J. Austral. Math. Soc. (Series A) 37 (1984), 258–268

BIORDERED SETS ARE BIORDERED SUBSETS OF IDEMPOTENTS OF SEMIGROUPS

DAVID EASDOWN

(Received 22 October 1982)

Communicated by T. E. Hall

Abstract

A new arrow notation is used to describe biordered sets. Biordered sets are characterized as biordered subsets of the partial algebras formed by the idempotents of semigroups. Thus it can be shown that in the free semigroup on a biordered set factored out by the equations of the biordered set there is no collapse of idempotents and no new arrows.

1980 *Mathematics subject classification (Amer. Math. Soc.)*: 20 M 05, 20 M 10, 20 M 30, 06 A 99.

1. Introduction

The concept of a biordered set was introduced by Nambooripad in [2], where he proves (Theorem 1.1) that the idempotents of a semigroup form a partial algebra in a natural way which satisfies the axioms for a biordered set. The question then is whether all biordered sets arise from semigroups in this way, for then the biordered set concept will be a characterization of those partial algebras of idempotents of semigroups.

In this paper we show that all biordered sets arise as biordered subsets of the idempotents of semigroups. To do this we use a construction introduced in [1] which is the biordered set analogue of an idempotent-separating representation of semigroups.

The author does not know whether all biordered sets are the biordered sets of semigroups. A natural approach is to consider the free semigroup on the elements of a given biordered set factored out by all the relations of the biordered set, and then attempt to show the biordered set of the semigroup obtained in this way coincides with the original. This breaks up into three problems: showing (i) there is no collapse of elements of the original biordered set under the factoring out by relations; (ii) no new arrows (see below) are created in the biordered set; (iii) no new idempotents are created. The main result of this paper enables (i) and (ii) to be proved; it is still not known whether (iii) occurs in general.

We begin by describing biordered sets using an arrow notation, $\rightarrow$ and $>\!\!-$, to replace the more commonly used ω^r and ω^l.

2. Preliminaries

Let E be a set with a partial multiplication with domain $D_E \subseteq E \times E$. Define relations $\rightarrow$ and $>\!\!- \subseteq E \times E$ by

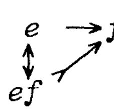
$$e \rightarrow f \text{ if and only if } (f, e) \in D_E \text{ and } fe = e,$$
$$e >\!\!- f \text{ if and only if } (e, f) \in D_E \text{ and } ef = e.$$

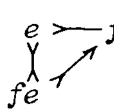
Note that $e \rightarrow f$ and $e -\!\!< f$ together imply $e = f$.

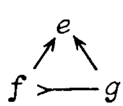
Suppose E satisfies the following axioms.

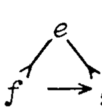
(B1) $\rightarrow$ and $>\!\!-$ are quasi-orders and

$$D_E = \rightarrow \cup >\!\!- \cup (\rightarrow \cup >\!\!-)^{-1},$$

(B21) $e \rightarrow f \Rightarrow$  ,

(B21)* $e >\!\!- f \Rightarrow$  ,

(B22)  $\Rightarrow fe >\!\!- ge$,

(B22)*  $\Rightarrow ef \rightarrow eg$,

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

(B31) $e \to f \to g \Rightarrow (eg)f = ef$,

(B31)* $e \succ\!\!- f \succ\!\!- g \Rightarrow f(ge) = fe$,

(B32) $$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & \succ\!\!- & g \end{array} \Rightarrow (gf)e = (ge)(fe),$$

(B32)* $$\begin{array}{ccc} & e & \\ \diagup & & \diagdown \\ f & \longrightarrow & g \end{array} \Rightarrow e(fg) = (ef)(eg).$$

Note that successive axioms rely on previous axioms for their sense: for example in (B21), if $e \to f$ then by (B1) $(e, f) \in D_E$ so that ef is defined; in (B31), if $e \to f \to g$ then $e \to g$ by the transitivity of $\to$, due to (B1), so by (B21) $eg \leftrightarrow e \to f$, so $eg \to f$, so that $(eg)f$ is defined. Each axiom can be made self-contained by requiring the appropriate products to be defined in E, but this is not necessary since E will always be assumed to satisfy all these axioms.

Note that a duality exists, obtained by interchanging arrowtypes, $\to$ with $\succ\!\!-$, and reversing the orders of products, so that (B1) is self-dual and an axiom superscripted by $*$ is the dual of the axiom without the superscript.

For $e, f \in E$ define
$$M(e, f) = \{g \in E : g \succ\!\!- e \text{ and } g \to f\}.$$
Define
$$\mathcal{S}(e, f) = \{h \in M(e, f) : \forall g \in M(e, f)\, eg \to eh \text{ and } gf \succ\!\!- hf\},$$
the *sandwich set* of e and f. We call E a *biordered set* if, in addition to the above axioms, E also satisfies

(B4) $$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & & g \end{array} \text{ and } fe \succ\!\!- ge$$
$$\Rightarrow \exists\ f' \text{ such that } \begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f' & \succ\!\!- & g \end{array} \text{ and } f'e = fe.$$

(B4)* $$\begin{array}{ccc} & e & \\ \diagup & & \diagdown \\ f & & g \end{array} \text{ and } ef \to eg$$
$$\Rightarrow \exists\ f' \text{ such that } \begin{array}{ccc} & e & \\ \diagup & & \diagdown \\ f' & \longrightarrow & g \end{array} \text{ and } ef' = ef.$$

If E and E' are biordered sets and $\theta: E \to E'$ is a mapping, then θ is called a *morphism* if θ is a morphism of the partial algebras, that is, if θ satisfies

(M) $\quad (e, f) \in D_E \Rightarrow (e\theta, f\theta) \in D_{E'}$ and $(ef)\theta = e\theta f\theta$.

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

A subset E' of E is called a *biordered subset* of E if E' is a partial subalgebra of E, in the sense that if $e, f \in E'$ and $(e, f) \in D_E$ then $(e, f) \in D_{E'}$, and further E' satisfies all the biordered set axioms with respect to the restrictions of $\rightarrow$ and $\succ\!-$ to E'. It is easy to see that if E' is a partial subalgebra of a biordered set E then E' is a biordered subset if and only if E' satisfies (B4) and (B4)*.

The following is straightforward and proved in [2] as part of Theorem 1.1.

THEOREM 1. *Let S be a semigroup, and $E(S)$ the set of idempotents of S. Then $E(S)$ forms a biordered set by restricting the semigroup multiplication to*

$$D_{E(S)} = \{(e, f) \in S \times S \colon ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

PROPOSITION 1 [2, Proposition (2.4)]. *Let E satisfy* (B1), (B21), (B21)*, (B22), (B22)*, (B31), (B31)*, (B32) *and* (B32)*. *Then E satisfies* (B4) *if and only if for all $e, f, g \in E$*

(B4′) $$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & & g \end{array} \Rightarrow S(f,g)e = S(fe,ge).$$

E satisfies (B4)* *if and only if for all $e, f, g \in E$*

(B4′)* $$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ f & & g \end{array} \Rightarrow eS(f,g) = S(ef,eg).$$

In the next section we use (B4′) and its dual rather than (B4) and its dual.

LEMMA 1. (Part of Proposition 2.3 of [2]). *Let E be a biordered set and $e, f, g \in E$. Then*

$$e \rightarrow f \rightarrow g \Rightarrow (ef)g = e(fg).$$

LEMMA 2. *Let E be a biordered set and $e, f, g \in E$. Then*

$$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & \succ\!\!-\!\!-\!\!- & g \end{array} \quad \textit{and} \quad fe \succ\!\!-\!\!\prec ge \quad \textit{together imply} \quad f \succ\!\!-\!\!\prec g.$$

PROOF.

$$\begin{aligned} gf &= (gf)g && (\text{since } gf \succ\!- g \text{ by (B21)*}) \\ &= [(gf)e]g && (\text{by (B31), since } gf \rightarrow g \rightarrow e) \\ &= [(ge)(fe)]g && (\text{by (B32)}) \\ &= (ge)g && (\text{since } ge \succ\!- fe) \\ &= g && (\text{since } ge \leftrightarrow g \text{ by (B21)}). \end{aligned}$$

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

3. The construction and the main result

Let E be a biordered set. Observe that $\rightarrowtail\!\!\!\!\leftarrow$ and $\leftrightarrow$ are equivalence relations on E, which we call $\mathfrak{L}$ and $\mathfrak{R}$ respectively, in accordance with the usage of [2]. Let $E/\mathfrak{L}$ $[E/\mathfrak{R}]$ denote the set of $\mathfrak{L}$ $[\mathfrak{R}]$-classes of E. Let L $[R]$ denote an arbitrary member of $E/\mathfrak{L}\,[E/\mathfrak{R}]$, and $L_e\,[R_e]$ the $\mathfrak{L}\,[\mathfrak{R}]$-class containing $e \in E$.

Denote the full transformation semigroup on a set X by $\mathfrak{T}(X)$, and the dual transformation semigroup by $\mathfrak{T}^*(X)$. If $\alpha \in \mathfrak{T}(X)$, then α^* denotes the corresponding element of $\mathfrak{T}^*(X)$. Suppose $\infty \notin E/\mathfrak{L} \cup E/\mathfrak{R}$. Put $A = E/\mathfrak{L} \cup \{\infty\}$ and $B = E/\mathfrak{R} \cup \{\infty\}$.

Define

(i) $\rho\colon E \to \mathfrak{T}(A)$ where

$$\rho_e\colon L \mapsto \begin{cases} L_{xe} & \text{if } x \to e \text{ for some } x \in L, \\ \infty & \text{otherwise,} \end{cases}$$
$$\infty \mapsto \infty.$$

(ii) $\lambda\colon E \to \mathfrak{T}(B)$ where

$$\lambda_e\colon R \mapsto \begin{cases} R_{ex} & \text{if } x \rightarrowtail e \text{ for some } x \in R, \\ \infty & \text{otherwise,} \end{cases}$$
$$\infty \mapsto \infty.$$

(iii) $\phi\colon E \to \mathfrak{T}(A) \times \mathfrak{T}^*(B)$ where

$$e \mapsto \phi_e = (\rho_e, \lambda_e^*).$$

Note that ρ and λ are well-defined by (B22) and (B22)*.

REMARK. ρ is the analogue of a semigroup representation. Let S be a semigroup, $\mathfrak{L}$ and $\mathfrak{R}$ the usual Green's relations on S. Denote the set of regular $\mathfrak{L}$-classes of S by X. define $\bar{\rho}\colon S \to \mathfrak{T}(X \cup \{\infty\})$ by

$$\bar{\rho}_s\colon L \mapsto \begin{cases} L_{xs} & \text{if } x \mathfrak{R} xs \text{ for some } x \in L, \\ \infty & \text{otherwise,} \end{cases}$$
$$\infty \mapsto \infty.$$

Then it is straightforward to prove $\bar{\rho}$ is a representation of S (see [1]). Dually λ is the analogue of a semigroup anti-representation.

By the following lemma, ρ and λ^* are morphisms of E into $E(\mathfrak{T}(A))$ and $E(\mathfrak{T}^*(B))$ respectively.

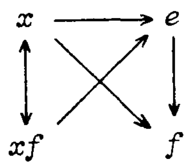
LEMMA 3. *Let E be a biordered set and $e, f \in E$. Then*

$$(e, f) \in D_E \Rightarrow \rho_{ef} = \rho_e \rho_f \quad \textit{and} \quad \lambda_{ef}^* = \lambda_e^* \lambda_f^*.$$

PROOF. Suppose $(e, f) \in D_E$. The argument falls into two cases, each with two subcases.

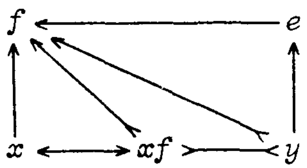
(i) Suppose $e \to f$. We first show $\rho_e(= \rho_{fe}) = \rho_f\rho_e$. Let $L \in E/\mathcal{L}$.

Suppose $L\rho_e \neq \infty$, so $x \to e$ for some $x \in L$. Then $L\rho_e = L_{xe}$ and $L\rho_f\rho_e = L_{(xf)e}$, since by transitivity of $\to$ and by (B21) we have

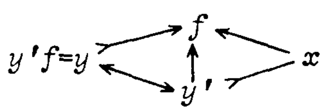


But $(xf)e = xe$ by (B31), so $L\rho_e = L\rho_f\rho_e$.

Suppose $L\rho_f\rho_e \neq \infty$, so $x \to f$ for some $x \in L$ and $y \to e$ for some $y \in L_{xf}$, so that by (B21) and transitivity of the arrows we have



In particular $y \rightarrowtail f \leftarrow x$ and $y = yf \leftrightarrow xf$, so by (B4′) and (B21) we have, for some $y' \in E$

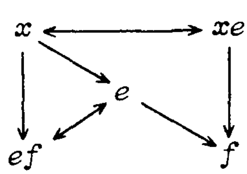


Since $y'f \rightarrowtail\!\!\!\!< xf$, we have $y' \rightarrowtail\!\!\!\!< x$, by Lemma 2. But

$$y' \leftrightarrow y'f = y \to e,$$

so $y' \to e$, and so $L\rho_e = Ly'e \neq \infty$. Hence if $L\rho_e = \infty$ then $L\rho_f\rho_e = \infty$, and hence $\rho_e = \rho_f\rho_e$.

Now we show $\rho_{ef} = \rho_e\rho_f$. Suppose $L\rho_{ef} \neq \infty$, so $x \to ef$ for some $x \in L$. Then

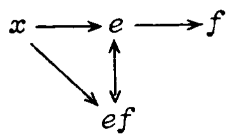


https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

so

$$L\rho_e\rho_f = L_{xe}\rho_f = L_{(xe)f} = L_{x(ef)} \quad \text{(by Lemma 1)}$$
$$= L\rho_{ef}.$$

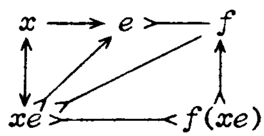
Suppose $L\rho_e\rho_f \neq \infty$, so $x \to e$ for some $x \in L$, and so



,

so that $L\rho_{ef} \neq \infty$. Hence $\rho_{ef} = \rho_e\rho_f$.

(ii) Suppose $e \succ\!\!- f$. We first show $\rho_e (= \rho_{ef}) = \rho_e\rho_f$. Suppose $L\rho_e \neq \infty$, so $x \to e$ for some $x \in L$. Hence

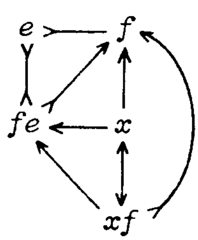


,

so $L\rho_e\rho_f = L_{xe}\rho_f = L_{f(xe)} = L_{xe} = L\rho_e$.

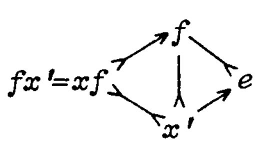
If $L\rho_e\rho_f \neq \infty$ then $L\rho_e \neq \infty$. Hence $\rho_e = \rho_e\rho_f$.

Now we show $\rho_{fe} = \rho_f\rho_e$. Suppose $L\rho_{fe} \neq \infty$, so $x \to fe$ for some $x \in L$. Then



.

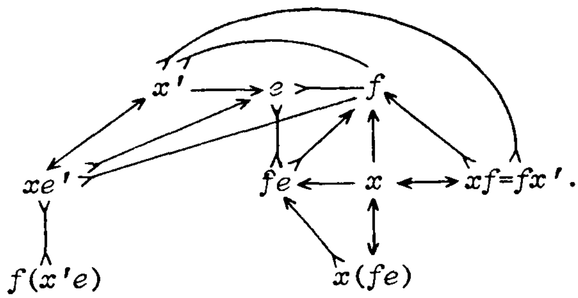
In particular $xf \succ\!\!\to f \leftarrow e$ and $xf = f(xf) \to fe$, so by (B4′)* and (B21)* we have for some $x' \in E$



.

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

Hence

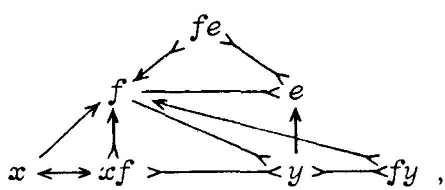


But

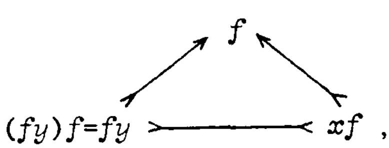
$$\begin{aligned} f(x'e) &= (fx')(fe) \qquad \text{(by (B32)*)} \\ &= (xf)(fe) \\ &= x(fe) \qquad \text{(by (B31))} \end{aligned}$$

so $L\rho_f\rho_e = L_{xf}\rho_e = L_{x'e} = L_{f(x'e)} = L_{x(fe)} = L\rho_{fe}$.

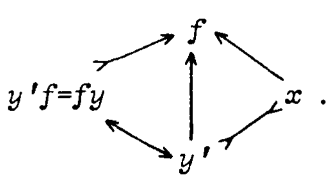
Suppose $L\rho_f\rho_e \neq \infty$, so $x \to f$ for some $x \in L$ and $y \to e$ for some $y \in L_{xf}$. Hence



so in particular

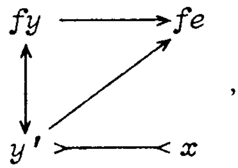


so by (B4′), (B21) and Lemma 2 we have for some $y' \in E$



https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

By (B22)* $fy \to fe$, so



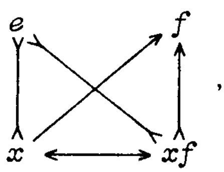
and so $L\rho_{fe} \neq \infty$. Hence $\rho_{fe} = \rho_f \rho_e$.

Thus we have shown $(e, f) \in D_E \Rightarrow \rho_{ef} = \rho_e \rho_f$. By the dual argument we also have $(e, f) \in D_E \Rightarrow \lambda_{ef} = \lambda_f \lambda_e$, that is $\lambda^*_{ef} = \lambda^*_e \lambda^*_f$.

Lemma 4. *Let E be a biordered set and $e, f \in E$. Then*

$$\phi_e \phi_f = \phi_e \Rightarrow e \succ\!- f \quad \textit{and} \quad \phi_f \phi_e = \phi_e \Rightarrow e \to f.$$

Proof. Suppose $\phi_e \phi_f = \phi_e$, so $\rho_e \rho_f = \rho_e$. Then $L_e = L_e \rho_e = L_e \rho_e \rho_f = L_e \rho_f$, so $x \to f$ for some $x \in L_e$, and $L_{xf} = L_e$. Hence



so $e \succ\!- f$.

If $\phi_f \phi_e = \phi_e$ then $\lambda^*_f \lambda^*_e = \lambda^*_e$, so $\lambda_e \lambda_f = \lambda_e$, and by dual reasoning we have $e \to f$.

Theorem 2. *ϕ is an injective morphism from E into $E(\mathfrak{T}(A) \times \mathfrak{T}^*(B))$. $E\phi$ is a biordered subset of $E(\mathfrak{T}(A) \times \mathfrak{T}^*(B))$ and $E \cong E\phi$ as biordered sets.*

Proof. By Lemma 3, ϕ is a morphism from E into $E(\mathfrak{T}(A) \times \mathfrak{T}^*(B))$. If $\phi_e = \phi_f$ then by Lemma 4, $e \succ\!\!\prec f$ and $e \leftrightarrow f$, so $e = f$. Hence ϕ is injective.

We show $E\phi$ is a partial subalgebra of $E(\mathfrak{T}(A) \times \mathfrak{T}^*(B))$.

Let $\phi_e, \phi_f \in E\phi$. If $\phi_e \succ\!- \phi_f$ then, by Lemma 4, $e \succ\!- f$, so by Lemma 3, $\phi_f \phi_e = \phi_{fe} \in E\phi$. Similarly if $\phi_e \to \phi_f$ then $\phi_e \phi_f \in E\phi$. Thus $E\phi$ is a partial subalgebra.

Let ϕ^{-1}: $E\phi \to E$ denote the map $\phi_e \mapsto e$. If $(\phi_e, \phi_f) \in D_{E\phi}$ then by Lemma 4, $(e, f) \in D_E$, so by Lemma 3, $(\phi_e \phi_f)\phi^{-1} = (\phi_{ef})\phi^{-1} = ef = (\phi_e)\phi^{-1}(\phi_f)\phi^{-1}$, so ϕ^{-1}

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

is a morphism of partial algebras, that is ϕ is an isomorphism of partial algebras. Hence $E\phi \cong E$ as partial algebras, so $E\phi$ is a biordered set, and hence a biordered subset of $E(\mathfrak{T}(A) \times \mathfrak{T}^*(B))$.

Thus we have the following characterization of biordered sets.

COROLLARY 1. *Biordered sets are biordered subsets of the biordered sets of semigroups.*

The author does not know whether a stronger result holds, namely whether biordered sets are precisely biordered sets of semigroups. Consider the free semigroup on the elements of a given biordered set, factored out by the relations of the biordered set; is the biordered set of this semigroup isomorphic to the original biordered set? If the answer is yes then the stronger result will be true, and for any given biordered set we will have produced the freest semigroup with that biordered set.

Let E be a biordered set and F the free semigroup on elements of E. Let $\rho = \{(z, xy): (x, y) \in D_E \text{ and } xy = z\}$, and denote the smallest congruence on F containing ρ by $\rho^\#$. To show $E \cong E(F/\rho^\#)$ is would be sufficient to show three things:

(i) $e, f \in E$ and $e\rho^\# f \Rightarrow e = f$ (no collapse);

(ii) $e \to f \Leftrightarrow e\rho^\# fe$, and $e \succ\!\!- f \Leftrightarrow e\rho^\# ef$ (no new arrows);

(iii) $w \in F$ and $w^2\rho^\# w \Rightarrow w\rho^\# e \exists e \in E$ (no new idempotents).

Corollary 1 gives us

COROLLARY 2. *With notation as in the preceding paragraph, conditions* (i) *and* (ii) *hold.*

PROOF. By Corollary 1, E is a biordered subset of $E(S)$, for some semigroup S, where we can suppose $S = \langle E \rangle$, so $S \cong F/\sigma$ for some congruence σ. If $(x, y) \in D_E$ and $xy = z$, then $z\sigma xy$ in F, so $\sigma \supseteq \rho$. Thus $\sigma \supseteq \rho^\#$. Hence if $e, f \in E$ with $e\rho^\# f$ then $e\sigma f$, in F, so $e = f$, which proves (i). If $e\rho^\# fe$ then $e\sigma fe$, in F, so $e = fe$, and so $e \to f$. Likewise if $e\rho^\# ef$ then $e \succ\!\!- f$, which proves (ii).

The question remains whether (iii) holds. In other words, are no new idempotents created upon taking the free semigroup on a biordered set factored out by the relations of the biordered set? If (iii) holds then E will be the biordered set of $F/\rho^\#$, and $F/\rho^\#$ will coincide with the free construction E^* of Pastijn [3].

If $e_1, \ldots, e_n \in E$ and the product $e_1 \cdots e_n$ is defined in E with a given bracketing, then, since (i) above is true, any other bracketing yielding a defined

https://doi.org/10.1017/S1446788700022072 Published online by Cambridge University Press

product gives the same product. The proof of this relies on Theorem 2, whose proof uses the fact that E satisfies (B4′) and its dual, or equivalently (B4) and its dual. Without requiring (B4′) or its dual, it is straightforward to show that if $(e_1e_2)e_3$ and $e_1(e_2e_3)$ are defined products in E then $(e_1e_2)e_3 = e_1(e_2e_3)$. Note that since E is a partial algebra this does not imply associative products where $n > 3$. The author has looked at some special cases with $n > 3$ and, with some difficulty, shown associativity in these cases without relying on (B4′) or its dual. A question then is whether associativity can be shown for any n in a partial algebra E satisfying only (B1), (B21), (B22), (B31), (B32) and their duals. If not, precisely what role do (B4′) and (B4′)* play?

Acknowledgement

The question of characterizing biordered sets was first posed to the author by his supervisor, Professor G. B. Preston.

The author gratefully acknowledges the support of a Monash Graduate Scholarship whilst undertaking this research.

References

[1] D. Easdown and T. E. Hall, 'Reconstructing some idempotent-generated semigroups from their biordered sets', *Semigroup Forum*, to appear.

[2] K. S. S. Nambooripad, 'Structure of regular semigroups I', *Mem. Amer. Math. Soc.* **22** (1979), no. 224.

[3] F. Pastijn, 'The Biorder on the partial groupoid of idempotents of a semigroup', *J. Algebra* **65** (1980), 147–187.

Department of Mathematics
Monash University
Clayton, Victoria
Australia 3168