp$ by [15, Proposition 4.2], which is not the case.
(iv) Towards a contradiction, suppose that $\mathfrak{A}$ is pseudo Connes amenable. Then $(\ell^1)^\sharp$, the image of $\mathfrak{A}$ under $\mathcal{P}_{(\ell^1)^\sharp}$, must be pseudo Connes amenable by [14, Proposition 4.5]. But, it follows from [14, Theorems 3.2 and 5.1] that $(\ell^1)^\sharp$ is not pseudo Connes amenable.
(v) Weak amenability of ℓ^1 implies that of $(\ell^1)^\sharp$. By Theorem 3.6, $(\ell^1)^\sharp$ is weakly Connes amenable. Then, using Theorem 3.7, weak Connes amenability of $\mathfrak{A}$ is a consequence of Theorem 4.3. $\qquad \square$

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