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A GENERALISATION OF THE LOWER RADICAL CLASS

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In this work we demonstrate that the lower radical class construction on a homomorphically closed class of associative rings generates a radical class for any class of associative rings. We also give a new description of the upper radical class using the construction on an appropriate generating class.

1. Background Preliminaries.

All rings in this work are associative and $\mathbb{A}$ denotes the class of all associative rings. If $\mathcal{P}$ is a class of rings with $\mathcal{P} \subseteq \mathbb{A}$, then a ring A is a $\mathcal{P}$–ring if $A \in \mathcal{P}$; an ideal I of A, denoted $I \triangleleft A$, is a $\mathcal{P}$–ideal of A if $I \in \mathcal{P}$. For a radical class $\mathcal{R}$, the largest $\mathcal{R}$-ideal of a ring A is denoted by $\mathcal{R}(A)$.

A subring B of a ring A is called an *accessible subring* of A if there exists a finite sequence $C_1, \dots, C_n$ of subrings such that

$$B = C_1 \triangleleft C_2 \triangleleft \dots \triangleleft C_n = A.$$

For a detailed exposition of the introductory concepts of the radical theory of associative rings we refer the reader to works by Divinsky [1] or Wiegandt [7].

In brief we mention the Theorem of Anderson, Divinsky and Sulinski (the ADS Theorem) which states that if I is an ideal of a ring A, then $\mathcal{R}(I)$ is an ideal of A.

$\mathcal{R}$ is a *hereditary radical class* if for $A \in \mathcal{R}$ and $I \triangleleft A$, $I \in \mathcal{R}$.

For any radical class $\mathcal{R}$, $\mathcal{S}(\mathcal{R}) = \{A \in \mathbb{A} \mid \mathcal{R}(A) = 0\}$ is a *semisimple* class. A ring A is called *strongly $\mathcal{R}$-semisimple* for a radical class $\mathcal{R}$ if every homomorphic image of A is an $\mathcal{R}$-semisimple ring and we denote the class of strongly $\mathcal{R}$-semisimple rings by $\widehat{\mathcal{S}(\mathcal{R})}$. The intersection of semisimple classes is semisimple and the intersection of radical classes is a radical class. A class is called a *radical-semisimple class* if it is a radical class as well as a semisimple class.

Any class $\mathcal{P}$ satisfying *every nonzero ideal of a ring A can be homomorphically mapped onto a nonzero $\mathcal{P}$-ring* is called a *regular* class. Define the *upper radical* operator

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$\mathcal{U}$ acting on an arbitrary class $\mathcal{M}$ by $\mathcal{U}(\mathcal{M}) = \{A \in \mathbb{A} \mid A$ has no nonzero homomorphic image in $\mathcal{M}\}$ and recall that if $\mathcal{M}$ is a regular class then $\mathcal{U}(\mathcal{M})$ is a radical class. Further, when $\mathcal{M}$ is regular, the *semisimple closure* of $\mathcal{M}$, denoted by $\overline{\mathcal{M}}$, is well defined since semisimple classes of associative rings are hereditary.

Let $\mathcal{X}$ be a homomorphically closed class. Then the class of rings for which every nonzero homomorphic image has a nonzero accessible subring in $\mathcal{X}$ is a radical class called the *lower radical class determined by* $\mathcal{X}$ and denoted $\mathcal{L}(\mathcal{X})$. $\mathcal{L}(\mathcal{X})$ is the smallest radical class containing $\mathcal{X}$.

Snider [**6**] showed that under suitable conditions a lattice of radical classes may be formed. Denote by L_{AR} the lattice of all radical classes and by L_{AHR} the sublattice of all hereditary radical classes. The meet $(\wedge)$ of two radical classes is the largest radical class common to both and the join $(\vee)$ is the smallest radical class containing both. For radical classes $\mathcal{R}_1$ and $\mathcal{R}_2$,

$$\mathcal{R}_1 \vee \mathcal{R}_2 = \mathcal{L}(\mathcal{R}_1 \cup \mathcal{R}_2) = \mathcal{U}\big(\mathcal{S}(\mathcal{R}_1) \cap \mathcal{S}(\mathcal{R}_2)\big).$$
$$\mathcal{R}_1 \wedge \mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{R}_2.$$

A lattice L is *Brouwerian* if for any two elements $A, B \in L$, there exists a largest element C such that $A \wedge C \leqslant B$. All Brouwerian lattices are distributive. In the special case where $B = 0$, C is called the *pseudocomplement* of A.

Puczyłowski [**5**] demonstrated that, for any radical class $\mathcal{R}$, the class of rings for which every nonzero accessible subring is an $\mathcal{R}$-ring is the largest hereditary radical subclass of $\mathcal{R}$. We denote this subradical by $\mathcal{T}_{\mathcal{R}}$.

PROPOSITION 1. [**4**] *Any nonzero, hereditary and homomorphically closed class of rings $\mathcal{X}$ contains a simple ring.*

COROLLARY 2. *A nonzero class $\mathcal{X}$ contains a hereditary, homomorphically closed subclass if and only if $\mathcal{X}$ contains a simple ring.*

The lower radical class determined by a homomorphically closed class $\mathcal{X}$ has the same collection of simple rings as $\mathcal{X}$.

2. THE BASE RADICAL CLASS.

This work was motivated by the following example in which a radical class is constructed using the lower radical class construction but starting with a nonhomomorphically closed class.

Let $\mathcal{X}$ be any class of associative rings and define $\widetilde{\mathcal{X}} = \{A \in \mathbb{A} \mid A$ has a nonzero homomorphic image in $\mathcal{X}\}$. $\widetilde{\mathcal{X}}$ is called the *homomorphic cover* of $\mathcal{X}$ [**4**].

The class $\mathcal{H}_{\mathcal{X}} = \{A \in \mathbb{A} \mid$ every nonzero accessible subring of A does not map homomorphically onto a nonzero $\mathcal{X}$-ring$\}$ is clearly hereditary. Let us now consider the upper radical class determined by $\mathcal{H}_{\mathcal{X}}$. If $A \in \mathcal{U}(\mathcal{H}_{\mathcal{X}})$, then A has no nonzero homomorphic image in $\mathcal{H}_{\mathcal{X}}$ and so every nonzero homomorphic image of A has an accessible subring which maps homomorphically onto a nonzero $\mathcal{X}$-ring. That is, every nonzero homomorphic image of A has a nonzero accessible subring in $\widetilde{\mathcal{X}}$.

The description of $\mathcal{U}(\mathcal{H}_{\mathcal{X}})$ is similar to the usual lower radical determined by a homomorphically closed class. Interestingly, for any class $\mathcal{X}$ with $\widetilde{\mathcal{X}} \neq \mathbb{A}$, $\widetilde{\mathcal{X}}$ is never homomorphically closed since if $A \in \widetilde{\mathcal{X}}$ and $B \notin \widetilde{\mathcal{X}}$, $A \oplus B$ has a nonzero homomorphic image in $\mathcal{X}$ so that $A \oplus B \in \widetilde{\mathcal{X}}$ but $A \oplus B$ also has a nonzero homomorphic image which is not in $\widetilde{\mathcal{X}}$. More surprisingly, $\widetilde{\mathcal{X}}$ may not be contained in the radical class generated.

For suppose $\mathcal{X}$ is the class $\{Z_3, Z_4\}$. Clearly, $\{Z_3, Z_4\}$ is not homomorphically closed and $Z_4 \subseteq \widetilde{\{Z_3, Z_4\}}$. $\mathcal{U}(\mathcal{H}_{\{Z_3,Z_4\}}) = \{A \in \mathbb{A} \mid$ every nonzero homomorphic image of A has an accessible subring in $\widetilde{\{Z_3, Z_4\}}\}$ but $Z_4 \notin \mathcal{U}(\mathcal{H}_{\{Z_3,Z_4\}})$ since Z_4 maps homomorphically onto $Z_2 \in \mathcal{H}_{\{Z_3,Z_4\}}$. Hence $\widetilde{\{Z_3, Z_4\}}$ is not contained in $\mathcal{U}(\mathcal{H}_{\{Z_3,Z_4\}})$.

Now let $\mathcal{L}_b(\mathcal{X}) = \{A \in \mathbb{A} \mid$ every nonzero homomorphic image of A has a nonzero accessible subring in $\mathcal{X}\}$. $\mathcal{L}_b(\mathcal{X})$ will be the lower radical class determined by $\mathcal{X}$ when $\mathcal{X}$ is homomorphically closed. We generalise this result by demonstrating

Theorem 3. *For any class of rings $\mathcal{X}$, $\mathcal{L}_b(\mathcal{X})$ is a radical class.*

Proof: Let $\mathcal{X}$ be a class of associative rings and let $A \in \mathcal{L}_b(\mathcal{X})$ with $I \triangleleft A$ such that $A/I \neq 0$. Since every nonzero homomorphic image of A has a nonzero accessible subring in $\mathcal{X}$, then every nonzero homomorphic image of A/I has a nonzero accessible subring in $\mathcal{X}$, $\mathcal{L}_b(\mathcal{X})$ is homomorphically closed and $\mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}(\mathcal{L}_b(\mathcal{X}))$, the lower radical class determined by $\mathcal{L}_b(\mathcal{X})$.

$\mathcal{L}(\mathcal{L}_b(\mathcal{X}))$ is homomorphically closed and for $A \in \mathcal{L}(\mathcal{L}_b(\mathcal{X}))$ with $I \triangleleft A$ and $A/I \neq 0$, $A/I \in \mathcal{L}(\mathcal{L}_b(\mathcal{X}))$. Therefore A/I has a nonzero accessible subring in $\mathcal{L}_b(\mathcal{X})$, say B, and B has a nonzero accessible subring in $\mathcal{X}$, say B'. Therefore, B' is a nonzero accessible subring of A/I and $A \in \mathcal{L}_b(\mathcal{X})$. Hence $\mathcal{L}(\mathcal{L}_b(\mathcal{X})) \subseteq \mathcal{L}_b(\mathcal{X})$ and $\mathcal{L}_b(\mathcal{X})$ is a radical class. $\square$

We call $\mathcal{L}_b(\mathcal{X})$ the *base radical class determined by* $\mathcal{X}$. Some straightforward examples illustrate the versatility and variability of the radical class generated. If $\mathcal{X} = \{Z_3\}$, then $\mathcal{L}_b(\mathcal{X}) \cap \mathcal{X} = \mathcal{X}$ since $\{Z_3\}$ is homomorphically closed. $\mathcal{L}_b(Z_3)$ is the lower radical class determined by Z_3. If $\mathcal{X} = \{Z_4\}$, then $\mathcal{L}_b(\mathcal{X}) \cap \mathcal{X} = 0$ since Z_4 maps homomorphically onto Z_2 and Z_2 does not have Z_4 as an accessible subring. As noted earlier, if $\mathcal{X} = \{Z_3, Z_4\}$, $\mathcal{L}_b(\mathcal{X}) \cap \mathcal{X} \neq 0$ and $\mathcal{L}_b(\mathcal{X}) \cap \mathcal{X} \neq \mathcal{X}$ since $Z_3 \in \mathcal{L}_b(\{Z_3, Z_4\})$ and $Z_4 \notin \mathcal{L}_b(\{Z_3, Z_4\})$.

From this point we shall develop the work in terms of the base radical class $\mathcal{L}_b(\mathcal{X})$, using the lower radical class notation $\mathcal{L}(\mathcal{X})$ when the homomorphic closure of $\mathcal{X}$ is certain.

Proposition 4. *For classes $\mathcal{X}$ and $\mathcal{Y}$:*

(i) *If $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}_b(\mathcal{Y})$.*

(ii) *A simple ring $S \in \mathcal{X}$ if and only if $S \in \mathcal{L}_b(\mathcal{X})$.*

(iii) $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(\mathcal{L}_b(\mathcal{X}))$.

PROOF: *(i)* and *(ii)* are clear. For *(iii)*, since $\mathcal{L}_b(\mathcal{X})$ is a radical class for any class $\mathcal{X}$, $\mathcal{L}_b(\mathcal{X})$ is homomorphically closed and $\mathcal{L}_b(\mathcal{L}_b(\mathcal{X})) = \mathcal{L}(\mathcal{L}_b(\mathcal{X})) = \mathcal{L}_b(\mathcal{X})$. ∎

For any class $\mathcal{X}$, let $\mathcal{X}^{+h} = \{A \in \mathbb{A} \mid A$ is the homomorphic image of an $\mathcal{X}$-ring$\}$. Denote the largest homomorphically closed subclass of $\mathcal{X}$ by $\mathcal{X}_h = \{A \in \mathbb{A} \mid$ every homomorphic image of A is in $\mathcal{X}\}$.

COROLLARY 5. *For all classes $\mathcal{X}$, $\mathcal{L}(\mathcal{X}_h) \subseteq \mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}(\mathcal{X}^{+h})$.*

PROOF: As a consequence of Proposition 4*(i)*, for all classes $\mathcal{X}$, $\mathcal{L}(\mathcal{X}_h) = \mathcal{L}_b(\mathcal{X}_h) \subseteq \mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}_b(\mathcal{X}^{+h}) = \mathcal{L}(\mathcal{X}^{+h})$. ∎

PROPOSITION 6. *$\mathcal{X}$ is contained in $\mathcal{L}_b(\mathcal{X})$ if and only if $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}(\mathcal{X}^{+h})$.*

PROOF: We have from Corollary 5 that $\mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}(\mathcal{X}^{+h})$. If $\mathcal{X}$ is contained in $\mathcal{L}_b(\mathcal{X})$, $\mathcal{L}(\mathcal{X}^{+h}) \subseteq \mathcal{L}_b(\mathcal{X})$ since $\mathcal{L}(\mathcal{X}^{+h})$ is the smallest radical class containing $\mathcal{X}$. Conversely, if $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}(\mathcal{X}^{+h})$, $\mathcal{X}$ is contained in $\mathcal{L}_b(\mathcal{X})$ as $\mathcal{X}$ is contained in $\mathcal{L}(\mathcal{X}^{+h})$. ∎

The semisimple class associated with $\mathcal{L}_b(\mathcal{X})$ is $\mathcal{S}(\mathcal{L}_b(\mathcal{X})) = \{A \in \mathbb{A} \mid A$ has no nonzero accessible subring in $\mathcal{L}_b(\mathcal{X})\}$. If $A \in \mathcal{S}(\mathcal{L}_b(\mathcal{X}))$, then every nonzero accessible subring of A has a nonzero homomorphic image C such that C has no nonzero accessible subring in $\mathcal{X}$.

PROPOSITION 7. *If $\mathcal{X} \cap \mathcal{S}(\mathcal{L}_b(\mathcal{X})) = 0$ then $\mathcal{S}(\mathcal{L}_b(\mathcal{X}))$ is the largest semisimple class having zero intersection with $\mathcal{X}$.*

PROOF: Let $\mathcal{P}$ be a radical class such that $\mathcal{S}(\mathcal{P})$ properly contains $\mathcal{S}(\mathcal{L}_b(\mathcal{X}))$ and assume that $\mathcal{S}(\mathcal{P})$ is the largest semisimple class with $\mathcal{X} \cap \mathcal{S}(\mathcal{P}) = 0$. Further, let $A \in \mathcal{S}(\mathcal{P}) \backslash \mathcal{S}(\mathcal{L}_b(\mathcal{X}))$ so that A has an $\mathcal{L}_b(\mathcal{X})$-ideal but no $\mathcal{P}$-ideal. There exists then $0 \neq I \triangleleft A$ such that $I \in \mathcal{L}_b(\mathcal{X})$, and consequently I has a nonzero accessible subring in $\mathcal{X}$, say C. C is a nonzero accessible subring of A. But $C \in \mathcal{S}(\mathcal{P})$ since $\mathcal{S}(\mathcal{P})$ is hereditary. Therefore $C \in \mathcal{X} \cap \mathcal{S}(\mathcal{P})$, a contradiction, and $\mathcal{S}(\mathcal{L}_b(\mathcal{X}))$ must be the largest semisimple class having zero intersection with $\mathcal{X}$. ∎

If $\mathcal{X}$ is contained in $\mathcal{L}_b(\mathcal{X})$, for example if $\mathcal{X}$ is homomorphically closed, then $\mathcal{X} \cap \mathcal{S}(\mathcal{L}_b(\mathcal{X})) = 0$ and $\mathcal{S}(\mathcal{L}_b(\mathcal{X}))$ is the largest semisimple class having zero intersection with $\mathcal{X}$. This confirms that for any radical class $\mathcal{R}$, $\mathcal{S}(\mathcal{R})$ is the largest semisimple class having zero intersection with $\mathcal{R}$.

PROPOSITION 8. *If $\mathcal{R}$ is an hereditary radical class, then $\mathcal{R} \cap \mathcal{L}_b(\mathcal{S}(\mathcal{R})) = 0$.*

PROOF: If $\mathcal{R}$ is an hereditary radical class, then for $A \in \mathcal{R}$ and C a nonzero accessible subring of A, $C \in \mathcal{R}$. It is clear then that $A \notin \mathcal{L}_b(\mathcal{S}(\mathcal{R}))$. ∎

Large radical classes were introduced by Gardner and Liang [2]. A radical class $\mathcal{R}$ is *large* if $\mathcal{R} \wedge \mathcal{T} \neq 0$ for every radical class $\mathcal{T} \neq 0$. If $\mathcal{R}$ is hereditary and large in the lattice of all radical classes, then $\mathcal{L}_b(\mathcal{S}(\mathcal{R})) = 0$. It follows that there are no rings B such that every nonzero homomorphic image of B has a nonzero $\mathcal{R}$-semisimple accessible subring and hence there are no strongly $\mathcal{R}$-semisimple rings. It is clear that whenever nonzero strongly $\mathcal{R}$-semisimple rings exist, for example if $\mathcal{R}$ does not contain all simple rings, then these are contained in the base radical determined by $\mathcal{S}(\mathcal{R})$.

PROPOSITION 9. *For a class of associative rings $\mathcal{X}$, $\mathcal{X} \cap \mathcal{L}_b(\mathcal{X}) = 0$ if and only if $\mathcal{X}$ satisfies*

(1^*) $A \in \mathcal{X} \Rightarrow$ there exists a nonzero homomorphic image C of A such that C has no nonzero accessible subring in $\mathcal{X}$.

The proof is clear. Any class $\mathcal{X}$ satisfying (1^*) is not homomorphically closed. Also, since $\mathcal{X}$ does not contain a simple ring then $\mathcal{X}$ contains no hereditary, homomorphically closed subclass by Corollary 2.

3. ON CLASSES GENERATING $\mathcal{L}_b(\mathcal{X})$.

We now give some results about the generating class for $\mathcal{L}_b(\mathcal{X})$.

PROPOSITION 10. *If there exists a nonzero ring $A_1 \in \mathcal{X}$ such that, for all $A \in \mathcal{X}$, A_1 is an accessible subring of A, then $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(A_1)$.*

PROOF: Suppose there exists a nonzero ring $A_1 \in \mathcal{X}$ such that, for all $A \in \mathcal{X}$, A_1 is an accessible subring of A. If $A \in \mathcal{L}_b(\mathcal{X})$ and $I \lhd A$ with $A/I \neq 0$, A/I has a nonzero accessible subring in $\mathcal{X}$, say C_1, and consequently C_1 has A_1 as a nonzero accessible subring. Therefore, A/I has A_1 as a nonzero accessible subring and $A \in \mathcal{L}_b(A_1)$. By Proposition 4*(i)*, $\mathcal{L}_b(A_1) \subseteq \mathcal{L}_b(\mathcal{X})$, since $A_1 \subseteq \mathcal{X}$, and hence $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(A_1)$. ∎

For every ring B in a hereditary, homomorphically closed class $\mathcal{X}$, define $B^* = \{A \in \mathbb{A} \mid A$ is a nonzero accessible subring of $B\}$. Let $A^* \subseteq \mathcal{X}$ be a class of rings such that $A^* \cap B^* \neq 0$ for all B^*. $A^* = 0$ if and only if $\mathcal{X} = 0$.

PROPOSITION 11. *For $\mathcal{X}$ and A^* described above, $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(A^*)$.*

PROOF: For all $A \in \mathcal{X}$, A has a nonzero accessible subring in A^*. $A^* \subseteq \mathcal{X} \Rightarrow \mathcal{L}_b(A^*) \subseteq \mathcal{L}_b(\mathcal{X})$. Now, as in Proposition 10, for $A \in \mathcal{L}_b(\mathcal{X})$ and $I \lhd A$ with $A/I \neq 0$, A/I has a nonzero accessible subring in $\mathcal{X}$, say C_1, and consequently C_1 has an A^*-ring as a nonzero accessible subring. Therefore, A/I has an A^*-ring as a nonzero accessible subring and $A \in \mathcal{L}_b(A^*)$. ∎

Since $\mathcal{X}$ is hereditary and homomorphically closed here, then $\mathcal{X}$ contains a simple ring S by Proposition 1. (In fact, A^* contains all the simple $\mathcal{X}$-rings.) If $\mathcal{X}$ is a class of rings for which every ring has a simple accessible subring (for example, a class of subdirectly irreducible rings with idempotent hearts) then these simple accessible subrings suffice as a generating class for $\mathcal{L}_b(\mathcal{X})$. It follows that if $\mathcal{X}$ is a class of simple rings, then $\mathcal{X}$ is the smallest class generating $\mathcal{L}_b(\mathcal{X})$.

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For any class $\mathcal{X}$, define $\mathcal{X}^{\triangleleft} = \{A \in \mathbb{A} \mid A$ has a nonzero accessible subring in $\mathcal{X}\}$, called the *accessible cover* of $\mathcal{X}$. $\mathcal{X} \subseteq \mathcal{X}^{\triangleleft}$ and $\mathcal{X}^{\triangleleft} = 0$ if and only if $\mathcal{X} = 0$. The accessible cover is the dual notion of the homomorphic cover mentioned earlier.

PROPOSITION 12. *Let $\mathcal{X}$ be a class of associative rings.*

(i) *A simple ring $S \in \mathcal{X}^{\triangleleft}$ if and only if $S \in \mathcal{X}$.*

(ii) *For any class of rings $\mathcal{X}$, $\mathcal{L}_b(\mathcal{X}^{\triangleleft}) = \mathcal{L}_b(\mathcal{X})$.*

(iii) *For any class of rings $\mathcal{X}$, $\mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}_b(\mathcal{X})^{\triangleleft} \subseteq \mathcal{X}^{\triangleleft}$.*

(iv) *$\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(\mathcal{X})^{\triangleleft} = \mathcal{X}^{\triangleleft}$ if and only if $\mathcal{X}^{\triangleleft}$ is homomorphically closed.*

PROOF: *(i)* is clear. *(ii)*: For any class of rings $\mathcal{X}$, $\mathcal{X} \subseteq \mathcal{X}^{\triangleleft}$ and so $\mathcal{L}_b(\mathcal{X}) \subseteq \mathcal{L}_b(\mathcal{X}^{\triangleleft})$. If $A \in \mathcal{L}_b(\mathcal{X}^{\triangleleft})$ and $I \triangleleft A$ with $A/I \neq 0$, then A/I has a nonzero accessible subring in $\mathcal{X}^{\triangleleft}$ and consequently A/I has a nonzero accessible subring in $\mathcal{X}$. Therefore $A \in \mathcal{L}_b(\mathcal{X})$ and $\mathcal{L}_b(\mathcal{X}) = \mathcal{L}_b(\mathcal{X}^{\triangleleft})$. *(iii)* follows immediately from the definition of $\mathcal{L}_b(\mathcal{X})$ and $\mathcal{X}^{\triangleleft}$. *(iv)* : If $\mathcal{X}^{\triangleleft}$ is homomorphically closed, then for $A \in \mathcal{X}^{\triangleleft}$ and $I \triangleleft A$ with $A/I \neq 0$, A/I has a nonzero accessible subring in $\mathcal{X}$ and so $A \in \mathcal{L}_b(\mathcal{X})$. The converse is clear. $\square$

PROPOSITION 13. *Let $\mathcal{R}$ be a radical class. $\mathcal{R}^{\triangleleft} = \mathbb{A}$ if and only if $\mathcal{R} = \mathbb{A}$.*

PROOF: If $\mathcal{R}^{\triangleleft} = \mathbb{A}$, then every ring $A \in \mathbb{A}$ has a nonzero accessible subring in $\mathcal{R}$ and consequently A has a nonzero $\mathcal{R}$-ideal by the ADS Theorem. Therefore $\mathcal{S}(\mathcal{R}) = 0$ and $\mathcal{R} = \mathbb{A}$. The converse is clear. $\square$

This confirms that there exists a nonzero $\mathcal{R}$-semisimple ring whenever a radical class $\mathcal{R}$ is not the class of all rings.

For any radical class $\mathcal{R}$, the rings having no nonzero accessible subring in $\mathcal{R}$ are the $\mathcal{R}$-semisimple rings. Let $\neg\mathcal{S}(\mathcal{R}) = \{A \in \mathbb{A} \mid A$ is not an $\mathcal{R}$-semisimple ring$\}$. This is clearly the accessible cover $\mathcal{R}^{\triangleleft}$ and it follows from Proposition 12*(ii)* that

COROLLARY 14. *For all radical classes $\mathcal{R}$, $\mathcal{L}_b(\neg\mathcal{S}(\mathcal{R})) = \mathcal{R}$.*

EXAMPLE. The prime radical class β can be expressed as the lower radical determined by the class of zerorings $\mathcal{Z}$. $\mathcal{S}(\beta)$ is the class of semiprime rings. $\neg\mathcal{S}(\beta) = \{A \in \mathbb{A} \mid A$ is not a semiprime ring$\} = \{A \in \mathbb{A} \mid \beta(A) \neq 0\}$. By Corollary 14, $\mathcal{L}_b(\neg\mathcal{S}(\beta)) = \beta$ $(= \mathcal{L}(\mathcal{Z}) = \mathcal{L}_b(\mathcal{Z}))$. We may then construct β from a class properly containing β and from a class properly contained in β.

More interestingly, for any radical class $\mathcal{R}$ with semisimple class $\mathcal{S}(\mathcal{R})$, we have $\mathcal{R} = \mathcal{L}_b(\neg\mathcal{S}(\mathcal{R})) = \mathcal{U}(\mathcal{S}(\mathcal{R}))$ which may be extended in the following way.

THEOREM 15. *For any regular class $\mathcal{X}$, let $\neg\overline{\mathcal{X}}$ denote the class of rings that are not in the semisimple closure of $\mathcal{X}$. Then $\mathcal{U}(\mathcal{X}) = \mathcal{L}_b(\neg\overline{\mathcal{X}})$.*

PROOF: For a regular class $\mathcal{X}$, $\mathcal{U}(\mathcal{X}) = \mathcal{U}(\overline{\mathcal{X}})$. If $A \in \mathcal{U}(\overline{\mathcal{X}})$, then every nonzero homomorphic image of A is in $\neg\overline{\mathcal{X}}$, $A \in \mathcal{L}_b(\neg\overline{\mathcal{X}})$ and $\mathcal{U}(\mathcal{X}) \subseteq \mathcal{L}_b(\neg\overline{\mathcal{X}})$. If $A \in \mathcal{L}_b(\neg\overline{\mathcal{X}})$, then A has no nonzero homomorphic image in $\overline{\mathcal{X}}$ (else this image would have no nonzero accessible subring in $\neg\overline{\mathcal{X}}$ as $\overline{\mathcal{X}}$ is hereditary). Hence $A \in \mathcal{U}(\overline{\mathcal{X}}) = \mathcal{U}(\mathcal{X})$ and $\mathcal{L}_b(\neg\overline{\mathcal{X}}) \subseteq \mathcal{U}(\mathcal{X})$. $\square$

We have from [3] that if a radical class $\mathcal{R}$ is regular, then $\mathcal{R}$ is pseudocomplemented by $\mathcal{U}(\mathcal{R})$ (and hence by $\mathcal{L}_b(\neg\overline{\mathcal{R}})$) in L_{AR}.

To conclude, let $\neg\mathcal{X} = \{A \in \mathbf{A} \mid A \text{ is not an } \mathcal{X}\text{-ring}\}$.

Proposition 16. *If $\mathcal{X}$ is hereditary, $\mathcal{U}(\mathcal{X}) = \mathcal{L}_b(\neg\mathcal{X})$.*

Proof: Suppose a class $\mathcal{X}$ is hereditary. Then for all $A \in \mathcal{X}$, A has no nonzero accessible subring in $\neg\mathcal{X}$ and $\mathcal{X} \cap \mathcal{L}_b(\neg\mathcal{X}) = 0$. If $A \in \mathcal{U}(\mathcal{X})$, then every nonzero homomorphic image of A is contained in $\neg\mathcal{X}$ and hence $A \in \mathcal{L}_b(\neg\mathcal{X})$. $\mathcal{L}_b(\neg\mathcal{X}) \subseteq \mathcal{U}(\mathcal{X})$ since $\mathcal{U}(\mathcal{X})$ is the largest radical class having zero intersection with $\mathcal{X}$. Therefore $\mathcal{U}(\mathcal{X}) = \mathcal{L}_b(\neg\mathcal{X})$. $\square$

If $\mathcal{R}$ is a hereditary radical class, then $\mathcal{R}$ is pseudocomplemented by $\mathcal{L}_b(\neg\mathcal{R})$ in L_{AR}. $\mathcal{L}_b(\neg\mathcal{R})$ is not a pseudocomplement for all radical classes $\mathcal{R}$ though as the following illustrates.

Proposition 17. *A radical class $\mathcal{R}$ contains a simple ring if and only if $\mathcal{L}_b(\neg\mathcal{R}) \neq \mathbb{A}$.*

Proof: Suppose a radical class $\mathcal{R}$ contains a simple ring S. Then $S \notin \mathcal{L}_b(\neg\mathcal{R})$ and $\mathcal{L}_b(\neg\mathcal{R}) \neq \mathbb{A}$. If $\mathcal{L}_b(\neg\mathcal{R}) \neq \mathbb{A}$, then there exists a ring with nonzero homomorphic image A such that A has no nonzero accessible subring in $\neg\mathcal{R}$. Therefore every nonzero accessible subring of A is an $\mathcal{R}$-ring and hence $\mathcal{T}_{\mathcal{R}}$, the largest hereditary subradical class, is nonzero. $\mathcal{T}_{\mathcal{R}}$ contains a simple ring by Proposition 1 and hence $\mathcal{R}$ contains a simple ring. $\square$

It is known that the radical class of divisible torsion rings $\mathcal{DT}$ contain no simple rings and hence $\mathcal{L}_b(\neg\mathcal{DT}) = \mathbb{A}$, clearly too large to be a pseudocomplement in L_{AR}.

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