INTERVAL FUNCTIONS AND NON-DECREASING FUNCTIONS

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1. Introduction. In a previous paper the author **(1)** has shown the following theorem.

THEOREM A. *If each of H and K is a real-valued bounded function of subintervals of the number interval* $[a, b]$ *and m is a real-valued non-decreasing function on* $[a, b]$ *such that each of the integrals* (Section 2)

$$\int_{[a,b]} H(I)\, dm \qquad \text{and} \qquad \int_{[a,b]} K(I)\, dm$$

exists, then the integral

$$\int_{[a,b]} H(I)K(I)\, dm$$

exists.

In this paper we prove the following generalization of Theorem A.

THEOREM 4. *If each of H and K is a real-valued bounded function of subintervals of the number interval* $[a, b]$ *and each of r and s is a real-valued non-decreasing function on* $[a, b]$ *such that*

$$\int_{[a,b]} H(I)\, dr \qquad \text{or} \qquad \int_{[a,b]} H(I)\, ds$$

exists and

$$\int_{[a,b]} K(I)\, dr \qquad \text{or} \qquad \int_{[a,b]} K(I)\, ds$$

exists, then the integral

$$\int_{[a,b]} H(I)K(I)[dr]^p[ds]^{1-p}$$

exists for each number p such that $0 < p < 1$.

2. Preliminary theorems and definitions. Throughout this paper all integrals considered are Hellinger **(3)** type limits of the appropriate sums; that is to say, if H is a real-valued function of subintervals of the number interval $[a, b]$, then $\int_{[a,b]} H(I)$ denotes the limit, for successive refinements of subdivisions, of sums $\sum_D H(I)$, where D is a subdivision of $[a, b]$ and the sum is taken over all intervals I of D.

The definitions, theorems, and proofs of this paper can be extended to "many-valued" interval functions.

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Suppose $[a, b]$ is a number interval and each of H and K is a real-valued function of subintervals of $[a, b]$.

We see that $\int_{[a,b]} K(I)$ exists if and only if for each subinterval $[u, v]$ of $[a, b]$, $\int_{[u,v]} K(J)$ exists, so that if $a \leqslant u \leqslant w \leqslant v \leqslant b$, then

$$\int_{[u,w]} K(J) + \int_{[w,v]} K(J) = \int_{[u,v]} K(J).$$

We state the following theorems.

THEOREM K (Kolmogoroff **4**). *If* $\int_{[a,b]} K(I)$ *exists, then*

$$\int_{[a,b]} |K(I) - \int_I K(J)| = 0.$$

COROLLARY K. *If* H *is bounded and* $\int_{[a,b]} K(I)$ *exists, then*

$$\int_{[a,b]} |H(I)||K(I) - \int_I K(J)| = 0,$$

so that if $[u, v]$ *is a subinterval of* $[a, b]$, *then*

$$\int_{[u,v]} H(I)K(I)$$

exists if and only if

$$\int_{[u,v]} H(I) \int_I K(J)$$

exists, in which case equality holds.

THEOREM P (Appling **2**). *If each of* H *and* K *is non-negative valued and each of* $\int_{[a,b]} H(I)$ *and* $\int_{[a,b]} K(I)$ *exists and* p *is a number such that* $0 < p < 1$, *then*

$$\int_{[a,b]} H(I)^p K(I)^{1-p} = \int_{[a,b]} [\int_I H(J)]^p [\int_I K(J)]^{1-p}.$$

3. Some preliminary theorems. In this section we prove some elementary facts about interval functions and non-decreasing functions.

Suppose $[a, b]$ is a number interval, K is a real-valued function of subintervals of $[a, b]$, m is a real-valued non-decreasing function on $[a, b]$ and $\int_{[a,b]} K(I)\, dm$ exists.

COROLLARY A. *If* K *is bounded and* n *is a positive integer, then* $\int_{[a,b]} K(I)^n\, dm$ *exists.*

Suppose p is a number such that $0 < p < 1$.

THEOREM 1. *If* K *is non-negative valued, then* $\int_{[a,b]} K(I)^p\, dm$ *exists.*

Proof. By Theorem P,

$$\int_{[a,b]} [\int_I K(J)\, dm]^p [dm]^{1-p} = \int_{[a,b]} [K(I) dm]^p [dm]^{1-p},$$

which is equal to $\int_{[a,b]} K(I)^p\, dm$.

THEOREM 2. *If* K *is non-negative valued and bounded and* q *is a positive number, then* $\int_{[a,b]} K(I)^q\, dm$ *exists.*

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Proof. There is a positive integer n such that $0 < q/n < 1$, so that by Theorem 1, $\int_{[a,b]} K(I)^{q/n}\, dm$ exists. By Corollary A $\int_{[a,b]} [K(I)^{q/n}]^n\, dm$ exists and is equal to $\int_{[a,b]} K(I)^q\, dm$.

Suppose each of r and s is a real-valued non-decreasing function on $[a, b]$.

THEOREM 3. *If K is bounded and $\int_{[a,b]} K(I)\, dr$ exists, then*

$$\int_{[a,b]} K(I)[dr]^p[ds]^{1-p}$$

exists.

Proof. It is sufficient to prove the theorem for K non-negative valued. By Theorem P, $\int_{[a,b]} [K(I)dr]^p[ds]^{1-p}$ exists. By Corollary K it is equal to $\int_{[a,b]} K(I)^p \int_I [dr]^p[ds]^{1-p}$. By Theorem 2 $\int_{[a,b]} [K(I)^p]^{1/p} \int_I [dr]^p[ds]^{1-p}$ exists and is equal to $\int_{[a,b]} K(I) \int_I [dr]^p[ds]^{1-p}$, which, by Corollary K, is equal to

$$\int_{[a,b]} K(I)[dr]^p[ds]^{1-p}.$$

4. The main theorem. We now prove Theorem 4, which is quoted in the introduction.

Proof. By Theorem 3, each of

$$\int_{[a,b]} H(I)[dr]^p[ds]^{1-p} \quad \text{and} \quad \int_{[a,b]} K(I)[dr]^p[ds]^{1-p}$$

exists, so that by Corollary K, each of

$$\int_{[a,b]} H(I) \int_I [dr]^p[ds]^{1-p} \quad \text{and} \quad \int_{[a,b]} K(I) \int_I [dr]^p[ds]^{1-p}$$

exists. Therefore by Theorem A, $\int_{[a,b]} H(I)K(I) \int_I [dr]^p[ds]^{1-p}$ exists. By Corollary K, it is equal to $\int_{[a,b]} H(I)K(I)[dr]^p[ds]^{1-p}$.

REFERENCES

1. W. D. L. Appling, *Interval functions and the Hellinger integral*, Duke Math. J., *29* (1962), 515–520.
2. ——— *Infinite series and nonnegative valued interval functions*, Duke Math. J., *30* (1963, 107–112.
3. E. Hellinger, *Die Orthogonalinvarianten quadratischer Formen von unendlichvielen Variablen*, Diss., Göttingen (1907).
4. A. Kolmogoroff, *Untersuchung über den Integralbegriff*, Math. Ann., *103* (1930), 654–696.

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