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INFLECTION POINTS OF BESSEL FUNCTIONS OF NEGATIVE ORDER

To Tim Rooney on his 65th Birthday

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ABSTRACT. We consider the positive zeros $j''_{\nu k}$, $k = 1, 2, \ldots$, of the second derivative of the Bessel function $J_\nu(x)$. We are interested first in how many zeros there are on the interval $(0, j_{\nu 1})$, where $j_{\nu 1}$ is the smallest positive zero of $J_\nu(x)$. We show that there exists a number $\lambda = -0.19937078\ldots$ such that $j''_{\nu 1} < j''_{\nu 2} < j_{\nu 1}$ for $\lambda < \nu < 0$ and $j''_{\nu 1} > j_{\nu 1}$ for $-1 < \nu < \lambda$. For $\nu = \lambda$, $j''_{\nu 1} < j_{\nu 1} < j''_{\nu 2}$. Moreover, $j''_{\nu 1}$ decreases to 0 and $j''_{\nu 2}$ increases to j''_{01} as ν increases from λ to 0. Further, $j''_{\nu k}$ increases in $-1 < \nu < \infty$, for $k = 3, 4, \ldots$. Monotonicity properties are established also for ordinates, and the slopes at the ordinates, of the points of inflection when $-1 < \nu < 0$.

1. **Introduction.** Here, as in [10] and in [13], we consider the positive zeros $j''_{\nu k}$, $k = 1, 2, \ldots$, of the second derivative of the Bessel function $J_\nu(x)$. In [10] and [13], ν was supposed to be positive. Here we pay special attention to the case $-1 < \nu \leq 0$ and ask first how many zeros there are on the interval $(0, j_{\nu 1})$ where $j_{\nu 1}$ is the smallest positive zero of $J_\nu(x)$. We show that there exists a number $\lambda = -0.19937078\ldots$ such that $j''_{\nu 1} < j''_{\nu 2} < j_{\nu 1}$ for $\lambda < \nu < 0$ and $j''_{\nu 1} > j_{\nu 1}$ for $-1 < \nu < \lambda$. For $\nu = \lambda$, $j''_{\nu 1} < j_{\nu 1} < j''_{\nu 2}$. Moreover, $j''_{\nu 1}$ decreases to 0 and $j''_{\nu 2}$ increases to j''_{01} as ν increases from λ to 0. Further, $j''_{\nu k}$ increases in $-1 < \nu < \infty$, for $k = 3, 4, \ldots$. Monotonicity properties are established also for ordinates, and the slopes at the ordinates, of the points of inflection when $-1 < \nu < 0$.

2. **Preliminaries.** As requisite preliminary information, we mention that the function $J_\nu(x)$ satisfies the differential equations

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \tag{1}$$

and [4, p. 13, (67)]

$$x^2(x^2 - \nu^2)y''' + x(x^2 - 3\nu^2)y'' + p_\nu(x)y' = 0, \tag{2}$$

where

$$p_\nu(x) = x^4 - (2\nu^2 + 1)x^2 + \nu^4 - \nu^2. \tag{3}$$

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It follows from (2) that any $j''_{\nu k}$, $-1 < \nu < 0$, $k = 1, 2, \ldots$, must give rise to a point of inflection except possibly when $p_\nu(j''_{\nu k}) = 0$, i.e., when

$$j''^{\,2}_{\nu k} = \mu_\nu{}^2 := \nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1}, \quad -1 < \nu < 0.$$

(Another situation arising from (2) and possibly not leading to a point of inflection is $y'(j''_{\nu k}) = y''(j''_{\nu k}) = 0$. But, using (1), this would imply $j''_{\nu k} = |\nu|$, a possibility ruled out by Lemma 5 below.) That $p_\nu(j''_{\nu k})$ can vanish will be established in Theorem 1. However, it can do so only once, when $\nu = \lambda = -0.19937078\ldots$ and $k = 1$. In this single instance $j''_{\nu k} = j''_{\nu 1}$ does not give rise to an inflection. We will also need the power series

$$\phi(x) = 2^\nu x^{2-\nu} J''_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+\nu)(2m+\nu-1)x^{2m}}{4^m m!\,\Gamma(\nu+m+1)}, \tag{4}$$

and the inequality [9, (5)]

$$j'_{\nu 1} > \max(j_{\nu 1}, |\nu|), \quad -1 < \nu < 0. \tag{5}$$

Further, we note that $J_\nu(0^+) = +\infty$, since $-1 < \nu < 0$, and so $J''_\nu(x) > 0$, $0 < x < \delta$. (The second assertion is a consequence of the first and of (1), or may be inferred from (4).)

Our first Lemma exhibits a contrast in the behaviour of $j''_{\nu 1}$ when $-1 < \nu < 0$ from that which occurs in $\nu > 1$ where $j''_{\nu 1} < \nu$ [12, p. 486, (2)]. The Lemma implies that $j''^{\,2}_{\nu 1} > \nu^2$, $-1 < \nu < 0$, a result which holds also for $0 \le \nu < 1$, but this is not needed here.

LEMMA 1. *If $-1 < \nu < 0$, then $j''^{\,2}_{\nu 1} > 2(\nu^2 - \nu)$.*

PROOF. The conclusion will follow once it is established, using (4), that $J''_\nu(x) > 0$, $x^2 \le 2(\nu^2 - \nu)$. The power series in (4) is alternating, with terms approaching 0. We show now that they decrease in absolute value when $x^2 \le 2(\nu^2 - \nu)$, at least from $m = 2$ on, $-1 < \nu < 0$. The sum of the first two terms then provides a lower bound for $2^\nu x^{2-\nu} J''_\nu(x) = \phi(x)$ which will be seen to be positive for $x^2 \le 2(\nu^2 - \nu)$, $-1 < \nu < 0$, justifying the conclusion.

The ratio of the absolute value of the $(m+1)$-st term to that of the m-th is

$$r_m(x) = \frac{1}{4}\,\frac{2m+\nu+2}{2m+\nu}\,\frac{2m+\nu+1}{2m+\nu-1}\,\frac{x^2}{(m+1)(m+\nu+1)}, \quad m = 0, 1, \ldots. \tag{6}$$

For fixed x and ν $(-1 < \nu < 0)$, the sequence $r_1(x), r_2(x), r_3(x) \ldots$, decreases. Further, $r_2(x) < 1$, $-1 < \nu < 0$, $x^2 \le 2(\nu^2 - \nu)$, so that

$$\phi(x) > \frac{\nu(\nu-1)}{\Gamma(\nu+1)} - \frac{(\nu+1)(\nu+2)x^2}{4\Gamma(\nu+2)}.$$

The expression $\phi_1(x)$ on the right is positive for $x^2 \leq 2(\nu^2 - \nu)$, $-1 < \nu < 0$, so that $\phi(x) > 0$ for these values as well, since

$$\begin{aligned}\Gamma(\nu+1)\phi_1(x) &= \nu(\nu-1) - \frac{1}{4}(\nu+2)x^2 \\ &\geq \Gamma(\nu+1)\phi_1([2(\nu^2-\nu)]^{1/2}) \\ &= -\frac{1}{2}\nu^2(\nu-1) > 0, \quad -1 < \nu < 0,\end{aligned}$$

as asserted. This completes the proof.

COROLLARY 1. *There are no zeros of* $J_\nu''(x)$, $-1 < \nu < 0$, *in* $j_{\nu k} < x < j_{\nu k}'$, $k = 1, 2, \ldots$.

PROOF. Applying the consequence $j_{\nu 1}''^{\,2} > \nu^2$ of Lemma 1 to (1) with $x = j_{\nu k}'' \geq j_{\nu 1}''$ shows that $J_\nu'(j_{\nu k}'')$ and $J_\nu(j_{\nu k}'')$ have opposite signs. (This follows also from [10, Lemma 2.1].)

COROLLARY 2. *If* $j_{\nu k}'' > j_{\nu 1}$, *then* $p_\nu(j_{\nu k}'') > 0$, $-1 < \nu < 0$, *so that* $J_\nu'''(j_{\nu k}'') \neq 0$ *and* $j_{\nu k}''$ *gives rise to a point of inflection.*

PROOF. From Corollary 1 and [9, (5′)], $j_{\nu k}'' > j_{\nu 1}' > j_{11}' = 1.84118\ldots > 3^{1/2}$. On the other hand, $0 < \mu_\nu < 3^{1/2}$, $-1 < \nu < 0$. The conclusion is now evident in view of (2).

COROLLARY 3. *In* $j_{\nu k}' < x < j_{\nu, k+1}$, $k = 1, 2, \ldots$, $-1 < \nu < 0$, *there exists exactly one point of inflection of* $J_\nu(x)$.

PROOF. From (1), it follows that $J_\nu''(j_{\nu k}')$ and $J_\nu''(j_{\nu,k+1})$ have opposite signs so that at least one point of inflection is present. Were there a second, say at $x = \xi''$, it would follow from (2) that $p_\nu(\xi'') < 0$, contradicting Corollary 2.

REMARK. These Corollaries hold also for $\nu \geq 0$; cf. [10, Lemma 2.1]

LEMMA 2. *If* $j_{\nu 1}'' < j_{\nu 1}$ *yields a point of inflection of* $J_\nu(x)$, $-1 < \nu < 0$, *then* $j_{\nu 1}'' < j_{\nu 2}'' < j_{\nu 1}$. *Moreover, in this case,*

$$\nu^2 < 2(\nu^2 - \nu) < j_{\nu 1}''^{\,2} < \mu_\nu^2 \leq j_{\nu 2}''^{\,2} < j_{\nu 1}^{\,2}. \tag{7}$$

PROOF. That there must be an even number (perhaps 0) of points of inflection in $0 < x < j_{\nu 1}$ is clear, since $J_\nu''(0+) = +\infty$ and (1) implies that $J_\nu''(j_{\nu 1}) > 0$, $-1 < \nu < 0$.

Inasmuch as the first point of inflection, assumed to lie in $0 < x < j_{\nu 1}$, occurs where $J_\nu''(x)$ changes from positive to negative, it follows that $J_\nu'''(j_{\nu 1}'') \leq 0$. Putting $x = j_{\nu 1}''$ in (2), noting that $j_{\nu 1}''^{\,2} > \nu^2$ (from Lemma 1) and that $J_\nu'(j_{\nu 1}'') < 0$ (from (5)), we find that $p_\nu(j_{\nu 1}'') \leq 0$, i.e., that $j_{\nu 1}'' \leq \mu_\nu$. Suppose $j_{\nu 1}'' = \mu_\nu$. Differentiating (2) and putting $x = j_{\nu 1}'' = \mu_\nu$ in the result gives $J_\nu^{(4)}(j_{\nu 1}'') > 0$, since $p_\nu'(\mu_\nu) > 0$, while from (2) and

Lemma 1, $J_\nu'''(j_{\nu 1}'') = 0$ under these circumstances. But this means that $J_\nu'''(x)$ and hence also $J_\nu''(x)$ increase from 0 as x increases past $x = \mu_\nu = j_{\nu 1}''$ so that $J_\nu''(x)$ would be positive instead of negative, a contradiction.

There remains only to show that $j_{\nu 2}'' \geq \mu_\nu$. Suppose that $j_{\nu 2}'' < \mu_\nu$ and put $x = j_{\nu 2}''$ in (2). Then $J_\nu'''(j_{\nu 2}'') < 0$ since now $p_\nu(j_{\nu 2}'') < 0$ and also $J_\nu'(j_{\nu 2}'') < 0$. Hence $J_\nu''(x)$ decreases from 0 as x increases through $j_{\nu 2}''$ thus being (indeed remaining) negative in $j_{\nu 2}'' < x < j_{\nu 2}'' + \epsilon$ for sufficiently small $\epsilon > 0$. If, however, $x = j_{\nu 2}''$ yields a point of inflection this is impossible since $J_\nu''(x)$ would be changing from negative to positive as x passes through $j_{\nu 2}''$.

If, on the other hand, $x = j_{\nu 2}''$ does not yield a point of inflection, then $J_\nu'''(j_{\nu 2}'') = 0$ so that $p_\nu(j_{\nu 2}'') = 0$. Then $j_{\nu 2}'' = \mu_\nu$, also contradicting the assumption that $j_{\nu 2}'' < \mu_\nu$.

The Lemma is proved.

REMARK. The possibility that $x = \mu_\nu$ might give rise to an inflection-point, $-1 < \nu < 0$, is ruled out by Theorem 1 below where it is shown that $J_\nu''(\mu_\nu) = 0, -1 < \nu < 0$, can occur only for $\nu = \lambda = -0.1993707809\ldots$. For this value of ν, μ_ν is a double zero of $J_\nu''(x)$, as (2) shows, arising (Theorem 2) from the confluence of two points of inflection. However, for temporary convenience, we consider now what will later appear vacuous.

LEMMA 3. *If $\mu_\nu = j_{\nu 2}''$, then it gives a point of inflection, $-1 < \nu < 0$.*

PROOF. Differentiating (2) and putting $x = \mu_\nu = j_{\nu 2}''$ in the resulting equation gives $J_\nu^{(4)}(\mu_\nu) > 0$, since $p_\nu'(\mu_\nu) > 0$, $J_\nu'(j_{\nu 2}'') < 0$. Therefore $J_\nu'''(x)$ and hence also $J_\nu''(x)$ increase from 0 as x increases past μ_ν. Since $J_\nu''(x) < 0, j_{\nu 1}'' < x < j_{\nu 2}''$, it follows that $x = j_{\nu 2}'' = \mu_\nu$ gives a point of inflection.

LEMMA 4. *If $-1 < \nu < \infty$, then $j_{\nu 3}'' > j_{\nu 1}$.*

PROOF. Unless $j_{\nu 2}'' < j_{\nu 1}$ there is nothing to prove. In this case $j_{\nu 3}'' > \mu_\nu$, so that (2) implies $J_\nu'''(j_{\nu 3}'') > 0$. But this is a contradiction, since at a point of inflection on $(0, j_{\nu 1})$ which $j_{\nu 3}''$ would yield, $J_\nu''(x)$ would change from positive to negative.

REMARK. In case $\nu > 1$, the foregoing proof requires us to notice that $j_{\nu 2}''$ exceeds the larger of the two positive roots which $p_\nu(x)$ possesses when $\nu > 1$. That $j_{\nu 2}''$ does so is a consequence of $p_\nu(j_{\nu 1}'') < 0$.

Next we observe that $p_\nu(j_{\nu 1}'') \leq 0, -1 < \nu < 0$, if $j_{\nu 1}'' < j_{\nu 1}$, as may be seen from (2), since $J_\nu'''(j_{\nu 1}'') \leq 0$.

It is indeed possible that $J_\nu'''(j_{\nu 1}'') = 0$, say for $\nu = \lambda$. This is equivalent to $j_{\nu 1}''$ being a double zero of $J_\nu''(x)$, $\nu = \lambda$, a situation studied numerically by M. K. Kerimov and S. L. Skorokhodov in [6] and [7]. In this case,

$$j_{\lambda 1}''^{\,2} = \lambda^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\lambda^2 + 1}; \tag{8}$$

the right-hand side is the square of the (unique) positive root of $p_\lambda(x) = 0$.

If, on the other hand, $J_\nu'''(j_{\nu 1}'') < 0$, as can also occur, then $J_\nu(x)$ does have a point of inflection when $x = j_{\nu 1}''$, and conversely.

LEMMA 5. *If $\nu > -1$, then $j_{\nu 1} - \mu_\nu$ is an increasing function of ν. There exists α, $-\frac{2}{3} < \alpha < -\frac{1}{2}$, such that*

$$\begin{cases} j_{\nu 1} < \mu_\nu, & -1 < \nu < \alpha, \\ j_{\alpha 1} = \mu_\alpha, & \\ j_{\nu 1} > \mu_\nu, & \alpha < \nu < \infty. \end{cases} \tag{9}$$

PROOF. For $-1 < \nu \leq 0$, all that is needed in the application to be made in this paper, the desired monotonicity is obvious, since $j_{\nu 1}$ increases for $\nu > -1$ [12, p. 508, (3)], while μ_ν decreases, $-1 < \nu < 0$. This is also sufficient to establish the bounds claimed for α, since $j_{-1/2,1} = \pi/2 > \mu_{-1/2} = (3/4 + \sqrt{3}/2)^{1/2} = 1.27122\ldots$, while [1] $j_{-2/3,1} = 1.2304\ldots < (17/18 + \sqrt{41}/6)^{1/2} = 1.41832\ldots = \mu_{-2/3}$.

To establish the Lemma for $\nu > 0$ (since it does add another inequality for $j_{\nu 1}$) we recall that $dj_{\nu 1}/d\nu > 1$ [3] and consider

$$\frac{d}{d\nu}(j_{\nu 1} - \mu_\nu) = \frac{dj_{\nu 1}}{d\nu} - \frac{\nu + 2\nu(8\nu^2 + 1)^{-1/2}}{(\nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1})^{1/2}}.$$

This equation, by the way, incorporates a proof for $-1 < \nu \leq 0$. To use it to complete a proof for $\nu > 0$, we need to show that

$$\nu + 2\nu(8\nu^2 + 1)^{-1/2} < (\nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1})^{1/2},$$

i.e.,

$$\nu^2 + \frac{4\nu^2}{\sqrt{8\nu^2 + 1}} + \frac{4\nu^2}{8\nu^2 + 1} < \nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1},$$

or

$$\frac{8\nu^2}{\sqrt{8\nu^2 + 1}} + \frac{8\nu^2}{8\nu^2 + 1} < \sqrt{8\nu^2 + 1} + 1,$$

which clearly holds.

REMARKS. 1. The upper bound for α can be reduced readily by using an inequality for $j_{\nu 1}$ due to Elbert [2], namely, $j_{\nu 1}^2 > (\nu + 1)(\nu + 5)$, $-1 < \nu < 0$, which in turn is at least

$$\nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1},$$

for $\nu \geq -10/17$. Thus $\alpha \leq -10/17 = -0.588235294\ldots$.

2. The value of α is approximately -0.60731, as may be inferred by using the Rayleigh upper and lower bounds for $j_{\nu 1}$ [12, p. 502]. These bounds are particularly sharp when $-1 < \nu < 0$.

3. **The existence and number of inflection points.** So far we have not established either the existence or non-existence in $0 < x < j_{\nu 1}$ of inflection points of $J_\nu(x)$ for any part of the ν-interval $-1 < \nu < 0$. This will be done now. It will be shown that $J_\nu(x)$ has two inflection points before its first positive zero when $-0.1993707809\ldots < \nu < 0$ and none in $0 < x < j_{\nu 1}$ for $-1 < \nu \le -0.1993707809\ldots$. Ifantis, Kokologiannaki and Kouris [5, Theorem 3.4 and Remark 3.6] have the result that there are no such zeros for $-1 < \nu \le -0.3934$.

THEOREM 1. *There exists a unique number* $\lambda = -0.1993707809\ldots$, *such that*

$$(10)\qquad \begin{cases} j''_{\nu 1} < j''_{\nu 2} < j_{\nu 1}, & \lambda < \nu < 0, \\ j''_{\nu 1} > j_{\nu 1}, & -1 < \nu < \lambda, \\ j''_{\lambda 1} = (\lambda^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\lambda^2+1})^{1/2} = 1.0553517724\ldots < j_{\lambda 1} < j''_{\lambda 2}. \end{cases}$$

PROOF. We break the proof into three parts, establishing respectively the uniqueness, the existence and the approximate value of λ. The method and ideas of part (i) of the proof will be used also in the proof of Theorem 2.

(i) *Uniqueness of* λ: We use the differential equation (1) and the recurrence relation [12, p. 45, (4)]

$$(11)\qquad xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x).$$

From these we see that the positive zeros of $J''_\nu(x)$ occur where

$$(12)\qquad \frac{J_{\nu+1}(x)}{J_\nu(x)} = \frac{x^2 - \nu^2 + \nu}{x}.$$

In view of the Mittag-Leffler partial fractions expansion [12, p. 498, (1)]

$$(13)\qquad \frac{J_{\nu+1}(x)}{J_\nu(x)} = \sum_{k=1}^{\infty} \frac{2x}{j_{\nu k}^2 - x^2},$$

the positive roots of $J''_\nu(x)$ are the same as those of the equation

$$(14)\qquad G_\nu(x) := 2\sum_{k=1}^{\infty} \frac{1}{j_{\nu k}^2 - x^2} + \frac{\nu^2 - \nu}{x^2} = 1.$$

It is clear that

$$\lim_{x\to 0^+} G_\nu(x) = \lim_{x\to j_{\nu 1}^-} G_\nu(x) = +\infty, \qquad -1 < \nu < 0.$$

Consequently the graph of $y = G_\nu(x)$ is U-shaped, with a unique minimum, on $0 < x < j_{\nu 1}$, $-1 < \nu < 0$, since $G''_\nu(x) > 0$, $0 < x < j_{\nu 1}$, $-1 < \nu < 0$. To verify this we write

$$G''_\nu(x) = 4\sum_{k=1}^{\infty} \frac{j_{\nu k}^2 + 3x^2}{(j_{\nu k}^2 - x^2)^3} + \frac{6(\nu^2 - \nu)}{x^4}, \qquad x \ne 0,\ j_{\nu k}.$$

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That $G'_\nu(x)$ (see (17)) and $G''_\nu(x)$ exist and have the representations stated follows from the uniform convergence over compact subsets of $(0,j_{\nu 1})\cup(j_{\nu 1},j_{\nu 2})\cup\cdots\cup(j_{\nu k},j_{\nu,k+1})\cup\ldots$ of both differentiated series.

Also, for each fixed x in the interval $0<x<j_{\nu 1}$, $G_\nu(x)$ is a decreasing function of ν, $-1<\nu\leq 0$. More precisely, for $-1<\nu<\nu+\epsilon\leq 0$, we have $G_{\nu+\epsilon}(x)<G_\nu(x)$, $0<x<j_{\nu 1}$, since [12, p. 508] each zero $j_{\nu k}$ is an increasing function of ν.

The zeros of $J''_\nu(x)$ occur where the U-shaped graph of $y=G_\nu(x)$ crosses the horizontal line $y=1$. Now it is clear from the consequence

$$\mu_\nu^2>2(\nu^2-\nu)$$

of Lemma 2 that there are no zeros for ν close to -1; there are no crossings for these values of ν. However as ν increases, the U-shaped curve referred to above becomes lower and if it meets the line $y=1$ will do so for a unique value λ. (Recall the uniqueness of the minimum of the U-shaped graph of $y=G_\nu(x)$.)

(ii) *Existence of* λ: The existence of such a λ is established as follows. Suppose that no such λ exists. In that case we would have for all ν satisfying $-1<\nu<0$,

$$G_\nu(x)>1,\quad 0<x<j_{\nu 1}.$$

Taking the limit as $\nu\longrightarrow 0^-$, we would get

$$G_0(x)\geq 1,\quad 0<x<j_{01}.$$

Now

$$G_0(x)=2\sum_{n=1}^{\infty}\frac{1}{j_{0n}^2-x^2}$$

is continuous on $[0,j_{\nu 1})$ so we would get $G_0(0)\geq 1$. But [12, p. 502]

$$G_0(0)=2\sum_{n=1}^{\infty}\frac{1}{j_{0n}^2}=1/2.$$

Hence we have a contradiction and so λ exists as asserted.

(iii) *Evaluation of* λ: In view of Lemma 2, the actual value of λ follows from a determination of the sign of $J''_\nu(\mu_\nu)$ for $\lambda<\nu<0, \nu=\lambda, -1<\nu<\lambda$. For those ν for which that sign is negative there are two points of inflection preceding the first positive zero. When the sign is positive, there are no such points and when it is zero there is a double zero of $J''_\nu(x)$ which does not yield a point of inflection.

This sign, the same as that of $\phi(\mu_\nu)$, will be determined by a study of the infinite series (4). Here, from (6),

$$r_2(\mu_\nu)=\frac{(\nu+5)(\nu+6)(\nu^2+\frac{1}{2}+\frac{1}{2}\sqrt{8\nu^2+1})}{12(\nu+3)^2(\nu+4)}<1,\quad -1<\nu<0.$$

Thus the terms in the alternating series in (4), evaluated at $x=\mu_\nu$, decrease to zero beginning with $m=2$, when, as here, $-1<\nu<0$. If we write

$$\phi_n(\mu_\nu)=\sum_{m=0}^{n}\frac{(-1)^m(2m+\nu)(2m+\nu-1)\mu_\nu^{2m}}{4^m m!\,\Gamma(\nu+m+1)}$$

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for the partial sums of $\phi(\mu_\nu)$, we have, for $-1 < \nu < 0$,

$$\phi_8(\mu_\nu) > \phi(\mu_\nu) > \phi_7(\mu_\nu).$$

To determine when $\phi(\mu_\nu)$, and hence also $J''_\nu(\mu_\nu)$, changes sign, we consider equivalently the polynomials $\Gamma(\nu+7)\phi_7(\mu_\nu)$ and $\Gamma(\nu+7)\phi_8(\mu_\nu)$. The former changes from negative to positive only when ν decreases from -0.19937078099 to -0.199370781, the latter only when ν decreases from -0.19937078098 to -0.19937078099. Thus $J''_\nu(\mu_\nu) = 0$ when $\nu = \lambda = -0.1993707809\ldots$. This conforms with the value recorded in [5, p. 104] and with the calculation made independently by Mr. Edgar Smart of the Department of Computer Science, University of Toronto.

4. **Monotonicity of the abscissas with respect to ν.** In [10] and [13] it was established that for all $\nu > 0$, the abscissas of the points of inflection of $J_\nu(x)$ are increasing functions of ν. The corresponding result for $-1 < \nu < 0$ is somewhat different.

THEOREM 2. *With the notation of Theorem 1, $j''_{\nu 1}$ decreases to* 0 *and $j''_{\nu 2}$ increases to $j''_{01} = 1.841184\ldots$, as ν increases on the interval $(\lambda, 0)$, so that $j''_{\nu 2} - j''_{\nu 1}$ increases from* 0 *to j''_{01} as ν increases from λ to* 0.

PROOF. Here we use the method which was described in part (i) of the proof of Theorem 1. As we pointed out there, the zeros of $J''_\nu(x)$ occur where the U-shaped graph of $y = G_\nu(x)$ crosses the horizontal line $y = 1$. We noted that as ν increases, this U-shaped curve becomes lower and meets the line $y = 1$ when $\nu = \lambda$. Beyond this point it meets the line in two points which move apart, the lefthand one, $j''_{\nu 1}$, moving left and the righthand one, $j''_{\nu 2}$, moving right as ν is further increased.

That $j''_{\nu 1}$ decreases to 0 follows from the inequalities

$$2(\nu^2 - \nu) < {j''_{\nu 1}}^2 < -4\nu, \quad \nu_0 \le \nu < 0, \tag{15}$$

where $\nu_0 = -0.17848262\ldots$.

The first inequality in (15) is from Lemma 1. It is repeated here only for comparison, being unnecessary for the conclusion that $j''_{\nu 1}$ decreases to 0. We rely on the power series for $\phi(x)$ now with $x = 2\sqrt{-\nu}$. Its terms alternate in sign, approach 0, and, from $m = 1$ on, decrease in absolute value for $x^2 \le -4\nu$, $-3.6 \le \nu < 0$. Therefore

$$\begin{aligned} 2\Gamma(\nu+3)\phi(2\sqrt{-\nu}) &< 2\Gamma(\nu+3)\phi_2(2\sqrt{-\nu}) \\ &= \nu\left[5\nu^3 + 21\nu^2 + 26\nu + 4\right] \le 0, \quad \nu_0 \le \nu < 0. \end{aligned}$$

Hence, $J''_\nu(2\sqrt{-\nu}) < 0$, $\nu_0 \le \nu < 0$, showing that ${j''_{\nu 1}}^2 < -4\nu$, $\nu_0 \le \nu < 0$, since $J''_\nu(x) > 0$ for all sufficiently small $x > 0$.

We have shown that $j''_{\nu 2}$ increases. It remains to show that it increases to j''_{01}. The U-shaped graph of $G_\nu(x)$, $-1 < \nu < 0$ lies above the strictly increasing graph of $G_0(x)$ on the interval $0 < x < j_{\nu 1}$ so $j''_{\nu 2} < j''_{01}$ and so there exists Λ such that $j''_{\nu 2} \uparrow \Lambda \le j''_{01}$ as ν increases from λ to 0. Thus, $J''_\nu(\Lambda) > 0$, $\lambda < \nu < 0$. But $J''_\nu(x)$ is continuous (even analytic) in ν so that $0 < J''_\nu(\Lambda) \to J''_0(\Lambda)$ as $\nu \to 0^-$. Hence $J''_0(\Lambda) \geq 0$. Therefore $\Lambda \geq j''_{01}$, since $J''_0(x) < 0$, $0 < x < j''_{01}$, and so $\Lambda = j''_{01}$, as asserted.

The approximate value of j''_{01} was calculated by noting that, from (12), it is the first positive zero of $J_1(x) - xJ_0(x)$, and using the IMSL routine BSJS to evaluate the latter function.

This completes the proof of the Theorem.

The next result overlaps with [10], especially Theorem 5.1, but its main import is to establish monotonicity for $j''_{\nu k}$ in $-1 < \nu < 0$ when the zeros of $J''_\nu(x)$, larger than $j_{\nu 1}$, are at issue.

THEOREM 3. *If $j''_{\nu\kappa} > j_{\nu 1}$, then $j''_{\nu k}$ increases, $-1 < \nu \leq 1/2$, $k = \kappa, \kappa + 1, \ldots$. Further, $j''_{\nu k}$ increases, $-1 < \nu < \infty$, $k = 3, 4, \ldots$.*

PROOF. With $G_\nu(x)$ defined as in (14), we have

$$G_\nu(j_{\nu k}{}^+) = -\infty, \quad G_\nu(j_{\nu k}{}^-) = +\infty, \quad k = 2, 3, \ldots .$$

Further,

(16) $\qquad G_\nu(x)$ increases for $j_{\nu k} < x < j_{\nu,k+1}, \quad k = 1, 2, \ldots .$

To see this we use

$$G'_\nu(x) = 4\sum_{k=1}^{\infty} \frac{x}{(j_{\nu k}^2 - x^2)^2} - \frac{2(\nu^2 - \nu)}{x^3}, \quad x \neq 0, j_{\nu k}. \tag{17}$$

Combining the isolated term with the first term of the infinite series always yields here a positive result, since

$$2\left(\frac{x^2}{x^2 - j_{\nu 1}^2}\right)^2 > 2 > \nu^2 - \nu, \quad -1 < \nu \leq 1.$$

Thus $G'_\nu(x) > 0$, $x > j_{\nu 1}$, $x \neq j_{\nu k}$, $k = 1, 2, \ldots$ and so (16) holds. We also have

(18) $\qquad G_\nu(x)$ decreases for fixed x as ν increases.

This follows from

$$\frac{\partial G_\nu(x)}{\partial \nu} = -4\sum_{1}^{\infty} \frac{j_{\nu k}\partial j_{\nu k}/\partial \nu}{(j_{\nu k}^2 - x^2)^2} + \frac{2\nu - 1}{x^2}$$

and this is negative for $-1 < \nu \leq 1/2$, since $j_{\nu k}$ increases for each k, $\nu > -1$ [12, p. 508]. The proof of the first sentence now concludes as for Theorem 2.

Combined with the results of [10] and [13], Theorem 3 establishes that $j''_{\nu k}$ is an increasing function of ν, $-1 < \nu < \infty$, for each fixed $k = \kappa, \kappa + 1, \ldots$, when $j''_{\nu\kappa} > j_{\nu 1}$. This condition is satisfied when $\kappa = 3$, since $j''_{\nu 3} > j_{\nu 1}$ even when $j''_{\nu 2} < j_{\nu 1}$, for all $-1 < \nu < \infty$ (Lemma 2). The second part of the theorem is now also proved.

REMARK. When the interval $-1 < \nu < \infty$ is considered *in toto*, κ cannot be taken any smaller, since two points of inflection come into existence when $\lambda < \nu < 0$,

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before the others and one of them disappears when $\nu \geq 0$ (Theorem 1). One of these increases, but the other decreases (Theorem 2), so that a mere change in notation would not abbreviate the statement of results. If attention is restricted either to $-1 < \nu < \lambda$ or to $0 < \nu < \infty$, then for each of these intervals κ could be chosen to be 1, even though the statement relative to $0 < \nu < \infty$ conceals a change in notation: $j''_{\nu 1}$ for $0 < \nu \leq 1$ becomes $j''_{\nu 2}$ once $\nu > 1$ since a new (and preceding) point of inflection comes into being when $\nu > 1$ [12, p. 486].

5. **An upper bound for $j''_{\nu 1}$.** The upper bound for $j''^{\,2}_{\nu 1}$, given in (15), adequate for the purpose it served, is valid for the subinterval $\nu_0 < \nu < 0$ of $\lambda < \nu < 0$. A similar upper bound can be established valid over the entire interval $\lambda < \nu < 0$, exact at both endpoints, but at the cost of being larger.

THEOREM 4. *If $\lambda < \nu < 0$, then $j''^{\,2}_{\nu 1} < -\beta\nu$, where $-\lambda\beta = \lambda^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\lambda^2 + 1}$ so that $\beta = 5.586412241$.*

PROOF. The conclusion of the theorem follows from the statement that $J''_\nu(\sqrt{-\beta\nu}) < 0$, $\lambda < \nu < 0$, since $J''_\nu(x) > 0$ for all sufficiently small $x > 0$, $-1 < \nu < 0$. Using the notation defined in (4), this is the same as proving that $\phi(\sqrt{-\beta\nu}) < 0$, $\lambda < \nu < 0$, also as establishing

$$f(\nu) := \sum_{m=0}^{\infty} \frac{(2m+\nu)(2m+\nu-1)\beta^m \nu^{m-1}}{4^m m!\,\Gamma(\nu+m+1)} > 0, \quad \lambda < \nu < 0. \tag{19}$$

To prove this last inequality, we write

$$f(\nu) = \sum_{m=0}^{2n} + \sum_{m=2n+1}^{\infty} := f_{2n}(\nu) + F_{2n+1}(\nu), \quad n = 1, 2, \ldots . \tag{20}$$

The infinite series defining $F_{2n+1}(\nu)$ is an alternating series with positive first term, since $\lambda < \nu < 0$. Its terms decrease (to 0) already for $n = 1$. Hence

$$F_3(\nu) > F_5(\nu) > \cdots > F_{2n+1}(\nu) > \cdots > 0, \quad -1 < \nu < 0, \tag{21}$$

and

$$f_2(\nu) < f_4(\nu) < \cdots < f_{2n}(\nu) < \cdots < f(\nu), \quad -1 < \nu < 0, \quad n = 1, 2, \ldots . \tag{22}$$

We shall show that each $f_{2n}(\nu)$ has a zero, ρ_n, in $\lambda < \nu < 0$, that $f_{2n}(\nu) > 0$, $\rho_n < \nu < 0$ and that ρ_n decreases as n increases, from which it will follow, as required, that $f(\nu) > 0$, $\lambda < \nu < 0$, since $f(\lambda) = 0$, from Theorem 1.

We examine $f_2(\nu)$ by writing

$$\begin{aligned} 32\Gamma(\nu+3) f_2(\nu) = (\beta^2 + 8\beta + 32)\nu^3 + (7\beta^2 + 40\beta + 64)\nu^2 \\ + 4(3\beta^2 + 16\beta - 8)\nu + 32(\beta - 2). \end{aligned}$$

It is easily seen that this polynomial and hence $f_2(\nu)$ vanish in $-1 < \nu < 0$ at $\rho_2 = -0.188623404232$; indeed this is the polynomial's only real root. The precise numerical

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value is a matter of indifference. However, it is of importance to note that $f_2(\nu) > 0$, $\rho_2 < \nu < 0$.

Hence, from (22), $f_4(\nu) > 0$, $\rho_2 \le \nu < 0$, so that any root, ρ_4, of $f_4(\nu)$ in $-1 < \nu < 0$ must precede ρ_2. If there were no root, ρ_4, in $-1 < \nu < 0$, then of course this would conclude the proof. But there must be such a root, since $f(\lambda) = 0$, and it must lie in $\lambda < \nu < 0$ in view of (22).

This applies to all $f_{2n}(\nu)$ for all $n = 1, 2, \ldots$, as well. It is not necessary to establish that there is only one such root in $\lambda < \nu < 0$. We define ρ_n to be the largest such root.

Repeating the reasoning which established that $\lambda < \rho_4 < \rho_2 < 0$, we find that

$$\lambda < \cdots < \rho_{2n+2} < \rho_{2n} < \cdots < \rho_4 < \rho_2 < 0,$$

and that $f(\nu) > 0$, $\rho_{2n} \le \nu < 0$, $n = 1, 2, \ldots$. Hence ρ_{2n} approaches a limit $\lambda' \ge \lambda$, as $n \longrightarrow \infty$ and, with $f(\nu) > 0$, $\lambda' < \nu < 0$. (The fact that $f_{2n}(\nu) \longrightarrow f(\nu)$ uniformly, $\lambda \le \nu \le 0$ follows from the fact that the alternating series for $F_{2n+1}(\nu)$ approaches 0 uniformly in ν, in any interval $-1 + \epsilon \le \nu \le 0$.) Therefore, $0 = f_{2n}(\rho_{2n}) \longrightarrow f(\lambda')$. But, from Theorem 1, $f(\nu) = 0$, $-1 < \nu < 0$, only when $\nu = \lambda$ and so $\lambda' = \lambda$, completing the proof.

REMARK. The exactness at $\nu = \lambda$ of the upper bound given here follows from Theorem 1 since $j''_{\lambda 1}$ is the unique double root of $J''_\nu(x)$, $-1 < \nu < 0$. For $-1 < \nu < \lambda$, ${j''_{\nu 1}}^2 > {j_{\nu 1}}^2 > \nu^2 + 6\nu + 5$.

6. Monotonicity properties of ordinates and slopes at points of inflection.

As when $\nu > 0$, we are able to establish in the case $-1 < \nu < 0$ certain monotonicity properties of the ordinates and slopes at the points of inflection of $J_\nu(x)$, for fixed order and changing rank.

THEOREM 5. *If $-1 < \nu < 0$, then the sequences $\{|J_\nu(j''_{\nu k})|\}$, $k = 1, 2, \ldots$, and $\{|J'_\nu(j''_{\nu k})|\}$, $k = 2, 3, \ldots$ both decrease. For $-1 < \nu \le \lambda$, with λ as in Theorem 1, then also $|J'_\nu(j''_{\nu 1})| > |J'_\nu(j''_{\nu 2})|$ while for $\lambda < \nu < 0$, $|J'_\nu(j''_{\nu 1})| < |J'_\nu(j''_{\nu 2})|$.*

PROOF. These results will emerge chiefly, but not exclusively, from the Sonin-Pólya-Butlewski Theorem [11, p.166, footnote] of differential equations. The differential equation (2) for $y' = J'_\nu(x)$ can be put in the form

$$\big(g(x)y'\big)' + f(x)y = 0,$$

where

$$g(x) = \frac{x^3}{x^2 - \nu^2}, \quad f(x) = \frac{x p_\nu(x)}{(x^2 - \nu^2)^2}, \tag{23}$$

with $p_\nu(x)$ defined as in (3). The function $p_\nu(x)$ has exactly one positive zero, μ_ν, with $\mu_\nu^2 = \nu^2 + \frac{1}{2} + \frac{1}{2}\sqrt{8\nu^2 + 1} > \nu^2$, when, as here, $\nu^2 \le 1$.

The hypotheses of the Sonin-Pólya-Butlewski Theorem require that $f(x) > 0$, $g(x) > 0$, $D_x\{f(x)g(x)\} > 0$, for the x-interval in which it is to be applied.

We denote by $j''_{\nu\kappa}$ the first zero of $J''_\nu(x)$ which exceeds μ_ν and proceed to establish that the two sequences in question decrease for $k = \kappa, \kappa + 1, \ldots$.

It is clear that $f(x)$ and $g(x)$ are positive for $x \geq \mu_\nu > |\nu|$. To show that $D_x\{f(x)g(x)\} > 0$, $x > |\nu|$, when $\nu^2 < 1$, we write

$$\begin{aligned}\frac{1}{2}x^{-3}(x^2-\nu^2)^4\frac{d}{dx}\{f(x)g(x)\}\\ = x^6 - 4\nu^2x^4 + \nu^2(5\nu^2+4)x^2 - 2\nu^4(\nu^2-1) := \Phi_\nu(x).\end{aligned}$$

Now

$$\Phi'_\nu(x) = 2x[3x^4 - 8\nu^2x^2 + \nu^2(5\nu^2+4)],$$

a polynomial whose only real root is $x = 0$ when $\nu^2 < 12$ and is therefore positive for all $x > 0$. Hence, $\Phi_\nu(x)$ increases from $\Phi_\nu(0) = 2\nu^4(1-\nu^2) > 0$ and is therefore positive for all $x \geq 0$ when $\nu^2 < 1$.

The Sonin-Pólya-Butlewski Theorem therefore enables us to conclude that,

(24) $\quad$ for $j''_{\nu\kappa} > \mu_\nu$, $\{|J'_\nu(j''_{\nu k})|\}$ decreases, $k = \kappa, \kappa+1, \ldots, \quad -1 < \nu < 0.$

The same information for the other sequence now follows, i.e.,

(25) $\quad$ for $j''_{\nu\kappa} > \mu_\nu$, $\{|J_\nu(j''_{\nu k})|\}$ decreases, $k = \kappa, \kappa+1, \ldots, \quad -1 < \nu < 0.$

The transition is made by putting $x = j''_{\nu k}$ in the differential equation (1). This yields

$$J_\nu^2(j''_{\nu k}) = \left(\frac{j''_{\nu k}}{j''^2_{\nu k} - \nu^2}\right)^2 J'^2_\nu(j''_{\nu k}).$$

When $j''^2_{\nu k} > \nu^2$ as for $k \geq \kappa$, the factor of $J'^2_\nu(j''_{\nu k})$ decreases as k increases. Together with (24) this yields (25).

Now we divide into three cases:

(i) $\lambda < \nu < 0$. Here we may take $\kappa = 2$, since, from Lemma 2 and the Remark following it, $j''_{\nu 2} > \mu_\nu$. Moreover, as we have seen in § 3, $j''_{\nu 1} < j''_{\nu 2} < j_{\nu 1}$, $\lambda < \nu < 0$, while, from (5), $j'_{\nu 1} > j_{\nu 1}$, $-1 < \nu < 0$. Thus

$$J'_\nu(j''_{\nu 1}) > J'_\nu(j''_{\nu 2}), \quad \lambda < \nu < 0$$

as asserted in the statement of the Theorem.

Finally in this case, we note that $J''_\nu(0+) = +\infty$, $-1 < \nu < 0$, so that $J'_\nu(x)$ increases, $0 < x < j''_{\nu 1}$, and decreases, $j''_{\nu 1} < x < j''_{\nu 2} < j_{\nu 1}, \lambda < \nu < 0$. Thus $0 > J'_\nu(j''_{\nu 1}) > J'_\nu(j''_{\nu 2})$ so that $|J'_\nu(j''_{\nu 1})| < |J'_\nu(j''_{\nu 2})|$ as stated in the theorem.

This concludes case (i).

(ii) $-1 < \nu < \lambda$. Here, recalling (5), $j''_{\nu 1} > j'_{\nu 1} > |\nu|$. Putting $x = j''_{\nu 1}$ in (2) we find that $p_\nu(j''_{\nu 1}) > 0$, so that $j''_{\nu 1} > \mu_\nu$, $-1 < \nu < \lambda$. Thus we may take $\kappa = 1$ in (24) and (25), all there is to verify in this case.

(iii) $\nu = \lambda$. Here, (24) and (25) hold for $\kappa = 2$, and it remains to establish that

$$J_\lambda'^{\,2}(j_{\lambda 1}'') > J_\lambda'^{\,2}(j_{\lambda 2}'').$$

This will yield as before also the corresponding inequality for $J_\lambda^2(x)$.

We appeal to [8, Lemma 4.1 (4.2), p. 355] which, in this application, reads

$$J_\lambda'^{\,2}(j_{\lambda 2}'') - J_\lambda'^{\,2}(j_{\lambda 1}'') = l + \int_{j_{\lambda 1}''}^{j_{\lambda 2}''} [g(t)J_\lambda'(t)]^2 D_t\{[f(t)g(t)]^{-1}\}\, dt, \tag{26}$$

where $f(t)$, $g(t)$ are as in (23) and $j_{\lambda 1}'' = \mu_\lambda$.

We need to verify the existence of l and to evaluate it. It is, by definition,

$$\begin{aligned} l &= \lim_{x \to \mu_\lambda} \frac{g(x)J_\lambda''^{\,2}(x)}{f(x)} \\ &= \lim_{x \to \mu_\lambda} \frac{x^2(x^2-\lambda^2)J_\lambda''^{\,2}(x)}{p_\lambda(x)} \\ &= \mu_\lambda^2(\mu_\lambda^2 - \lambda^2) \lim_{x \to \mu_\lambda} \frac{J_\lambda''^{\,2}(x)}{p_\lambda(x)}. \end{aligned}$$

The existence and value of the last limit follow from l'Hospital's rule, since $p_\lambda(\mu_\lambda) = J_\lambda''(\mu_\lambda) = J_\lambda''(j_{\lambda 1}'') = 0$. Thus

$$\begin{aligned} \lim_{x \to \mu_\lambda} \frac{J_\lambda''^{\,2}(x)}{p_\lambda(x)} &= \lim_{x \to \mu_\lambda} \frac{2J_\lambda''(x)J_\lambda'''(x)}{p_\lambda'(x)} \\ &= \frac{2J_\lambda''(j_{\lambda 1}'')J_\lambda'''(j_{\lambda 1}'')}{p_\lambda'(j_{\lambda 1}'')} = 0, \end{aligned}$$

since $p_\lambda'(\mu_\lambda) = 2\mu_\lambda(8\lambda^2+1)^{1/2} > 0$.

Thus, $l = 0$, and so (26) implies

$$|J_\lambda'(j_{\lambda 2}'')| < |J_\lambda'(j_{\lambda 1}'')|,$$

since $f(x)g(x)$ increases in $j_{\lambda 1}'' < x < j_{\lambda 2}''$. This completes the proof of Theorem 5.

REMARK. Some of the assertions of Theorem 5 may be extended to the inflection points of arbitrary solutions $C_\nu(x)$ of the Bessel equation. We can assert, for example, that for $c_{\nu\kappa}'' > \mu_\nu$, $\{|C_\nu'(c_{\nu k}'')|\}$ and $\{|C_\nu(c_{\nu k}'')|\}$ both decrease, $k = \kappa, \kappa+1, \ldots$, $-1 < \nu < 0$, where now $c_{\nu\kappa}''$ is the smallest zero of $C_\nu''(x)$ which exceeds μ_ν. The reason is that the argument for the corresponding part of Theorem 5 depends only on the Bessel differential equation and not on any special properties of the solution $J_\nu(x)$.

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ADDED IN PROOF. The problems treated in [10] and [13] have been dealt with recently by A. McD. Mercer, *The zeros of* $az^2J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z)$ *as functions of order*, Internat. J. Math. and Math. Sci. No. 1 **15**(1992).

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