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ON NONINNER AUTOMORPHISMS OF SOME FINITE P-GROUPS

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Abstract

We settle the noninner automorphism conjecture for finite p-groups ($p > 2$) with certain conditions. Also, we give an elementary and short proof of the main result of Ghoraishi ['On noninner automorphisms of finite nonabelian p-groups', *Bull. Aust. Math. Soc.* **89**(2) (2014) 202–209].

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1. Introduction

There is a famous conjecture known as the noninner automorphism conjecture, listed in the book 'Unsolved Problems in Group Theory: The Kourovka Notebook', which states that *every finite nonabelian p-group admits an automorphism of order p which is not an inner automorphism* [11, Problem 4.13].

A sharpened version of the conjecture states that every finite nonabelian p-group G has a noninner automorphism of order p which fixes the Frattini subgroup $\Phi(G)$ element-wise. This conjecture was first attacked by Liebeck [12]. He proved that for an odd prime p, every finite p-group G of nilpotency class 2 has a noninner automorphism of order p fixing $\Phi(G)$ element-wise. For 2-groups of class 2, the conjecture was settled by Abdollahi [1] in 2007. In 2013, Abdollahi *et al.* [3, Theorem 4.4] proved that every finite p-group G of odd order and of nilpotency class 3 has a noninner automorphism of order p that fixes $\Phi(G)$ element-wise. In 2013, Shabani-Attar [15] proved that if G is a finite nonabelian p-group of order p^m and exponent p^{m-2}, then G has a noninner automorphism of order p. In 2014, Abdollahi *et al.* [4] showed that every finite p-group G of co-class 2 has a noninner automorphism of order p leaving $Z(G)$ element-wise fixed. For more such results, see [5–10].

If there is a maximal subgroup M of a finite p-group G with $|G| > p$ and $Z(M) \subseteq Z(G)$, then there exists a noninner automorphism of G of order p (see [13, Lemma 9.108]). In 2002, Deaconescu and Silberberg [5] proved that if G is

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a finite nonabelian p-group such that $C_G(Z(\Phi(G))) \neq \Phi(G)$, then G has a noninner automorphism of order p which fixes $\Phi(G)$ element-wise. This reduces the verification of the conjecture to the special case in which $C_G(Z(\Phi(G))) = \Phi(G)$. If the conjecture is false for a finite p-group G, then it follows from [5, Remark 2] that $Z(G) < Z(M)$ for all maximal subgroups M of G. This suggests the following natural question.

QUESTION 1.1. Given a finite p-group G with $Z(G) < Z(M)$ for all maximal subgroups M of G, can the conjecture hold?

In Theorem 2.1, we prove that every finite p-group G ($p > 2$) of nilpotency class n with $\exp(\gamma_{n-1}(G)) = p$, $|\gamma_n(G)| = p$ and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, has a noninner automorphism of order p which fixes $\Phi(G)$ element-wise. As a consequence, in Corollaries 2.2 and 2.3, we give an affirmative answer to the above question under some conditions. In 2017, Ruscitti *et al.* [14] confirmed the conjecture for finite p-groups of co-class 3 with $p \neq 3$. We also validate the conjecture for some nonabelian finite 3-groups of co-class 3 in Corollary 2.4. In [8, Theorem 1.1], Ghoraishi improved the reduction given by Deaconescu and Silberberg [5] by proving that if a finite nonabelian p-group G fails to fulfil the condition $Z_2^\star(G) \leq C_G(Z_2^\star(G)) = \Phi(G)$, where $Z_2^\star(G)$ is the pre-image of $\Omega_1(Z_2(G)/Z(G))$ in G, then G has a noninner automorphism of order p which fixes $\Phi(G)$ element-wise. At the end of Section 2, we provide an elementary and short proof of this result.

Throughout, p denotes a prime number. For a group G, by $Z_m(G)$, $\gamma_m(G)$, $d(G)$ and $\Phi(G)$, we denote the mth term of the *upper central series* of G, the mth term of the *lower central series* of G, the *minimum number of generators* of G and the *Frattini subgroup* of G, respectively. The *nilpotency class* and the *exponent* of a finite group G are denoted by $\mathrm{cl}(G)$ and $\exp(G)$, respectively. A finite p-group G of order p^n with $\mathrm{cl}(G) = n - c$ is said to be of *co-class* c. For a finite p-group G, we write $\Omega_1(G) = \langle g \in G \mid g^p = 1\rangle$. All other unexplained notation, if any, are standard.

2. Main results

Since the conjecture is true for all finite p-groups G having nilpotency class 2 and 3, we consider only finite p-groups G with $\mathrm{cl}(G) \geq 4$.

THEOREM 2.1. *Let G be a finite p-group ($p > 2$) of class n such that $|\gamma_n(G)| = \exp(\gamma_{n-1}(G)) = p$ and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Then, G has a noninner automorphism of order p that fixes $\Phi(G)$ element-wise.*

PROOF. Since $n = \mathrm{cl}(G) \geq 4$ and $\exp(\gamma_{n-1}(G)) = p$, there exists an element $x \in \gamma_{n-1}(G) \setminus Z(G)$ of order p. Thus, $[x, G] \subseteq \gamma_n(G)$ and, therefore, the order of the conjugacy class of x in G is p. It follows that $M = C_G(x)$ is a maximal subgroup of G. Let $g \in G \setminus M$. Then,

$$(gx)^p = g^p x^p [x, g]^{p(p-1)/2} = g^p.$$

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Consider the map β of G defined by $\beta(g) = gx$ and $\beta(m) = m$ for all $m \in M$. The map β can be extended to an automorphism of G fixing $\Phi(G)$ element-wise and of order p. We claim that β is a noninner automorphism of G. For a contradiction, assume that $\beta = \theta_y$, the inner automorphism of G induced by some $y \in G$, which implies that $y \in C_G(M)$. If $y \notin M$, then $G = M\langle y\rangle$. It follows that $y \in Z(G)$, which is a contradiction. Therefore, $y \in Z(M)$. Since $\beta = \theta_y$, we have $g^{-1}\theta_y(g) = [g, y] = x$. Now, by the hypothesis that $Z(C_G(x)) \le \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, we have $y \in \gamma_{n-1}(G)$. Therefore,

$$x = [g, y] \in \gamma_n(G) \le Z(G),$$

which contradicts our choice of x in G. Hence, G has a noninner automorphism of order p that fixes $\Phi(G)$ element-wise. □

Let G be a finite p-group such that $|Z(G)| = p$. Let M be any maximal subgroup of G. Since $Z(M)$ is a characteristic subgroup of M and M is a normal subgroup of G, it follows that $Z(M)$ is a normal subgroup of G. Thus, $Z(G) \le Z(M)$ for all maximal subgroups M of G. We obtain the following corollary from Theorem 2.1.

COROLLARY 2.2. *Let G be a finite p-group $(p > 2)$ of class n such that $|Z(G)| = exp(\gamma_{n-1}(G)) = p$ and $Z(C_G(x)) \le \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Then, G has a noninner automorphism of order p that fixes $\Phi(G)$ element-wise.*

COROLLARY 2.3. *Let G be a finite p-group $(p > 2)$ of class n such that $|Z(G)| = p$ and $Z(M) = \gamma_{n-1}(G)$ is of exponent p for all maximal subgroups M of G. Then, G has a noninner automorphism of order p that fixes $\Phi(G)$ element-wise.*

PROOF. Given that $\mathrm{cl}(G) = n$. It follows that $\gamma_n(G) \le Z(G)$. Consequently, $|\gamma_n(G)| = p$. Considering the hypothesis that $Z(M) = \gamma_{n-1}(G)$ is of exponent p for all maximal subgroups M of G and the proof of Theorem 2.1, we deduce that $Z(C_G(x)) \le \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Hence, it follows from Theorem 2.1 that G possesses a noninner automorphism of order p that fixes $\Phi(G)$ element-wise. □

COROLLARY 2.4. *Let G be a finite 3-group of order 3^n and of co-class 3 such that $Z(M) = \gamma_{n-4}(G)$ is of exponent 3 for all maximal subgroups M of G. Then, G has a noninner automorphism of order 3.*

PROOF. Assume that G does not possess any noninner automorphism of order 3. Then, it follows from [2, Corollary 2.3] that

$$d(Z_2(G)/Z(G)) = d(Z(G))\, d(G). \tag{2.1}$$

Since G is of co-class 3, we have $p^i \le |Z_i(G)| \le p^{i+2}$ for all $1 \le i \le n-4$. Thus, by (2.1), $d(Z(G)) = 1$. Now, if $|Z(G)| = p^3$, then $|Z_2(G)| = p^4$, which contradicts (2.1). Further, $|Z(G)|$ cannot be p^2 according to [14, Theorem 4.3]. Finally, assume that $|Z(G)| = p$. In this case, the conclusion follows from Corollary 2.3. □

The following example of a 3-group of order 3^7 supports Theorem 2.1.

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EXAMPLE 2.5. Consider the group: $G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7\rangle$ with relations:

$$f_3 = [f_2, f_1], \quad f_4 = f_1^3, \quad f_5 = [f_3, f_1], \quad f_6 = [f_3, f_2], \quad f_7 = [f_5, f_1], \quad f_7^2 = [f_4, f_2],$$
$$f_2^3 = f_3^3 = f_4^3 = f_5^3 = f_6^3 = f_7^3 = [f_4, f_1] = [f_6, f_1] = [f_7, f_1] = [f_5, f_2] = [f_6, f_2]$$
$$= [f_7, f_2] = [f_4, f_3] = [f_5, f_3] = [f_6, f_3] = [f_7, f_3] = [f_5, f_4] = [f_6, f_4] = [f_7, f_4]$$
$$= [f_6, f_5] = [f_7, f_5] = [f_7, f_6] = 1.$$

Then:

- $|G| = 3^7$;
- the nilpotency class of G is 4;
- $Z(G) = \langle f_6, f_7\rangle$;
- $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7\rangle$;
- $\gamma_3(G) = \langle f_5, f_6, f_7\rangle$;
- $\gamma_4(G) = \langle f_7\rangle$.

Let $x = f_5$ and $M = C_G(x)$. Then, $x \in \gamma_3(G) \setminus Z(G)$ is of order 3 and $Z(M) = \langle f_5, f_6, f_7\rangle = \gamma_3(G)$. Consider the automorphism defined by

$$\alpha(f_1 f_4^2 f_5 f_6^2 f_7^2) = f_1 f_4^2 f_5^2 f_6^2 f_7^2,$$
$$\alpha(f_1 f_3 f_5 f_6^2) = f_1 f_3 f_5^2 f_6^2,$$
$$\alpha(f_1^2 f_2^2 f_3 f_4^2 f_6) = f_1^2 f_2^2 f_3 f_4^2 f_5^2 f_6 f_7.$$

By using the relators of G, we find $\alpha(f_i) = f_i$ for all $i \in \{2, 3, 4, 5, 6, 7\}$ and $\alpha(f_1) = f_1 f_5$. It is easy to verify that α is a noninner automorphism of order 3 which fixes $\Phi(G)$ element-wise.

We conclude the paper by giving an elementary and short proof of [8, Theorem 1.1].

THEOREM 2.6. *Let G be a finite nonabelian p-group. If G fails to fulfil the condition $Z_2^\star(G) \leq C_G(Z_2^\star(G)) = \Phi(G)$, where $Z_2^\star(G)/Z(G) = \Omega_1(Z_2(G)/Z(G))$, then G has a noninner automorphism of order p leaving $\Phi(G)$ element-wise fixed.*

PROOF. Let G be a finite nonabelian p-group such that at least one of the following holds: $Z_2^\star(G) \nleq C_G(Z_2^\star(G))$ or $C_G(Z_2^\star(G)) \neq \Phi(G)$. Assume that G does not possess any noninner automorphism of order p that fixes $\Phi(G)$ element-wise. Observe that $Z_2^\star(G) = \{z \in Z_2(G) \mid z^p \in Z(G)\}$. Let $x \in Z_2^\star(G)$, $y \in G'$ and $z^p \in G^p$. Then, $[x, y] = 1$ and $[x, z] \in Z(G)$. Now,

$$[x, yz^p] = [x, z^p][x, y][x, y, z^p] = [x, z^p] = [x^p, z] = 1.$$

Therefore, it follows from [5, Theorem] that $Z_2^\star(G) \leq C_G(\Phi(G)) = Z(\Phi(G))$. This implies that $Z_2^\star(G) \leq C_G(Z_2^\star(G))$.

Next, we prove that $C_G(Z_2^\star(G)) = \Phi(G)$. Let M be a maximal subgroup of G and let $g \in G - M$. Let $z \in Z(G) \cap M$ be of order p. Then, the map $\alpha : G \to G$, defined

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by $\alpha(mg^i) = mg^i z^i$ for all $m \in M$, is easily seen to be an automorphism of order p that fixes $\Phi(G)$ element-wise. By assumption, $\alpha = \theta_{a_M}$, the inner automorphism induced by some $a_M \in G$. It is easy to check that $a_M \in Z_2^\star(G)$ and $M = C_G(a_M)$. Since $[Z_2^\star(G), \Phi(G)] = 1$, we have $\Phi(G) \leq C_G(Z_2^\star(G))$. It follows that

$$\Phi(G) \leq C_G(Z_2^\star(G)) \leq \bigcap_M C_G(a_M) = \bigcap_M M = \Phi(G).$$

Hence, $Z_2^\star(G) \leq C_G(Z_2^\star(G)) = \Phi(G)$, which is a contradiction. □

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