Bazin's Matrix and other allied Matrices

By J. WILLIAMSON, The Johns Hopkins University.

(*Received* 22*nd December* 1930. *Read* 16*th January* 1931.)

Introduction. In a paper entitled "On differentiating a Matrix"[1] H. W. Turnbull deduced some interesting and elegant results by the use of a matrix operator Ω, a matrix whose elements were the partial differential operators with respect to the elements of a square matrix X. Throughout the present paper the differential operator Ω is used, or rather a matrix operator, which is the product of Ω and another square matrix Y.

By means of this operator in §§ 1 and 2 Bazin's[2] matrix and Reiss's matrix are considered from the standpoint of matrices as distinct from that of determinants. Reiss's matrix is shown to be a constant times a compound of Bazin's matrix; and the latent roots of Reiss's matrix are immediately determined in terms of the latent roots of Bazin's matrix. From this result a theorem, discovered by Deruyts, is deduced as well as a more general theorem.

In § 3 a series of matrices intimately connected with Bazin's matrix is considered; and in § 4 the connection between this paper and Turnbull's is clearly shown.

§ 1. *Bazin's Matrix.* Let X be a matrix whose transposed[3]

$$X' = [x_i^j] \qquad (i, j = 1, 2, \ldots, n) \tag{1}$$

is an n-rowed square matrix, whose elements x_i^j are n^2 independent variables. We may also write

$$X' = [x^1 x^2 \ldots\ldots x^n], \tag{2}$$

[1] H. W. Turnbull, "On Differentiating a Matrix," *Proc. Edin. Math. Soc.*, Ser. 2, Vol. 1, Part 2, pp. 111-128 (1928). This paper will be referred to as Turnbull's paper.

[2] On Bazin (1851) see Muir, "Theory of Determinants" II, 206-208, and on Reiss (1867) see Muir, III, 181, 189, and also on Picquet, see III, 198-199, and on Deruyts, IV, 15. The work of these authors was purely determinantal.

[3] We define the matrix X' instead of the matrix X for convenience; in particular it conforms more closely with the notation of Turnbull's paper, which has already been cited.

where x^j denotes the n-rowed matrix of one column, whose element in the ith row is x_i^j. We denote the determinant of the matrix X by the notation

$$|X| = |X'| = (x^1 x^2 \dots x^n). \tag{3}$$

If Y is the n-rowed square matrix whose transposed is

$$Y' = [y_i^j] = [y^1 y^2 \dots y^n] \quad (i, j = 1, 2, \dots n), \tag{4}$$

where y^j is a column matrix similar to x^j, then Bazin's matrix is the matrix $M = [m_i^j]$ defined by

$$m_i^j = (x^1 x^2 \dots x^{j-1} y^i x^{j+1} \dots x^n). \tag{5}$$

In other words the element in the ith row and jth column of Bazin's matrix is the determinant of the matrix obtained by replacing the jth column of X' by the ith column of Y'. In particular, if $n = 4$, M is the matrix

$$\begin{vmatrix} (y^1 x^2 x^3 x^4) & (x^1 y^1 x^3 x^4) & (x^1 x^2 y^1 x^4) & (x^1 x^2 x^3 y^1) \\ (y^2 x^2 x^3 x^4) & (x^1 y^2 x^3 x^4) & (x^1 x^2 y^2 x^4) & (x^1 x^2 x^3 y^2) \\ (y^3 x^2 x^3 x^4) & (x^1 y^3 x^3 x^4) & (x^1 x^2 y^3 x^4) & (x^1 x^2 x^3 y^3) \\ (y^4 x^2 x^3 x^4) & (x^1 y^4 x^3 x^4) & (x^1 x^2 y^4 x^4) & (x^1 x^2 x^3 y^4) \end{vmatrix}$$

If now Ω is the matrix[1] composed of the partial differential operators with respect to x_i^j, that is if

$$\Omega = \left[\frac{\partial}{\partial x_i^j}\right], \tag{6}$$

then by actual multiplication

$$Y \Omega |X| = M, \tag{7}$$

for the element in the ith row and jth column of the matrix $Y \Omega |X|$ is

$$y_\alpha^i \frac{\partial}{\partial x_\alpha^j} |X| = m_i^j \text{ by } (5)^2.$$

Similarly

$$X \Omega |X| = |X| E, \tag{8}$$

where E is the unit matrix. If X is non-singular,[3] from (7) and (8) we have the result

$$Y X^{-1} \Delta = M, \text{ where } \Delta = |X|. \tag{9}$$

[1] Turnbull. *Loc. cit.*, page 112.

[2] Throughout this paper repeated Greek suffixes will denote summation from 1 to n.

[3] Throughout this paper we shall take the matrices X and Y to be non-singular. If $\Delta = 0$, the matrix $X^{-1}\Delta$ in (9) must be replaced by the adjugate matrix of X.

[4] Bazin. *Loc. cit.*, page 148.

Taking the determinants of both sides of the matrix equation (9), we have Bazin's theorem[4]

$$|M| = |Y|\,|X|^{n-1}.$$

The result (9) is greatly simplified if $Y = X^r$, where r is an integer, and becomes

$$M = \Delta X^{r-1}. \tag{11}$$

Again if N is the matrix obtained from M by interchanging the rôles of the matrices X and Y, the result corresponding to (9) is

$$XY^{-1}|Y| = N. \tag{12}$$

Hence, by multiplying (9) and (12)

$$MN = |X|\,|Y|E,$$

or

$$\frac{M}{|X|}\,\frac{N}{|Y|} = E. \tag{13}$$

In other words $M/|X|$ and $N/|Y|$ are reciprocal matrices. But, if A and B are reciprocal matrices, each minor of order m that can be formed from the matrix A is proportional to the corresponding complementary minor of the matrix B' (the transposed of B), the ratio being the determinant of A. If we use $(A)_m$ and $(B')_{n-m}$ to denote two such complementary minors we have

$$(A)_m = (B')_{n-m}|A|,$$

and applying this formula to the reciprocal matrices $M/|X|$ and $N/|Y|$ we derive the result

$$|X|^{-m}(M)_m = |Y|^{m-n}(N')_{n-m}|Y|\,|X|^{-1},$$

or

$$(M)_m = (N')_{n-m}|X|^{m-1}|Y|^{m+1-n}. \tag{14}$$

This final result is due to Reiss.[1]

The matrix

$$Q = Y'(X')^{-1}|X| \tag{15}$$

is a matrix similar to the matrix M except that the elements of Q are obtained as the determinants of matrices formed by replacing the *rows* of the matrix X' by the *rows* of the matrix Y'.

---

[1] M. Reiss. *Loc. cit.* It is important to notice that the relationship given by (14) holds between a minor of M and the complementary minor of N'. This fact is not made clear in the statement of the theorem in Turnbull's "Determinants, Matrices, and Invariants," page 109.

Theorem 1. *The matrices Q and M have the same latent roots.*

If A and B are two n-rowed square matrices of which B is non-singular, the matrices AB and BA are connected by a similarity transformation $AB = B^{-1}(BA)B$, and therefore have the same latent roots. But the latent roots of a matrix are also the same as the latent roots of its transposed, and accordingly AB has the same latent roots as $B'A'$, and therefore as $A'B'$. If $A = Y$ and $B = X^{-1}|X|$, the latent roots of $YX^{-1}|X|$ are the same as those of $Y'(X^{-1})'|X|$, and this proves our theorem.

Corollary. If the matrices Y and X are permutable, that is if $XY = YX$, then the matrix Q is the transposed of the matrix M. For,

$$M' = (YX^{-1}|X|)' = |X|(X^{-1})'\,Y' = |X|\,Y'(X^{-1})' = Q.$$

§ 2. *Reiss's Matrix.* The square matrix of order $\binom{n}{r}$, whose elements in any one row are the determinants of the matrices obtained by replacing all sets of r columns of X' by a definite set of r columns of Y' and whose elements in any one column are the determinants of the matrices obtained by replacing one set of r columns of X' by the $\binom{n}{r}$ sets of r columns of Y' in turn, is called Reiss's Matrix. We shall denote this matrix by R_r. For example, if $n = 4$ and $r = 2$, the leading diagonal of R_2 is

$$(y^1y^2x^3x^4),\quad (y^1x^2y^3x^4),\quad (y^1x^2x^3y^4),\quad (x^1y^2y^3x^4),\quad (x^1y^2x^3y^4),\quad (x^1x^2y^3y^4).$$

In particular if $r = 1$, Reiss's matrix reduces to Bazin's matrix.

From any square matrix of order n a new matrix of order $\binom{n}{r}$ can be formed, whose elements are the $\binom{n}{r}\binom{n}{r}$ determinants of order r that can be formed from the original matrix. If the elements of this new matrix occurring in any row are the $\binom{n}{r}$ determinants, that can be formed from a definite set of r rows of the original matrix, and if the elements of any column are the $\binom{n}{r}$ determinants, that can be formed from a definite set of r columns of the original matrix, this new matrix is called the rth compound matrix of the original matrix or more shortly the rth compound.

https://doi.org/10.1017/S0013091500007793 Published online by Cambridge University Press

Theorem 2. *The rth compound of Bazin's matrix M is Reiss's matrix R_r, multiplied by the r-1th power of the determinant of X*[1].

If M_r, X_r, Y_r denote the rth compounds of M, X, and Y respectively, we wish to prove that

$$M_r = \Delta^{r-1} R_r. \tag{16}$$

By actual multiplication, or else by repeating the steps in the derivation of (9), with X and Y replaced by X_r and Y_r respectively and Ω by a differential matrix whose elements are the partial differential operators with respect to the elements of X_r', it can be shown that

$$Y_r X_r^{-1} | X | = R_r. \tag{17}$$

But by the theorem, that the rth compound of the product of two matrices is the product of the rth compounds of the two matrices[2]

$$Y_r X_r^{-1} | X |^r = M_r. \tag{18}$$

The formula (16) follows immediately from (17) and (18) and the theorem is proved.

Corollary. $$| R_r | = | X |^k | Y |^s, \tag{19}$$

where $k = \binom{n-1}{r}$, $s = \binom{n-1}{r-1}$.

For, by Jacobi's ratio theorem,

$$| M_r | = | M |^s, \tag{20}$$

and by (16)

$$| R_r | = | M_r | \, | X |^p, \; p = (1-r)\binom{n}{r}. \tag{21}$$

The result (19), known as Reiss's theorem,[1] follows at once from (10), (20), and (21).

If now S_r denotes the matrix obtained from R_r by interchanging the rôles of the matrices X and Y, the result corresponding to (16) is

$$N_r = | Y |^{r-1} S_r = \Theta^{r-1} S_r, \text{ where } \Theta = | Y | \tag{22}$$

and where N_r denotes the rth compound of N. But by (13)

$$M_r \Delta^{-r} N_r \Theta^{-r} = E,$$

or by (16) and (22)

$$R_r \Delta^{-1} S_r \Theta^{-1} = E, \tag{23}$$

[1] The elements of the two matrices R_r and M_r must naturally correspond; *i.e.* if an element of R_r is the determinant of the matrix obtained by replacing the columns $k_1, \ldots\ldots k_r$ of X' by the columns $s_1, \ldots\ldots, s_r$ of Y', then the corresponding element of M_r is the r-rowed minor of M formed from the rows $s_1, \ldots\ldots s_r$ and the columns $k_1, \ldots\ldots, k_r$ of M.

[2] E. Pascal, "Repertorium der höheren Mathematik," I, Analysis, page 139.

and $R_r \Delta^{-1}$ and $S_r \Theta^{-1}$ are reciprocal matrices. Accordingly any m-rowed minor of $R_r \Delta^{-1}$ is proportional to the corresponding complementary minor in $S_r' \Theta^{-1}$. But an m-rowed minor of $R_r \Delta^{-1}$ is Δ^{-m} times the corresponding minor of R_r. Hence any m-rowed minor of R_r is equal to q times the corresponding complementary minor in S_r', where

$$q = | R_r \Delta^{-1} | \Delta^m \Theta^{m - \binom{n}{r}}.$$

But $$| R_r \Delta^{-1} | = | R_r | \Delta^{-\binom{n}{r}},$$
and hence
$$q = \Delta^p \Theta^t,$$
where by (19)
$$p = \binom{n-1}{r} - \binom{n}{r} + m = m - \binom{n-1}{r-1},$$
$$t = m - \binom{n}{r} + \binom{n-1}{r-1} = m - \binom{n-1}{r}.$$

This final result is also due to Reiss,[1] but again it is worthy of notice that this relationship exists between complementary minors of R_r and S_r', the transposed of S_r.

Theorem 3. *If the latent roots of M are $\lambda_1, \lambda_2, \ldots, \lambda_n$, the latent roots of R_r are the $\binom{n}{r}$ products*

$$\Delta^{1-r} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r} (i_1, i_2, \ldots, i_r = 1, 2, 3, \ldots, n).$$

This theorem is a direct consequence of theorem 2 and Rado's theorem, namely that the latent roots of the rth compound of a matrix are the $\binom{n}{r}$ products of the latent roots of the original matrix taken r at a time.[2]

Since by theorem 1 the matrices M and Q have the same latent roots, we have as an immediate result the corollary: *The latent roots of Reiss's matrix T_r, where $T_r = \Delta^{1-r} Q_r$ and Q_r is the rth compound of Q, are the same as those of R_r.*

From this corollary we may deduce the theorem due to Deruyts[3] that the sum of the $\binom{n}{r}$ determinants obtained by replacing r columns of the square matrix X' by the corresponding r columns of the matrix

---

[1] M. Reiss. *Loc. cit.* This first theorem is sometimes known as Picquet's theorem. See Muir, "Theory of Determinants," Vol. III, page 198.

[2] Muir, "Theory of Determinants," IV, 215-217.

[3] *Cf.* Muir, IV, 15.

https://doi.org/10.1017/S0013091500007793 Published online by Cambridge University Press

Y' in every possible way, is equal to the sum of the $\binom{n}{r}$ determinants obtained by replacing r rows of the matrix X' by the corresponding r rows of the matrix Y' in every possible way. For the first sum is simply the spur or trace of R_r, that is the sum of the elements in the leading diagonal of R_r; and the second sum is the spur of S_r. But the spur of a matrix is equal to minus the sum of the latent roots of the matrix; and as R_r and S_r have the same latent roots, their spurs are equal. Moreover the sum of the m-rowed principal minors of T_r is equal to the sum of the m-rowed principal minors of R_r, for each sum is equal to plus or minus the elementary symmetric function of order m of the latent roots of R_r. This leads to a more general theorem of which Deruyt's is a special case.

Theorem 4. *Let $Y_1, Y_2, \ldots, Y_m$ represent m distinct sets of r columns of Y' and $X_1, X_2, \ldots, X_m$ the corresponding m sets of r columns of X' and let a determinant of order m be formed whose element in the ith row and jth column is the determinant of the matrix obtained by substituting Y_i for X_j in the matrix X'. Since m distinct sets of r columns of Y' can be chosen in $\binom{L}{m}$ ways, where $L = \binom{n}{r}$, there are $\binom{L}{m}$ such determinants. The sum of these $\binom{L}{m}$ determinants is equal to the corresponding sum of the $\binom{L}{m}$ determinants obtained by a similar process applied to the rows, instead of the columns, of the matrices X' and Y'.*

§3. *Further Results.* We shall now consider the effect of operating with the matrices $X\Omega$ and $Y\Omega$ on the spurs of the matrices R_r. If p_r denote the spur of R_r, then

$$p_r = \Sigma_r (z^1 z^2 \ldots z^n), \tag{24}$$

where Σ_r denotes the sum of the $\binom{n}{r}$ determinants obtained by replacing r of the z^i by y^i and the remaining $n-r$ of the z^i by x^i. In particular $p_0 = |X| = \Delta$, $p_n = |Y| = \Theta$. The result of operating on p_r by the matrix operator $X\Omega$ is a matrix, whose element in the ith row and jth column is

$$\begin{aligned} x_a{}^i \frac{\partial}{\partial x_a{}^j} p_r &= x_a{}^i \frac{\partial}{\partial x_a{}^j} \Sigma_r (z^1 z^2 \ldots z^n), \\ &= x_a{}^i \frac{\partial}{\partial x_a{}^j} \Sigma_r (z^1 z^2 \ldots z^{j-1}\, x^j\, z^{j+1} \ldots z^n), \\ &= \Sigma_r (z^1 z^2 \ldots z^{j-1}\, x^i z^{j+1} \ldots z^n), \end{aligned} \tag{25}$$

https://doi.org/10.1017/S0013091500007793 Published online by Cambridge University Press

where Σ_r denotes the sum of all possible determinants with r of the z^k replaced by y^k and the remaining $n-r-1$ of the z^k replaced by x^k. In fact

$$X\,\Omega\,p_r = A_r, \tag{26}$$

where A_r is a matrix, whose element in the ith row and the jth column consists of the sum of all possible determinants obtained by replacing the jth column of X' by the ith column of X' and r other columns of X' by the corresponding r columns of Y'. If $i \neq j$, these determinants in (25) with $z^i = x^i$ vanish, so that there are only $\binom{n-2}{r-1}$ non-vanishing determinants in the sum (25). If however $i = j$, that is for a diagonal element, none of the determinants in (25) vanish and their number is $\binom{n-1}{r}$. For example, if $n = 4$ and $r = 2$, the first two columns of A_2 are

$$\begin{array}{ll}
(x^1y^2y^3x^4) + (x^1y^2x^3y^4) + (x^1x^2y^3y^4), & (y^1x^1y^3x^4) + (y^1x^1x^3y^4) \\
(x^2y^2y^3x^4) + (x^2y^2x^3y^4), & (y^1x^2y^3x^4) + (y^1x^2x^3y^4) + (x^1x^2y^3y^4) \\
(x^3y^2y^3x^4) + (x^3x^2y^3y^4), & (y^1x^3y^3x^4) + (x^1x^3y^3y^4) \\
(x^4x^2y^3y^4) + (x^4y^2x^3y^4), & (y^1x^4x^3y^4) + (x^1x^4y^3y^4).
\end{array}$$

In particular $A_{n-1} = N$, $A_1 = p_1 E - M$, and $A_0 = \Delta E$. Corresponding to the matrix A_r is the matrix B_r obtained by interchanging the rôles of the matrices X and Y in the formation of A_r. We shall prove later that

$$B_{n-r} = p_r E - A_r, \tag{27}$$

a result that may be verified by actual consideration of the two matrices A_r and B_{n-r}.

Similarly the result of operating on p_r with the matrix operator $Y\,\Omega$ is a matrix $T = [t_i^j]$, where

$$\begin{aligned}
t_i^j &= y_\alpha{}^i \frac{\partial}{\partial x_\alpha^j} \Sigma_r (z^1 z^2 \dots z^n), \\
&= y_\alpha{}^i \frac{\partial}{\partial x_\alpha^j} \Sigma_r (z^1 z^2 \dots z^{j-1} x^j z^{j+1} \dots z^n), \\
&= \Sigma_r (z^1 z^2 \dots z^{j-1} y^i z^{j+1} \dots z^n), \\
&= - \Sigma_r (z^1 z^2 \dots z^{i-1} y^i z^{i+1} \dots z^{j-1} x^i z^{j+1} \dots z^n), \text{ if } i \neq j, \\
&= - \Sigma_{r+1} (z^1 z^2 \dots z^{i-1} z^i z^{i+1} \dots z^{j-1} x^i z^{j+1} \dots z^n), \\
&= - \text{element in } i\text{th row and } j\text{th column of } A_{r+1} \text{ by (25)}.
\end{aligned}$$

https://doi.org/10.1017/S0013091500007793 Published online by Cambridge University Press

If however $i = j$,

$$t_i^i = \Sigma_{r+1}(z^1 z^2 \ldots . z^i \ldots . z^n) - \Sigma_{r+1}(z^1 z^2 \ldots . z^{i-1} x^i z^{i+1} \ldots . z^n),$$
$$= p_{r+1} - \text{element in } i\text{th row and } i\text{th column of } A_{r+1}.$$

Hence,

$$Y \Omega p_r = p_{r+1} E - A_{r+1}. \tag{28}$$

But by (26)

$$\Omega p_r = X^{-1} A_r,$$

and therefore

$$Y \Omega p_r = Y X^{-1} A_r = M \Delta^{-1} A_r \text{ by (9)}.$$

Accordingly, by (28),

$$M A_r = \Delta (p_{r+1} E - A_{r+1}),\ r = 0,\ 1,\ \ldots,\ n-1, \tag{29}$$

where $\quad A_0 = p_0 E = \Delta E$ and $A_n = 0$.

Similarly

$$N B_r = \Theta (p_{n-r-1} E - B_{r+1}),\ r = 0,\ 1,\ \ldots,\ n-1. \tag{30}$$

Multiplying equation (30) by M we have

$$\Delta B_r = M (p_{n-r-1} E - B_{r+1}),$$

and hence, if $p_{n-r-1} E - B_{r+1} = A_{n-r-1}$, $\quad B_r = p_{n-r} E - A_{n-r}$ by (29). But $B_{n-1} = M = p_1 E - A_1$, and by induction (27) is true in general.

We immediately deduce the following theorem.

Theorem 5. *Each matrix* A_r $(r = 1, 2, \ldots, n-1)$ *may be expressed as a polynomial of degree r in the matrix M.*

In fact, by eliminating A_i, $i \leqq r-1$, from the set of equations (28),

$$\Delta^{r-1} A_r = \Delta^{r-1} p_r E - \Delta^{r-2} p_{r-1} M + \Delta^{r-3} p_{r-2} M^2 - \ldots + (-1)^{r-2} p_1 M^{r-1} + (-)^{r-1} M^r.$$

Accordingly $A_r = f_r(M)$, where $f_r(M)$ is the polynomial on the right of (31) multiplied by Δ^{1-r}.

From this theorem we derive the corollaries.

Corollary I. If $r = n$, *equation* (31) *becomes the Cayley-Hamilton equation satisfied by M.*

For $A_n = 0$ and p_r is the spur of R_r and therefore by (16) $\Delta^{r-1} p_r$ is the spur of M_r, which is the sum of the r-rowed principal minors of M.

Corollary II. Each matrix B_{n-r} $(r = 1, 2, \ldots . n-1)$ *is a polynomial of degree r in the matrix N.*

If in equation (31), A_r is replaced by B_{n-r}, p_r by p_{n-r}, Δ by Θ, and M by N, the actual expression for $\Theta^{r-1} B_{n-r}$ is obtained.

Corollary III. The matrix M is permutable with each of the matrices A_r *and* B_r, *and* A_r *is permutable with* A_k *and* B_s, *for* $r, s, k = 1, 2, \ldots n-1$.

Corollary IV. The latent roots of A_r *are* $f_r(\lambda_i)$, *where* λ_i *are the latent roots of M.*[1]

Corollary V. Any power M^r, $r = 1, 2, \ldots . n$, *of the matrix M can be expressed as a linear function of the matrices* $E, A_1, A_2, \ldots . A_r$ *or as a linear function of the matrices* $B_{n-1}, B_{n-2}, \ldots . , B_{n-r}$.

For the r equations (31), $r = 1, 2, \ldots, r$, can be solved for $M, M^2, \ldots, M^r$ in terms of the matrices $B_{n-1}, \ldots, B_{n-r}$.

If from A_r a new matrix C_r is formed by replacing each element of A_r by a similar element in which each column of X' and Y' is replaced by the corresponding row of X' and Y' respectively, then formulae similar to (29) and (31) are true in which A_r is replaced by C_r and M *by* Q. In particular the following theorem is valid;

Theorem 6. *The latent roots of* A_r *are the same as those of* C_r.

This theorem follows immediately from the fact that the latent roots of Q are the same as those of M (theorem 1).

Corollary I. The sum of the principal minors of order m of A_r *is equal to the sum of the principal minors of order m of* C_r.

This corollary, which is an immediate consequence of the theorem, is a generalisation of theorem 4 but an explicit statement of it would be very involved.

Since the spur of $M^t = s_t$ is the sum of the tth powers of the latent roots of M, and since the spur of A_r is $(n-r)\,p_r$, by taking the spurs of the matrices on both sides of equation (31), we have

$$\Delta^{r-1}(n-r)\,p_r = \sum_{i=0}^{r} (-1)^{i-1} \Delta^{i-1} p_i\, s_{r-i}. \tag{32}$$

But, as $\Delta^{i-1} p_i = (-1)^i$ (the ith elementary symmetric function of the latent roots λ_j of M), equation (32) is nothing else than Newton's well known formula connecting the sum of the powers of the roots of an equation with the elementary symmetric functions of the roots.

---

[1] This corollary is a special case of the Sylvester-Frobenius Theorem that if λ is a latent root of a matrix A, then $f(\lambda)$ is a latent root of the matrix polynomial $f(A)$.

https://doi.org/10.1017/S0013091500007793 Published online by Cambridge University Press

We shall now prove the lemma;

Lemma I. $$\Omega A_r = (r+1)\,\Omega p_r.$$

We shall prove this lemma by induction, assuming that

$$\Omega A_{r-1} = r\,\Omega p_{r-1} = rB_{n-r}. \quad \text{By (29)}$$
$$\Delta B_{n-r} = MA_{r-1} = A_{r-1}M, \text{ or}$$
$$B_{n-r}X = A_{r-1}Y \text{ by (9).}$$

By operating on both sides of the last equation by $Y\,\Omega$[1],

$$Y(\Omega B_{n-r})X + Y\,(\text{spur of } B_{n-r}) = Y\,\Omega\,(A_{r-1}Y),$$

or $$Y\,\Omega\,(p_r E - A_r)X + rp_r Y = rB_{n-r}Y$$

by assumption. Hence

$$\Delta\,(B_{n-r-1} - Y\,\Omega\,A_r) = -rMA_r \text{ by (28) and (9),}$$
$$= -rY\,\Omega\,p_r \text{ by (28) and (29).}$$

But $$B_{n-r-1} = Y\,\Omega\,p_r \text{ and accordingly}$$
$$Y\,\Omega\,A_r = (r+1)\,Y\,\Omega\,p_r.$$

The truth of our lemma follows when we notice that $\Omega A_0 = \Omega p_0$.

If for brevity we write Ω_y for the matrix operator $Y\,\Omega$, we are now in a position to prove the theorem;

Theorem 6. *The matrix* $B_{n-s-1} = \dfrac{(-1)^s}{s!}\,\Omega_y^{s+1}\,p_0.$

For by lemma I

$$\begin{aligned}\Omega_y^2\,p_r = \Omega_y(\Omega_y\,p_r) &= \Omega_y\,(p_{r+1}E - A_{r+1})\\ &= \Omega_y\,(p_{r+1}) - (r+2)\,\Omega_y\,p_{r+1},\\ &= -(r+1)\,\Omega_y\,p_{r+1}.\end{aligned} \qquad (33)$$

By repeated applications of formula (33) we derive

$$\Omega_y^{s+1}\,p_r = (-1)^s(r+1)(r+2)\,..\,(r+s)\,\Omega_y\,p_{r+s}, \qquad (34)$$

and if $r = 0$ in (34), (34) becomes

$$\Omega_y^{s+1}\,p_0 = (-1)^s s!\,\Omega_y\,p_s = (-1)^s s!\,B_{n-s-1},$$

and this last result proves our theorem.

§ 4. *Connection with Turnbull's paper.* If we now take Y to be the unit matrix, the matrix operator $Y\,\Omega = \Omega$ and formula (28) becomes

$$\Omega\,p_r = p_{r+1}E - A_{r+1}$$

which by (26) is equivalent to

$$\Omega\,p_r = p_{r+1}E - X\,\Omega\,p_{r+1}.$$

[1] Turnbull. *Loc. cit.* Formula (12) § 1 and formula (4) § 2.

But, with the necessary changes in notation, this is simply formula (1) of section (8) of Turnbull's paper. Similarly formula (31) corresponds to theorem IV of his paper. The correspondence is, however, not direct, since by (9) if $Y = E$, then $M = \Delta X^{-1}$. If this value be substituted for M in (31), and the resulting equation multiplied by X^r be subtracted from the Cayley-Hamilton equation satisfied by X, we get the actual results listed in Theorem IV. This suggests that to obtain Turnbull's theorem IV as a direct consequence of our results, we should use the formula similar to (31), which involves N instead of M. This is actually the case and in fact his theorems I, II, III and IV have immediate counterparts when the matrix N is used.

Let N be denoted by n_i^j, where $n_i^j = (y_1 \ldots . y^{j-1} x^i y^{j+1} \ldots . y^n)$, (formula (5)). If s_r denote the sum of the rth powers of the latent roots of N, then s_r is the spur of N^r and

$$s_r = n_{a_1}^{a_2} n_{a_2}^{a_3} \ldots . n_{a_{r-1}}^{a_r} n_{a_r}^{a_1}, \qquad (35)$$

$$= n_{a_1}^{a_2} K.$$

Therefore since (35) is symmetric in the a_i, the element in the ith row and the jth column of $\Omega_y s_r$ is

$$rKy_a^i \frac{\partial}{\partial x_a^j} n_{a_1}^{a_2},$$

$$= rK \,\Theta\, \delta_{a_1}^j \delta_{a_2}^i .$$

Hence

$$\Omega_y s_r = r \,\Theta\, N^{r-1}. \qquad (36)$$

Similarly the element in the ith row and the jth column of the matrix $\Omega_y N^r$ is

$$y_a^i \frac{\partial}{\partial x_a^{a_1}} n_{a_1}^{a_2} n_{a_2}^{a_3} \ldots . n_{a_{r-1}}^{a_r} n_{a_r}^j$$

$$= \Theta\, \delta_{a_1}^{a_k} \delta_{a_{k+1}}^i \sum_{k=1}^{r} (n_{a_1}^{a_2} n_{a_2}^{a_3} \ldots n_{a_{k-1}}^{a_k})(n_{a_{k+1}}^{a_{k+2}} \ldots n_{a_r}^j),$$

and therefore

$$\Omega_y N^r = \Theta \sum_{k=1}^{r} s_{k-1} N^{r-k}. \qquad (37)$$

If $Y = E$, $\Theta = 1$ and equations (36) and (37) reduce to the formulae of theorems II and III of Turnbull's paper. In a similar manner it may be shown that

$$\Omega_y N'^r = \Theta (N'^{r-1} + NN'^{r-2} + \ldots + N^i N'^{r-i-1} + \ldots + N^{r-1})$$

a result which corresponds to Turnbull's theorem I.