



RESEARCH ARTICLE

Measurable Vizing’s theorem

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Abstract

We prove a *full* measurable version of Vizing’s theorem for bounded degree Borel graphs, that is, we show that every Borel graph \mathcal{G} of degree uniformly bounded by $\Delta \in \mathbb{N}$ defined on a standard probability space (X, μ) admits a μ -measurable proper edge coloring with $(\Delta + 1)$ -many colors. This answers a question of Marks [Question 4.9, *J. Amer. Math. Soc.* 29 (2016)] also stated in Kechris and Marks as a part of [Problem 6.13, *survey* (2020)], and extends the result of the author and Pikhurko [Adv. Math. 374, (2020)], who derived the same conclusion under the additional assumption that the measure μ is \mathcal{G} -invariant.

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1. Introduction

Vizing’s theorem is a fundamental result in graph theory that relates the number of colors needed to properly color edges of a given graph G , the so-called *chromatic index* $\chi'(G)$ of G , with its maximum degree; it states that if the maximum degree of G is $\Delta \in \mathbb{N}$, then $\chi'(G) \leq \Delta + 1$. Together with König’s *line coloring theorem* (that states that $\chi'(G) = \Delta$ under the additional assumption that G is bipartite), these classical results laid the foundation of edge-coloring, an important and active area of graph theory; see, for example, the recent book on edge-coloring by Stiebitz, Scheide, Toft and Favrhøldt [SSTF12].

In this paper, we study Vizing’s theorem from the perspective of *measurable graph combinatorics*, a subfield of descriptive set theory that lies at the intersection of measure theory, random processes,

dynamics, group theory, combinatorics and distributed computing; a (nonexhaustive) sample of results related to the field (and to our investigation) includes [Lac90, MU17, GMP17, DF92, MU16, Gab00, KST99, Mar16, CM17, CJM⁺23, CGM⁺17, GP20, Ber23, BCG⁺24]. In its most abstract form, measurable graph combinatorics systematically studies the existence of *measurable* or *definable* solutions to various graph coloring problems defined on so-called *Borel graphs*, graphs where the vertex set (V, \mathcal{B}) is a *standard Borel space*, for example, the unit interval, and the edge relation E is a Borel subset of $[V]^2$, where $[V]^2$ denotes the set of unordered pairs endowed with the canonical Borel structure coming from V . This study originated in the seminal paper of Kechris, Solecki and Todorćević [KST99] and since then found many applications in various areas of central mathematics; see the surveys of Kechris and Marks [KM20] and Pikhurko [Pik21].

Basic questions about vertex colorings of bounded degree Borel graphs are well understood. Recall that the *Borel chromatic number* $\chi_{\mathcal{B}}(\mathcal{G})$ of a Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ is the smallest $k \in \mathbb{N}$ such that there is a decomposition $V = V_1 \sqcup \dots \sqcup V_k$, where V_i is a Borel subset of V that does not span any edge of \mathcal{G} for every $i \in [k]$. In the paper [KST99], Kechris, Solecki and Todorćević showed that there is a *Borel measurable* version of the classical greedy algorithm for proper vertex coloring, thus proved that $\chi_{\mathcal{B}}(\mathcal{G}) \leq \Delta + 1$ for every Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ of maximum degree $\Delta \in \mathbb{N}$. In the groundbreaking paper [Mar16], Marks found an example of acyclic Δ -regular Borel graph \mathcal{G} such that $\chi_{\mathcal{B}}(\mathcal{G}) = \Delta + 1$ for every $\Delta \in \mathbb{N}$, thus concluding that there is no Borel analogue of the *classical Brooks' theorem* from finite combinatorics. On the other hand, Conley, Marks and Tucker-Drob [CMTD16] proved that Brooks' theorem holds once *Borel measurability* is relaxed either in the sense of measure theory or topology. For example, if $\mathcal{G} = (V, \mathcal{B}, E)$ is a Borel graph of degree bounded by $\Delta \geq 3$ that does not contain a clique on $\Delta + 1$ vertices and μ is a *Borel probability measure on* (V, \mathcal{B}) , then there is a μ -null Borel set Y such that $\chi_{\mathcal{B}}(\mathcal{G} \upharpoonright (V \setminus Y)) = \Delta$. Recently, these results were refined by Bernshteyn [Ber23] and Brandt, Chang, the author, Grunau, Rozhoň and Vidnyánszky [BCG⁺24] using ideas from the theory of distributed computing.

In contrast, similar questions about *edge colorings* are not yet fully understood. A *Borel matching* of a Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ is a Borel subset $M \subseteq E$ that is a matching, that is, every two edges $e, f \in M$ are either equal (as unordered pairs) or do not share a vertex. The *Borel chromatic index* $\chi'_{\mathcal{B}}(\mathcal{G})$ is defined to be the smallest $k \in \mathbb{N}$ such that there is a decomposition $E = E_1 \sqcup \dots \sqcup E_k$, where E_i is a Borel matching for every $i \in [k]$. Equivalently, this can be stated in the language of edge colorings as finding the smallest $k \in \mathbb{N}$ such that there is a Borel map $c : E \rightarrow [k]$ that is a *proper edge coloring*, i.e., $c(e) \neq c(f)$ whenever $e \neq f \in E$ share a vertex. The greedy upper bound [KST99] implies that $\chi'_{\mathcal{B}}(\mathcal{G}) \leq 2\Delta - 1$ for every Borel graph \mathcal{G} of maximum degree bounded by $\Delta \in \mathbb{N}$. In the same paper [Mar16], Marks found an example of acyclic Δ -regular Borel graph \mathcal{G} such that $\chi'_{\mathcal{B}}(\mathcal{G}) = 2\Delta - 1$ for every $\Delta \in \mathbb{N}$. This shows that Borel analogue of Vizing's theorem does not hold. Similarly as in the case of vertex colorings, it is natural to ask if Vizing's theorem hold once Borel measurability is relaxed either in the sense of measure theory or topology. This question (for both measure and topology) is explicitly stated in the survey of Kechris and Marks [KM20, Problem 6.13]. Marks asked about the measurable relaxation for regular graphs [Mar16, Question 4.9], and, in fact, the same question for invariant measures, so-called *graphings*, was raised earlier by Abért [Abe10].

In this paper, we completely resolve the measurable case; see Theorem 1.1. Before we state the result, we formalize the definitions and discuss relevant results from recent years. Recall that we always assume that the Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ in question has degree uniformly bounded by $\Delta \in \mathbb{N}$. Given a *Borel probability measure* μ on (V, \mathcal{B}) , we define the μ -*chromatic index of* \mathcal{G} , $\chi'_{\mu}(\mathcal{G})$, to be the minimal $k \in \mathbb{N}$ such that there is a Borel map $c : E \rightarrow [k]$ that satisfies

$$\mu(\{v \in V : c \text{ is not proper at } v\}) = 0,$$

where c is *not proper at* $v \in V$ if there are edges $e \neq f \in E$ that are adjacent to v and $c(e) = c(f)$. In another words, c is a proper edge coloring at μ -almost every vertex $v \in V$. We say that μ is \mathcal{G} -*invariant*, if $\mu(g(C)) = \mu(C)$ for every $C \in \mathcal{B}$ and every Borel injection $g : C \rightarrow V$ that satisfies that x and $g(x)$

are in the same connected component of \mathcal{G} for every $x \in C$. In that case, the quadruple $\mathcal{G} = (V, \mathcal{B}, \mu, E)$ is called a *graphing*.

Csóka, Lippner and Pikhurko [CLP16] showed that $\chi'_\mu(\mathcal{G}) \leq \Delta + 1$ for a graphing $\mathcal{G} = (V, \mathcal{B}, \mu, E)$ that does not contain odd cycles and proved an upper bound of $\Delta + O(\sqrt{\Delta})$ colors for graphings in general. In a related result, Bernshteyn [Ber19, Theorem 1.3] proved that $\Delta + o(\Delta)$ colors are enough (even for the so-called list-coloring version) provided that the graphing factors to the shift action $\Gamma \curvearrowright [0, 1]^\Gamma$ of a finitely generated group Γ . Answering the question of Abért, the author and Pikhurko [GP20] proved a measurable version of Vizing's theorem for graphings, that is, $\chi'_\mu(\mathcal{G}) \leq \Delta + 1$ for any graphing $\mathcal{G} = (V, \mathcal{B}, \mu, E)$. Interestingly, the technique developed in [GP20] was greatly extended by Bernshteyn [Ber22] who found striking applications to the LOCAL model of distributed computing; see [Ber22, Chr23, BD23] for a current development in that direction.

Bernshteyn [Ber19], the author and Pikhurko [GP20], Tóth [T21] and the author [Gre22] investigated weaker notion of *approximate* edge colorings. In this setting, we require to find for each probability measure μ and $\epsilon > 0$ a coloring of edges that is not correct or undefined for at most ϵ fraction of all edges. Here, the analogues of Vizing's theorem as well as König's line coloring theorem hold; see [GP20] and [Gre22].

Bowen and Weilacher [BW23] investigated Vizing's theorem in the context of (Borel) asymptotic dimension introduced in [CJM⁺23]. As an application they derived that $\Delta + 1$ colors are enough for edge colorings of bipartite graphs in the topological relaxation sense and for measures that are hyperfinite. Stronger results in the special case of free Borel \mathbb{Z}^d -actions, that is, Borel edge coloring with $2d$ colors which is the analogue of König's line coloring theorem in this setting, were obtained independently around the same time in [BHT24, CU22, GR23, Wei21].

Recently, Qian and Weilacher [QW22] found connections of the topological relaxation to computable combinatorics which allowed them to derive an upper bound of $\Delta + 2$ colors for the Baire measurable analogue of Vizing's theorem, the full topological analogue, that is, $\Delta + 1$ colors only, remains an interesting open problem.

Next, we state our main result, which is the full analogue of Vizing's theorem in the measurable setting.

Theorem 1.1. *Let (V, \mathcal{B}) be a standard Borel space, $\Delta \in \mathbb{N}$, $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph of uniformly bounded degree by $\Delta \in \mathbb{N}$ and μ be a Borel probability measure on (V, \mathcal{B}) . Then $\chi'_\mu(\mathcal{G}) \leq \Delta + 1$, that is, there is a Borel map $c : E \rightarrow [\Delta + 1]$ that is a proper edge coloring at μ -almost every vertex $v \in V$.*

In the proof, we combine the technique of augmenting iterated Vizing chains introduced by the author and Pikhurko in [GP20] together with the following result, Theorem 1.2, that is interesting in its own right and might be useful for other applications as well.¹ Before we state the result, we need several definitions. Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and μ be a Borel probability measure on (V, \mathcal{B}) . Recall that μ is called \mathcal{G} -quasi-invariant if $\mu([A]_{\mathcal{G}}) = 0$, whenever $\mu(A) = 0$, where $[A]_{\mathcal{G}}$ is the union of all connected component of \mathcal{G} that have a nonempty intersection with A . It is a standard fact (see, e.g., [KM04, Chapter II, Section 8], that if μ is \mathcal{G} -quasi-invariant, then there is a function ρ_μ , called the *Radon–Nikodym cocycle of \mathcal{G}* that takes values in $(0, +\infty)$, is defined for every ordered pair of points $x, y \in V$ that are in the same connected component of \mathcal{G} and satisfies the following *mass transport principle*:

$$\mu(g(C)) = \int_C \rho_\mu(x, g(x)) d\mu$$

for every $C \subseteq V$ and an injective Borel map $g : C \rightarrow X$ that satisfies for every $x \in C$ that $g(x)$ is in the same connected component of \mathcal{G} as x . While in general ρ_μ can behave chaotically, the next result shows that one can always pass to an equivalent measure ν such that ρ_ν is bounded on edges of \mathcal{G} .

¹After the first version of this paper appeared on arXiv, we were informed by Gábor Elek that Gabriella Kuhn [Kuh94, Lemma 1] proved the same result in the context of group actions; see Remark 1.3.

Theorem 1.2 (see also [Kuh94] Lemma 1). *Let (V, \mathcal{B}) be a standard Borel space, $\Delta \in \mathbb{N}$, $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph of uniformly bounded degree by $\Delta \in \mathbb{N}$ and μ be a Borel probability measure on (V, \mathcal{B}) that is \mathcal{G} -quasi-invariant. Then there is an equivalent Borel probability measure ν on (V, \mathcal{B}) such that*

$$\frac{1}{4\Delta} \leq \rho_\nu(x, y) \leq 4\Delta$$

for every edge $(x, y) \in E$.

In particular, if $\text{dist}_{\mathcal{G}}(x, y) = k \in \mathbb{N}$, then $\rho_\nu(x, y) \leq (4\Delta)^k$, where $\text{dist}_{\mathcal{G}}$ is the graph distance on \mathcal{G} .

Remark 1.3. Kuhn [Kuh94, Lemma 1] showed that if a countable group Γ acts in a quasi-invariant fashion on a standard probability space (X, μ) , then there is an equivalent measure ν such that the cocycle $\rho_\nu(-, \gamma \cdot -)$ (as a function $X \rightarrow (0, \infty)$) is bounded for every fixed $\gamma \in \Gamma$. This result combined with the fact that every Borel graph \mathcal{G} of degree bounded by $\Delta(\mathcal{G}) < +\infty$ can be generated by $2\Delta(\mathcal{G}) - 1$ involutions (which follows from the Lusin–Novikov uniformization theorem [Kec95, Theorem 18.10]) implies Theorem 1.2 possibly with a constant larger than 4Δ .

Similar, stronger conditions on the cocycle was used by Conley and Tamuz [CT21]. They designed a procedure for solving the *unfriendly coloring* problem on graphs of degree bounded by Δ and showed that this procedure terminates off of a μ -null set for every quasi-invariant Borel probability measure μ under the assumption that $1 - 1/\Delta \leq \rho_\mu(x, y) \leq 1 + 1/\Delta$ for every edge $\{x, y\}$ (this includes, for example, the case when μ is invariant). Finding a measurable unfriendly coloring for a general Borel probability measure remains an interesting open problem. In general, it would be nice to investigate if our condition has another applications in the context of local graph coloring problems.

The paper is structured as follows: In Section 2, we set the notation and recall basic results; in Section 3, we define the augmenting chains that we consider in this paper, so-called 3-step Vizing chains, and estimate how many of them can be attached to an uncolored edge; in Section 4, we describe an infinite procedure that improves a given coloring so that it does not contain augmenting chains of small weight; in Section 5, we prove Theorem 1.2, that is, we show how to modify a given quasi-invariant measure to an equivalent measure that has bounded cocycle on edges; in Section 6, we describe the double counting argument that estimates the number of uncolored edges; and finally in Section 7, we combine all the results to prove Theorem 1.1.

2. Preliminaries

Our basic descriptive set theory reference is [Kec95]; see also [KM20, Pik21]. We mostly follow the notation developed in [GP20]. Let us point out that \mathbb{N} contains 0. If $k \in \mathbb{N} \setminus \{0\}$, then we write $[k] = \{1, 2, \dots, k\}$. If S is a set, then we write $[S]^k$ for the set of k -element subsets of S . In particular, $[S]^2$ is the set of unordered pairs of S . We follow a standard graph theoretic notation, that is, a graph is a pair $G = (V, E)$, where $V \subseteq [E]^2$, $\text{deg}_G(x)$ is the number of neighbors of x in G , $\text{dist}_G(x, y)$ denotes the graph distance of $x, y \in V$ in G , and $N_G(x)$ denotes the set of all edges of G that are adjacent to x , that is, $N_G(x) = \{e \in E : x \in e\}$.

Borel graphs. A Borel graph is a triplet $\mathcal{G} = (V, \mathcal{B}, E)$, where (V, \mathcal{B}) is a standard Borel space, (V, E) is a graph and E is a Borel subset of $[V]^2$ (endowed with the natural σ -algebra inherited from the product σ -algebra $\mathcal{B} \times \mathcal{B}$ on $V \times V$). We say that \mathcal{G} is of *bounded maximum degree* if there is $\Delta \in \mathbb{N}$ such that $\text{deg}_{\mathcal{G}}(x) \leq \Delta$ for every $x \in V$. We write $\Delta(\mathcal{G})$ for smallest such Δ . We denote the connectedness relation of \mathcal{G} as $F_{\mathcal{G}}$. That is, $F_{\mathcal{G}}$ is an equivalence relation on X , and we write $[x]_{\mathcal{G}}$ for the \mathcal{G} -connectivity component of $x \in V$. If $\Delta(\mathcal{G}) < \infty$, then $F_{\mathcal{G}}$ is a Borel subset of $V \times V$ and $[x]_{\mathcal{G}}$ is at most countable. Recall that an equivalence relation F on a standard Borel space (X, \mathcal{D}) that is Borel as a subset of $X \times X$ and each F -equivalence class is at most countable is called a *countable Borel equivalence relation (CBER)* on X ; see [Kec24]. In particular, if $\Delta(\mathcal{G}) < \infty$, then $F_{\mathcal{G}}$ is a CBER on V . Given $A \subseteq X$, we write $[A]_F$ for the F -saturation of A in F , that is, the smallest F -invariant subset of X

that contains A . If $[A]_F = A$, then we say that A is F -invariant. In case $F = F_G$, we write simply $[A]_G$ and say that A is G -invariant if it is F_G -invariant. Observe that if A is a Borel set, then so is $[A]_F$.

The *line graph* of a Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ is the Borel graph $\mathcal{E} = (E, \mathcal{C}, I_G)$, where \mathcal{C} is the σ -algebra on E inherited from $[V]^2$ and (E, I_G) is the line graph of (V, E) , that is, $\{e, f\} \in I_G$ if and only if e and f share exactly one vertex. Observe that if $\Delta(\mathcal{G}) < \infty$, then $\Delta(\mathcal{E}) \leq 2\Delta(\mathcal{G}) - 2$. Similarly as above, $F_{\mathcal{E}}$ is the CBER on E induced by the connectivity components of \mathcal{E} .

A *Borel chromatic number* $\chi_{\mathcal{B}}(\mathcal{G})$ of a Borel graph \mathcal{G} is the minimal $k \in \mathbb{N}$ such that there is a Borel proper vertex coloring $d : V \rightarrow [k]$ of \mathcal{G} . Note that the subscript $\chi_{\mathcal{B}}$ refers to the corresponding σ -algebra. A *Borel chromatic index* $\chi'_{\mathcal{B}}(\mathcal{G})$ is defined as $\chi_{\mathcal{C}}(\mathcal{E})$. That is $\chi'_{\mathcal{B}}(\mathcal{G}) = k$ if there is a Borel proper vertex coloring $c : E \rightarrow [k]$ of \mathcal{E} .

Measures. The set of all *Borel probability measures* on a standard Borel space (X, \mathcal{B}) is denoted as $\mathcal{P}(X)$. Let F be a CBER on X and $\mu \in \mathcal{P}(X)$. We say that μ is F -quasi invariant if $\mu([A]_F) = 0$, whenever $\mu(A) = 0$. If $\mathcal{G} = (V, \mathcal{B}, E)$ is a Borel graph, then we say that $\mu \in \mathcal{P}(V)$ is G -quasi-invariant if it is F_G -quasi-invariant

Proposition 2.1 (Propositions 3.1 and 3.2 in [GP20]). *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $\mathcal{E} = (E, \mathcal{C}, I_G)$ be its line graph and $\mu \in \mathcal{P}(V)$. Then there is $\hat{\mu} \in \mathcal{P}(E)$ that is \mathcal{E} -quasi-invariant and satisfies*

$$\mu(\{x \in V : \exists e \in A \ x \in e\}) = 0$$

for every $A \subseteq E$ that satisfy $\hat{\mu}(A) = 0$.

Proof. By [GP20, Proposition 3.2], we find $\tilde{\mu} \in \mathcal{P}(V)$ that is G -quasi invariant and satisfies $\mu([A]_G) = \tilde{\mu}([A]_G)$ for every $A \subseteq V$. By [GP20, Proposition 3.1], we find $\hat{\mu} \in \mathcal{P}(E)$ that is \mathcal{E} -quasi invariant and satisfies

$$\tilde{\mu}(\{x \in V : \exists e \in A \ x \in e\}) < \Delta(\mathcal{G})\epsilon$$

for every $A \subseteq E$ that satisfies $\hat{\mu}(A) < \epsilon$. In particular, if $\hat{\mu}(A) = 0$, then

$$\begin{aligned} \mu(\{x \in V : \exists e \in A \ x \in e\}) &\leq \mu(\{[x \in V : \exists e \in A \ x \in e]\}_G) \\ &= \tilde{\mu}(\{[x \in V : \exists e \in A \ x \in e]\}_G) = 0 \end{aligned}$$

as $\tilde{\mu}(\{x \in V : \exists e \in A \ x \in e\}) = 0$. □

A fundamental tool in the study of quasi-invariant measures is the Radon–Nikodym cocycle. Let (X, \mathcal{B}) be a standard Borel space, F be a CBER on X and $\mu \in \mathcal{P}(X)$ be F -quasi-invariant. Then the *Radon–Nikodym cocycle (of μ with respect to F)* is a Borel function $\rho_{\mu, F} : F \rightarrow \mathbb{R}_{>0}$ with the property that

$$\mu(g(C)) = \int_C \rho_{\mu, F}(x, g(x)) \ d\mu(x)$$

for every $C \in \mathcal{B}$ and injective Borel map $g : C \rightarrow X$ such that $(x, g(x)) \in F$. It is a standard fact that the Radon–Nikodym cocycle exists, and it is unique up to null-sets, that is, if ρ and ρ' are two Radon–Nikodym cocycles of μ with respect to F , then there is a μ -conull F -invariant set $A \subseteq X$ such that

$$\rho \upharpoonright (F \cap (A \times A)) = \rho' \upharpoonright (F \cap (A \times A));$$

see [KM04]. The following statement summarizes the properties of cocycles that we need.

Proposition 2.2 (Chapter II, Section 8 in [KM04]). *Let (X, \mathcal{B}) be a standard Borel space, F be a CBER on X and $\mu \in \mathcal{P}(X)$ be F -quasi-invariant. Then the Radon–Nikodym cocycle $\rho_{\mu, F} : F \rightarrow \mathbb{R}_{>0}$ satisfies:*

1. there is a μ -conull F -invariant set $A \in \mathcal{B}$ such that $\rho_{\mu,F}(x, y)\rho_{\mu,F}(y, z) = \rho_{\mu,F}(x, z)$ for any $x, y, z \in A$ such that $y, z \in [x]_F$,
2. (mass transport) we have

$$\int_X \sum_{x \in [y]_F} \mathbf{F}(x, y) d\mu(y) = \int_X \sum_{y \in [x]_F} \mathbf{F}(x, y)\rho_{\mu,F}(x, y) d\mu(x)$$

for any function $\mathbf{F} : F \rightarrow [0, \infty]$.

In case that $F = F_G$, we write $\rho_{\mu,G}$ instead of ρ_{μ,F_G} . If F is clear from context, then we write simply ρ_μ .

Definition 2.3. Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$ and $\mu \in \mathcal{P}(V)$ be \mathcal{G} -quasi-invariant. We say that μ is \mathcal{G} -bounded if

$$\frac{1}{4\Delta} \leq \rho_\mu(x, y) \leq 4\Delta$$

for every $\{x, y\} \in E$.

We show in Theorem 5.1 that every \mathcal{G} -quasi-invariant $\mu \in \mathcal{P}(V)$ is equivalent with a \mathcal{G} -bounded measure $\nu \in \mathcal{P}(V)$.

3. Edge colorings

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$. A *partial (Borel proper edge) coloring* of \mathcal{G} is a partial Borel (with respect to the σ -algebra \mathcal{C} on E , in particular, $\text{dom}(c) \in \mathcal{C}$) map $c : E \rightarrow [\Delta(\mathcal{G}) + 1]$ that assigns different colors to different edges that share a vertex. Usually we use lower case Greek letters for colors, for example, $\alpha, \beta \in [\Delta(\mathcal{G}) + 1]$. Given a partial coloring c , we define $m_c(x)$ to be the set of missing colors at $x \in V$, that is, $m_c(x) = [\Delta(\mathcal{G}) + 1] \setminus \{c(e) : e \in N_{\mathcal{G}}(x)\}$. We also write U_c for the set of uncolored edges, that is, $U_c = E \setminus \text{dom}(c)$.

In order to improve a given partial coloring c , we utilize an idea from the proof of Vizing's theorem, so-called *Vizing chains*. In general, given an uncolored edge $e \in U_c$, we want to find an injective augmenting sequence of edges $W_c(e) = (e_i)_{i \leq k}$ such that $e_0 = e$, $e_i \cap e_{i+1} \neq \emptyset$ for $i \leq k$ and $e_i \notin U_c$ for every $1 \leq i \leq k$ with the property that keeping the colors outside of $W_c(e)$ intact, but shifting the colors from e_{i+1} to e_i produces a different partial (proper) coloring c' such that $m_{c'}(z_0) \cap m_{c'}(z_1) \neq \emptyset$, where $\{z_0, z_1\} = e_k$ is the last edge of $W_c(e)$. Extending c' by assigning any color from $m_{c'}(z_0) \cap m_{c'}(z_1)$ to e_k then improves c as we decreased the number of uncolored edges. Observe that the difference between c and c' is contained in $W_c(e)$.

Various types of sequences $W_c(e)$ have been used in the literature. In order to prove classical Vizing's theorem, one chooses $W_c(e)$ to be a concatenation of a so-called *fan* and an *alternating path*, also known as a *Vizing chain*. To prove an analogue of Vizing's theorem for graphings, the author and Pikhurko [GP20] iterated this process two times. Namely, first we fix a Vizing chain, then truncate it at any edge on the alternating path, and then grow a second Vizing chain from that place; such a sequence is called an *iterated Vizing chain*. In order to devise efficient (both deterministic and randomized) local algorithms that produce a proper edge colorings with $\Delta + 1$ colors on finite graphs, Bernshteyn [Ber22] iterated this process $\log(n)$ times, where n is the number of vertices of the graph. Bernshteyn called the produced injective sequence a *multistep Vizing chain*. In this paper, being ideologically more closer to [GP20], we iterate the process three times. We follow the terminology of Bernshteyn and call this augmenting chain a *3-step Vizing chain*. The estimate for the number of 3-step Vizing chains that can be assigned to a given uncolored edge follows the computation from [GP20]. It seems plausible that one can get similar estimate by adapting the results from finite graphs, but we decided to follow the more direct path of generalizing [GP20].

3.1. 3-step Vizing chains

We recall the notation from [GP20, Section 2] and refer the reader there for basic results about this notation. Fix a Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ such that $\Delta(\mathcal{G}) < \infty$ and a partial edge coloring $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$. Set $\Delta = \Delta(\mathcal{G})$, and fix an ordering of the set of colors $[\Delta + 1]$.

A *chain* is a sequence $P = (e_0, \dots)$ of edges of \mathcal{G} such that for every index $i \in \mathbb{N}$ with e_i, e_{i+1} being in P we have $e_i \cap e_{i+1} \neq \emptyset$, that is, every two consecutive edges in P intersect. Let $l(P) = |P|$ denote the *length* of the chain P , that is, the number of edges in P . Note that a chain can be finite (possibly empty) or infinite; thus, $l(P) \in \mathbb{N} \cup \{\infty\}$ and, if P is finite, then $P = (e_0, \dots, e_{l(P)-1})$. The convention of labeling the first edge as e_0 allows us to write $P = (e_i)_{i < l(P)}$, regardless of whether P is finite or not. If $l(P) = \infty$, then we define $l(P) - 1 = \infty$ in order to avoid case by case statements in several places.

We call e_{i-1} the i -th edge of P . For an edge f that occurs exactly once in P , let its *index* $i(f)$ be $i \geq 1$ such that $f = e_{i-1}$, that is, the index of the i -th edge is i . Also, for $i \leq l(P)$, let $P_i = (e_j)_{j < i}$ denote the i -th *prefix* of P (which consists of the first i edges from P). We have, for example, that $P_{l(P)} = P$. For chains P and Q , we write $P \sqsubseteq Q$ if $P = Q_{l(P)}$, that is, P is a prefix of Q . If P is a finite chain with the last edge e and Q is a chain with the first edge f and $e \cap f \neq \emptyset$, then we write $P \frown Q$ for the chain that is the concatenation of P and Q .

Let us call a chain $P = (e_i)_{i < l(P)}$ a *path* if P is empty, or if every vertex $z \in V$ belongs to at most two edges from P and there is a vertex that belongs only to e_0 . (In other words, P is a finite path with a fixed direction on edges or an infinite one-sided ray, where no self-intersections are allowed.) Also, a chain P is called a *cycle* if P is nonempty and every vertex belongs to zero or two edges of P . (These are just finite cycles, having some edge and direction fixed.)

Definition 3.1 (Definition 2.2 in [GP20]). We say that a chain $P = (e_i)_{i < l(P)}$ is

1. *edge injective* if every edge appears at most once in P , that is, for every $0 \leq i < j < l(P)$ we have that $e_i \neq e_j$,
2. *c-shiftable* if $l(P) \geq 1$, P is edge injective, $e_0 \in U_c$ and $e_j \in \text{dom}(c)$ for every $1 \leq j < l(P)$ (that is, if P is nonempty with no edge repeated and e_0 is the unique uncolored edge of P);
3. *c-proper-shiftable* if P is c -shiftable and $c_P; E \rightarrow [\Delta + 1]$ is a partial coloring, where c_P is the *shift of c along P* (or P -*shift of c* for short) which is defined as
 - $\text{dom}(c_P) = \text{dom}(c) \cup \{e_0\} \setminus \{e_{l(P)-1}\}$ where we put $\{e_{l(P)-1}\} = \emptyset$ if $l(P) = \infty$,
 - $c_P(e_i) = c(e_{i+1})$ for every $i + 1 < l(P)$,
 - $c_P(f) = c(f)$ for every $f \in \text{dom}(c) \setminus P$;
4. *c-augmenting* if P is c -proper-shiftable and either $l(P) = \infty$ or P is finite with $m_{c_P}(x) \cap m_{c_P}(y) \neq \emptyset$, where $x \neq y$ are the vertices of the last edge $e_{l(P)-1}$ of P .

Next, we describe the building blocks that will be used to build 3-step Vizing chains:

Alternating path. Let $x \in V$ and $\alpha, \beta \in [\Delta + 1]$ be different colors such that $\beta \in m_c(x)$. Then there is a unique maximal chain $P = (e_i)_{i < l(P)}$ such that $x \in e_0$ if $l(P) > 0$, $x \notin e_1$ if $l(P) > 1$, and $c(e_i) = \alpha$ (resp. $c(e_i) = \beta$) for every $i < l(P)$ that is even (resp. odd). We call this unique maximal chain the *(alternating) α/β -path starting at $x \in V$* and denote it as $P_c(x, \alpha/\beta)$. If $P_c(x, \alpha/\beta)$ is finite and nonempty, then we call the unique $y \in V$ such that $|\{f \in P_c(x, \alpha/\beta) : y \in f\}| = 1$ and $y \neq x$ the *last vertex of $P_c(x, \alpha/\beta)$* . If $P_c(x, \alpha/\beta)$ is empty (which happens exactly when $\alpha \in m_c(x)$), then the *last vertex* is x . Whenever we write $P_c(x, \alpha/\beta)$, we always assume that the condition that $\beta \in m_c(x)$ is satisfied. Observe that the colors on the chain alternate between α and β (starting with α) and we never return to a vertex we have previously visited (and thus the edges in P form a path).

Fan. Let $e \in U_c$ and $x \in e$. We define the *maximal fan around x starting at e* , in symbols $F_c(x, e)$, as a (finite) chain $P = (e_0, e_1, \dots, e_k)$ such that $x \in e_i$ for every $i \leq k$ and if we denote the other vertex in e_i by v_i , then the following statements are satisfied

1. $e_0 = e$,
2. P is edge injective,

3. $c(e_{i+1}) \in m_c(v_i)$ for every $i < k$ and $c(e_{i+1})$ is the minimal color available in the i -th step, where we say that a color α is available in the i -th step if $\alpha \in m_c(v_i)$,
4. (e_0, \dots, e_k) is maximal with these properties.

Conditional fan. We generalize, but only formally, the following definition from [GP20, Section 2.5]. Take $P_c(x, \alpha/\beta)$ for some $x \in V$. Let $f \in P_c(x, \alpha/\beta)$ be a an edge that is not first nor last in $P_c(x, \alpha/\beta)$ and $y \in V$ be the last vertex of $P_c(x, e)_{i(f)}$. We define the maximal α/β -conditional fan starting at f , denoted as $F_c(x, \alpha/\beta, y)$, as a chain $P = (g_0, \dots, g_m)$ such that $y \in g_i$ for every $i \leq m$ and, if we denote the other vertex of g_i by u_i , then the following is satisfied

1. $g_0 = f$,
2. P is edge injective,
3. $c(g_{i+1}) \in m_c(u_i)$ and it is the minimal available color,
4. $\alpha, \beta \notin m_c(u_i)$ for every $i < m$,
5. if $\alpha, \beta \notin m_c(u_m)$, then (g_0, \dots, g_m) is maximal with the properties above.

Note that we should rather write u_i^f, g_i^f and y^f to stress that those objects depend on the choice of f . This will be, however, omitted in the cases when we work with only one f .

Now, we are ready to for the main definition.

Definition 3.2 (3-step Vizing chain). Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ be a partial edge coloring and $e \in U_c$. We say that a c -augmenting chain $W_c(e) = (e_i)_{i < l(W_c(e))}$, where $l(W_c(e)) \in \mathbb{N} \cup \{\infty\}$, is a 3-step Vizing chain (at e) if there are pairwise different vertices $y_1, y_2, y_3, z_1, z_2, z_3 \in V$, and colors $\alpha_i, \beta_i \in [\Delta(\mathcal{G}) + 1]$ for $i \in \{1, 2, 3\}$ such that

$$W_c(e) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3),$$

where

1. $F_c^1 \sqsubseteq F_c(y_1, e)$, $F_c^2 \sqsubseteq F_c(z_1, \alpha_1/\beta_1, y_2)$ and $F_c^3 \sqsubseteq F_c(z_2, \alpha_2/\beta_2, y_3)$,
2. $P_c^i \sqsubseteq P_c(z_i, \alpha_i/\beta_i)$ for every $i \in \{1, 2, 3\}$,
3. if $T \in \{F_c^i\}_{i=1}^3 \cup \{P_c^i\}_{i=1}^3$ satisfies $T = \emptyset$, then every S to the right from T in the definition of $W_c(e)$ is empty as well, that is, $W_c(e)$ is built by at most three iterations of the ‘Vizing chain’ construction,

3.2. Construction of 3-step Vizing chains

Let $e \in U_c$ and $x \in e$ be fixed. First, we describe a process that produces many chains of the form

$$W_c(e) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3)$$

that satisfies (1)–(3) in Definition 3.2, then we investigate how many of these chains are 3-step Vizing chains, that is, which ones are c -augmenting. We handle each iteration separately.

Iteration I. We start with [GP20, Section 2.4]. Recall that the Vizing chain $V_c(x, e)$ either consists of the fan $F_c(x, e)$, in case it is augmenting, or we have

$$V_c(x, e) = F_c(x, e)_{i+1} \frown P_c(v_i, \alpha/\beta),$$

where $i < l(F_c(x, e))$ (so-called first critical index) and $\{x, v_i\}$ is the last edge of the truncated fan $F_c(x, e)_{i+1}$.

Claim 3.3 (Proposition 2.9 in [GP20]). *If $F_c(x, e)$ is not c -augmenting, then there is $i < l(F_c(x, e))$ such that $F_c(x, e)_{i+1} \frown P_c(v_i, \alpha/\beta)$ is c -augmenting, where (x, v_i) is the last edge of the truncated fan*

$F_c(x, e)_{i+1}$ and β is the smallest missing color at v_i . In particular, the Vizing chain $V_c(x, e)$ is a 3-step Vizing chain.

Moreover, if nonempty, then $P_c(v_i, \alpha/\beta)$ does not use the vertex x .

Suppose that $P_c(v_i, \alpha/\beta)$ is nonempty. Define $F_c^1 = F_c(x, e)_{i+1}$, $y_1 = x$, $z_1 = v_i$ and $\alpha_1 = \alpha$, $\beta_1 = \beta$. This concludes the first iteration of the construction.

Iteration II. Recall that $f^1 \in P_c(z_1, \alpha_1/\beta_1)$ is *suitable* [GP20, Definition 2.10] if it is of graph distance ≥ 3 from F_c^1 , it is not the last edge of $P_c(z_1, \alpha_1/\beta_1)$ and $c(f^1) = \alpha_1$ (we remark that the last condition only helps with the notation and is otherwise irrelevant). Let y_2 be the last vertex of $P_c(z_1, \alpha_1/\beta_1)_{i(f^1)}$, and set $F_c(x, e \rightsquigarrow f^1) = F_c(z_1, \alpha_1/\beta_1, y_2)$, where $F_c(z_1, \alpha_1/\beta_1, y_2)$ is the maximal α_1/β_1 -conditional fan starting at f^1 .

Claim 3.4 (Proposition 2.12 in [GP20]). *Let $f^1 \in P_c(z_1, \alpha/\beta)$ be suitable. Then*

$$P = V_c(x, e)_{i(f^1)-1} \frown F_c(x, e \rightsquigarrow f^1)$$

is c -proper-shiftable.

According to the *type* of f^1 , as defined in [GP20, Section 2.5], it is possible to assign an injective sequence of edges Q such that either $Q = F_c(x, e \rightsquigarrow f^1)$, or

$$Q = F_c(x, e \rightsquigarrow f^1)_{m+1} \frown P_c(u_m, \gamma/\delta),$$

where $m < l(F_c(x, e \rightsquigarrow f^1))$ (the so-called second critical index), $\{y_2, u_m\}$ is the last edge of the truncated fan $F_c(x, e \rightsquigarrow f^1)_{m+1}$ and $\{\gamma, \delta\}$ is either equal to or disjoint from $\{\alpha_1, \beta_1\}$. Using the technical notion of *superb* edges, the following, again adapted to our terminology, is shown in [GP20].

Claim 3.5 (Propositions 2.15 and 2.17 in [GP20]). *Let $f^1 \in P_c(z_1, \alpha_1/\beta_1)$ be suitable and superb. The chain*

$$(F_c^1) \frown (P_c(z_1, \alpha_1/\beta_1)_{i(f^1)-1}) \frown Q$$

is a 3-step Vizing chain.

Suppose that $P_c(u_m, \gamma/\delta)$ is nonempty. Define $P_c^1 = P_c(z_1, \alpha_1/\beta_1)_{i(f^1)-1}$, $F_c^2 = F_c(x, e \rightsquigarrow f^1)_{m+1}$, $z_2 = u_m$ and $\alpha_2 = \gamma$, $\beta_2 = \delta$. This concludes the second iteration of the construction.

Iteration III. We say that $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ is *2-suitable* if

- o the graph distance of f and f^2 is at least 3 for every $f \in (F_c^1) \frown (P_c^1) \frown (F_c^2)$,
- o it is not the last edge of $P_c(z_2, \alpha_2/\beta_2)$,
- o $c(f^2) = \alpha_2$.

Let y_3 be the last vertex of $P_c(z_2, \alpha_2/\beta_2)_{i(f^2)}$, and set

$$F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2) = F_c(z_2, \alpha_2/\beta_2, y_3),$$

where $F_c(z_2, \alpha_2/\beta_2, y_3)$ is the maximal α_2/β_2 -conditional fan starting at f^2 .

Proposition 3.6. *Let $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ be 2-suitable. Then*

$$P = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c(z_2, \alpha_2/\beta_2)_{i(f^2)-1}) \frown F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$$

is c -proper-shiftable.

Proof. Suppose that c_P is not a partial coloring. By the definition, we find a vertex $v \in V$ such that $c_P \upharpoonright N_G(v)$ is not proper. By the fact that f^2 is 2-suitable together with [GP20, Proposition 2.12], we see that $v \notin f$ for any $f \in (F_c^1) \frown (P_c^1) \frown (F_c^2)$. Moreover, we must have $v \in f$ for some $f \in F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$ as the colors used around vertices that lie only on the path $P_c(z_2, \alpha_2/\beta_2)_{i(f^2)-1}$ do not change. Now, the definition of $F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$, as the maximal α_2/β_2 -conditional fan starting at f^2 , shows that no such $v \in V$ can exist, the argument is literally the same as in [GP20, Proposition 2.12]. \square

Suppose that f^2 is 2-suitable and set $P_c^2 = P_c(z_2, \alpha_2/\beta_2)_{i(f^2)-1}$. Next, we define various types of 2-suitable edges and the notion of an amazing edge. This is inspired by similar notions in [GP20, Section 2.5].

We say that a 2-suitable edge $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ is of *type (a)* if

$$P = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$$

is c -augmenting. Every edge of type (a) is said to be *amazing*. Define $F_c^3 = F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$, and set

$$W_c(e, f^1, f^2) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3).$$

Then the following is immediate from the definitions.

Proposition 3.7. *Let f^2 be of type (a). Then $W_c(e, f^1, f^2)$ is a 3-step Vizing chain.*

We say that a 2-suitable edge $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ is of *type (b)* if it is not of type (a) and in the construction of the conditional fan $F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$ we encountered α_2 or β_2 . Observe that as f^2 is not of type (a), we must have $\beta_2 \in m_c(w_n)$, where $\{y_3, w_n\}$ is the last edge in $F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$. Following the previous notation, we say that n is the *third critical index*. Define $F_c^3 = F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$, $y_3 = w_n$, $\alpha_3 = \alpha_2$, $\beta_3 = \beta_2$ and $P_c^3 = P_c(z_3, \alpha_3/\beta_3)$. We say that f^2 of type (b) is *amazing*, if

$$\circ P_{c_Q}(z_3, \alpha_3/\beta_3) = P_c(z_3, \alpha_3/\beta_3), \text{ where } Q = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3).$$

Proposition 3.8. *Let f^2 be of type (b) and amazing. Then*

$$W_c(e, f^1, f^2) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3)$$

is a 3-step Vizing chain.

Proof. It follows directly from the definitions that $W_c(e, f^1, f^2)$ satisfies (1)–(3) in Definition 3.2. It remains to show that $W_c(e, f^1, f^2)$ is c -augmenting. This can be done by the same argument as in [GP20, Proposition 2.15]. Namely, first observe that c_Q , where

$$Q = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3),$$

is a proper coloring by the same reasoning as in Proposition 3.6. Moreover, by the definition of 2-suitable edge, we must have $\alpha_2 \in m_{c_Q}(y_3)$. As $P_{c_Q}(z_3, \alpha_3/\beta_3) = P_c(z_3, \alpha_3/\beta_3)$, we have that $W_c(e, f^1, f^2)$ is edge injective and $\beta_2 \in m_{c_Q}(z_3)$. This shows that $\{y_3, z_3\} \frown P_c^3$ is c_Q -augmenting as y_3 cannot be the last vertex of P_c^3 (if it were, then $P_{c_Q}(z_3, \alpha_3/\beta_3) \neq P_c(z_3, \alpha_3/\beta_3)$ as $\alpha_2, \beta_2 \notin m_c(y_3)$). Hence, $W_c(e, f^1, f^2)$ is c -augmenting as desired. \square

We say that a 2-suitable edge $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ is of *type (c)* if it is not of type (a), or (b). Let γ be the smallest color in $m_c(y_3)$. The reason why we cannot extend $F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)$ is the same as when we build the standard Vizing chain; see [GP20, Proposition 2.8], or [GP20, Section 2.5: Type II edge]. Namely, there is a color δ and index

$$j < n = l(F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)) - 1$$

such that δ is the minimal color available in both $m_c(w_j)$ and $m_c(w_n)$. It is clear that $\gamma \neq \delta$ because f^2 is not of Type (a) and $\{\alpha_2, \beta_2\} \cap \{\gamma, \delta\} = \emptyset$ because f is not of Type (b).

Consider now the alternating γ/δ -paths $P_c(w_j, \gamma/\delta)$ and $P_c(w_n, \gamma/\delta)$. Our aim is to choose one of them, call it Q , and then define

$$W_c(e, f^1, f^2) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)_{\ell+1}) \frown Q,$$

where $\ell \in \{i, m\}$, depending on the choice of Q , is such that $W_c(x, f^1, f^2)$ is c -augmenting. As in the case of type (b), we need to rule out some edges. We say that a 2-suitable $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ of type (c) is *amazing* if, in the above notation, both of the following equalities hold

- $P_c(w_j, \gamma/\delta) = P_{c_R}(w_j, \gamma/\delta)$,
- $P_c(w_n, \gamma/\delta) = P_{c_R}(w_j, \gamma/\delta)$,

where $R = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2)$.

Let $f^2 \in P_c(z_2, \alpha_2/\beta_2)$ be of type (c) and *amazing*. We take $\ell \in \{j, n\}$ to be the index for which there is no $h \in P_c(u_j, \delta/\epsilon)$ such that $y_3 \in h$; see [GP20, Proposition 2.9]. If both indices j and n satisfy this, then we put $\ell = j$ for definiteness. We call this index ℓ the *third critical index*. We define $F_c^3 = F_c(x, e \rightsquigarrow f^1 \rightsquigarrow f^2)_{\ell+1}$, $z_3 = w_\ell$, $\alpha_3 = \gamma$, $\beta_3 = \delta$ and $P_c^3 = P_c(w_\ell, \alpha_3/\beta_3)$.

Proposition 3.9. *Let f^2 be of type (c) and amazing. Then*

$$W_c(e, f^1, f^2) = (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3)$$

is a 3-step Vizing chain.

Proof. It follows directly from the definitions that $W_c(e, f^1, f^2)$ satisfies (1)–(3) in Definition 3.2. It remains to show that $W_c(e, f^1, f^2)$ is c -augmenting. This can be done by the same argument as in [GP20, Proposition 2.17]. Namely, first observe that c_Q , where

$$Q = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3),$$

is c -proper shiftable by the same reasoning as in Proposition 3.6. In particular, Q is edge injective. As $y_3 \notin f$ for any $f \in P_c^3$ by the definition of the third critical index and $P_{c_R}(z_3, \alpha_3/\beta_3) = P_c(z_3, \alpha_3/\beta_3)$, we infer that $W_c(e, f^1, f^2)$ is edge injective. Similar argument shows that $P_{c_Q}(z_3, \alpha_3/\beta_3) = P_{c_R}(z_3, \alpha_3/\beta_3) = P_c(z_3, \alpha_3/\beta_3)$. Moreover, by the definition of 2-suitable edge, we must have $\alpha_3 \in m_{c_Q}(y_3)$. This shows that $\{y_3, z_3\} \frown P_c^3$ is c_Q -augmenting. Hence, $W_c(e, f^1, f^2)$ is c -augmenting as desired. □

Altogether, we just proved the following statement.

Theorem 3.10. *Let $e \in U_c$, $x \in e$, f^1 be superb and f^2 be amazing as defined above. Then $W_c(e, f^1, f^2)$ is a 3-step Vizing chain.*

3.3. How many 3-step Vizing chains are there

Let $e \in U_c$ and $x \in e$. We say that e is K -bad for c , where $K \in \mathbb{N}$, if every 3-step Vizing chain $W_c(e)$ at e satisfies $l(W_c(e)) \geq 2K + 2\Delta$. In the following claims, we use the notation from previous section.

Proposition 3.11. *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c and $x \in e$. Then*

- $l(P_c(v_i, \alpha/\beta)) \geq K$, where $P_c(v_i, \alpha/\beta)$ is the alternating path from the first iteration,
- $l(P_c(u_m, \gamma/\delta)) \geq \frac{K}{2}$ for every superb edge f^1 such that $i(f^1) \leq \frac{K}{2}$, where $P_c(u_m, \gamma/\delta)$ is the alternating path in the second iteration that corresponds to f^1 .

Proof. Both chains from Claim 3.3 and Claim 3.5 are 3-step Vizing chains. As e is K -bad for c and both chains contain at most two fans that each contain at most Δ edges, we conclude that

$l(P_c(v_i, \alpha/\beta)) \geq K$ and $l(P_c(u_m, \gamma/\delta)) \geq \frac{K}{2}$ under the assumption that f^1 is superb and satisfies $i(f^1) \leq \frac{K}{2}$. □

Claim 3.12 (Proposition 2.20 in [GP20]). *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c and $x \in e$. Then there are colors $\{\gamma, \delta\} \subseteq [\Delta + 1]$ and at least*

$$\frac{1}{3(\Delta + 1)^2} \left(\frac{K}{4} - \Delta^5 - 1 \right) - 2\Delta^3$$

many superb edges $f^1 \in P_c(v_i, \alpha/\beta)$ such that $i(f^1) \leq \frac{K}{2}$ and the alternating path in the second iteration that corresponds to f^1 is a γ/δ -path.

Definition 3.13. Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c , $x \in e$ and $\{\gamma, \delta\} \subseteq [\Delta + 1]$ be as in Claim 3.12. Define $\mathcal{V}_c(e)$ to be the set of pairs (f^1, f^2) such that $f^1 \in P_c(v_i, \alpha/\beta)$ is superb and satisfies $i(f^1) \leq \frac{K}{2}$ and $f^2 \in P_c(u_m, \gamma/\delta)$ is amazing and satisfies $i(f^2) \leq \frac{K}{2}$ (where the index $i(f^2)$ is taken with respect to $P_c(u_m, \gamma/\delta)$), where $P_c(u_m, \gamma/\delta)$ is the alternating path in the second iteration that corresponds to f^1 .

Our aim is to estimate the cardinality of $\mathcal{V}_c(e)$ as this gives a lower bound on the cardinality of the set of all 3-step Vizing chains at e . In fact, we bound the cardinality of the projection of $\mathcal{V}_c(e)$ to the second coordinate as this clearly gives a lower bound for the cardinality of $\mathcal{V}_c(e)$.

The following proposition gives a sufficient condition for an edge to be amazing.

Proposition 3.14. *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c , $x \in e$ and $\{\gamma, \delta\} \subseteq [\Delta + 1]$ be as in Claim 3.12. Suppose that $f^1 \in P_c(v_i, \alpha/\beta)$ is superb, $i(f^1) \leq \frac{K}{2}$, and pick f^2 on the the alternating path $P_c(u_m, \gamma/\delta)$ (that corresponds to f^1 in the second iteration). Assume that*

1. f^2 is 2-suitable,
2. there is no alternating path $P_c(w, \iota/\kappa)$ such that, simultaneously, $w \in V$ is of \mathcal{G} -distance 1 from $y_3 \in f^2$ and $P_c(w, \iota/\kappa)$ is of distance at most 3 from $(F_c^1) \frown (P_c^1) \frown (F_c^2)$, where f^1 is the first edge of F_c^2 .

Then f^2 is amazing. In particular, $(f^1, f^2) \in \mathcal{V}_c(e)$.

Proof. Suppose that the conditions are satisfied. If f^2 is of type (a), then it is amazing.

If f^2 is of type (b), then we need to verify that $P_{c_Q}(z_3, \alpha_3/\beta_3) = P_c(z_3, \alpha_3/\beta_3)$, where $Q = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3)$. As $\text{dist}_{\mathcal{G}}(y_3, z_3) = 1$, we have that $P_c(z_3, \alpha_3/\beta_3)$ avoids $(F_c^1) \frown (P_c^1) \frown (F_c^2)$. Hence, if $P_{c_Q}(z_3, \alpha_3/\beta_3) \neq P_c(z_3, \alpha_3/\beta_3)$ it must be the case that y_3 is covered by $P_c(z_3, \alpha_3/\beta_3)$. This can only happen if $P_c(z_3, \alpha_3/\beta_3)$ contains P_c^2 as $\alpha_3 = \alpha_2$ and $\beta_3 = \beta_2$. But that is not possible as P_c^2 is of distance 1 from $(F_c^1) \frown (P_c^1) \frown (F_c^2)$.

If f^2 is of type (c), then we need to verify that $P_c(w_j, \gamma/\delta) = P_{c_R}(w_j, \gamma/\delta)$ and $P_c(w_n, \gamma/\delta) = P_{c_R}(w_n, \gamma/\delta)$. This follows easily as $\{\gamma, \delta\} \cap \{\alpha_2, \beta_2\} = \emptyset$ and both $P_c(w_j, \gamma/\delta)$, $P_c(w_n, \gamma/\delta)$ avoid $(F_c^1) \frown (P_c^1) \frown (F_c^2)$. □

Proposition 3.15. *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c , $x \in e$ and $\{\gamma, \delta\} \subseteq [\Delta + 1]$ be as in Claim 3.12. Define $f^2 \in \mathcal{S}_c(e)$ if $c(f^2) = \gamma$, and there is $f^1 \in P_c(v_i, \alpha/\beta)$ such that f^1 is superb, $i(f^1) \leq \frac{K}{2}$, $f^2 \in P_c(u_m, \gamma/\delta)$ and $i(f^2) \leq \frac{K}{2}$ (the index is taken with respect to $P_c(u_m, \gamma/\delta)$ and $P_c(u_m, \gamma/\delta)$ corresponds to f^1 in the second iteration), and at least one of the items from Proposition 3.14 is not satisfied. Then we have*

$$|\mathcal{S}_c(e)| \leq 4(\Delta + 1)^4 \sum_{r=0}^4 (2\Delta)^r \left(\frac{K}{2} + \Delta \right).$$

Proof. Let f^1 and f^2 are as above and assume that item (1) from Proposition 3.14 is not satisfied. As $c(f^2) = \gamma$, we know that either f^2 is the last edge on $P_c(u_m, \gamma/\delta)$ or it is of distance at most 4 from $(F_c^1) \frown P_c(v_i, \alpha/\beta)_{\frac{K}{2}+1}$. There are at most $\frac{K}{2}$ many edges that satisfy the former condition, and we have

$$\left| \left\{ f \in E : \text{dist}_{\mathcal{E}}(f, (F_c^1) \cap P_c(v_i, \alpha/\beta)_{\frac{K}{2}+1}) \leq 4 \right\} \right| \leq \sum_{r=0}^4 (2\Delta)^r \left(\frac{K}{2} + \Delta \right) \tag{3.1}$$

which gives upper bound on the latter condition.

Suppose that (2) from Proposition 3.14 is not satisfied. That means that there is a path $P_c(w, \iota/\kappa)$, where w is of distance one from y_3 , that intersect

$$B = \left\{ f \in E : \text{dist}_{\mathcal{E}}(f, (F_c^1) \cap P_c(v_i, \alpha/\beta)_{\frac{K}{2}+1}) \leq 4 \right\}.$$

Every edge from B is an element of $2(\Delta + 1)^2$ many such paths. Together with Equation (3.1), we conclude that there are at most

$$2(\Delta + 1)^4 \sum_{r=0}^4 (2\Delta)^r \left(\frac{K}{2} + \Delta \right)$$

edges f^2 that do not satisfy (2) from Proposition 3.14.

Summing these three bounds gives the desired estimate. □

Proposition 3.16. *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c , $x \in e$ and $\{\gamma, \delta\} \subseteq [\Delta + 1]$ be as in Claim 3.12. Define $f^2 \in \mathcal{T}_c(e)$ if $c(f^2) = \gamma$, and there is $f^1 \in P_c(v_i, \alpha/\beta)$ such that f^1 is superb, $i(f^1) \leq \frac{K}{2}$, $f^2 \in P_c(u_m, \gamma/\delta)$ and $i(f^2) \leq \frac{K}{2}$ (the index is taken with respect to $P_c(u_m, \gamma/\delta)$ and $P_c(u_m, \gamma/\delta)$ corresponds to f^1 in the second iteration). Then we have*

$$|\mathcal{T}_c(e)| \geq \frac{K}{4\Delta^2} \left(\frac{1}{3(\Delta + 1)^2} \left(\frac{K}{4} - \Delta^5 - 1 \right) - 2\Delta^3 \right).$$

Proof. By Claim 3.12, there is $\{\gamma, \delta\} \subseteq [\Delta + 1]$ and at least

$$\frac{1}{3(\Delta + 1)^2} \left(\frac{K}{4} - \Delta^5 - 1 \right) - 2\Delta^3$$

many superb edges $f^1 \in P_c(v_i, \alpha/\beta)$ such that $i(f^1) \leq \frac{K}{2}$ and the alternating path in the second iteration that corresponds to f^1 is a γ/δ -path. For each such f^1 , there are at least $\frac{K}{4}$ edges of the corresponding $P_c(u_m, \gamma/\delta)$ that have color γ . This shows that there are at least

$$\frac{K}{4} \left(\frac{1}{3(\Delta + 1)^2} \left(\frac{K}{4} - \Delta^5 - 1 \right) - 2\Delta^3 \right)$$

pairs (f^1, f^2) that satisfy the conditions above. To estimate the size of $\mathcal{T}_c(e)$, we need to compute the number of pairs (f^1, f^2) to which a given edge f^2 contributes.

Every f^2 can reach f_1 by following the γ/δ path in one of its two directions, and then there are Δ^2 many choices for f^1 . Altogether,

$$|\mathcal{T}_c(e)| \geq \frac{K}{4\Delta^2} \left(\frac{1}{3(\Delta + 1)^2} \left(\frac{K}{4} - \Delta^5 - 1 \right) - 2\Delta^3 \right)$$

as needed. □

Finally, observe that the projection of $\mathcal{V}_c(e)$ to the second coordinate contains $\mathcal{T}_c(e) \setminus \mathcal{S}_c(e)$ by Proposition 3.14. Hence, the combination of Propositions 3.15 and 3.16, together with a trivial modification gives the main estimate.

Theorem 3.17. *Let $K \in \mathbb{N}$, $e \in U_c$ be K -bad for c and $x \in e$. Then we have*

$$|\mathcal{V}_c(e)| \geq \frac{K^2}{(8\Delta)^4} - (16\Delta)^5 K.$$

In particular, for every $e \in U_c$ that is K -bad and $x \in e$ there are at least $\left(\frac{K^2}{(8\Delta)^4} - (16\Delta)^5 K\right)$ many pairs (f^1, f^2) such that $W_c(e, f^1, f^2)$ is a 3-step Vizing chain.

4. Improving colorings

In this section, we describe one step of the algorithm that will, in Section 7, produce the desired $\Delta(\mathcal{G}) + 1$ edge coloring μ -almost everywhere. The step, and therefore the whole algorithm, can be run on any Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ of degree bounded by $\Delta(\mathcal{G}) < \infty$ endowed with an \mathcal{E} -quasi-invariant Borel probability measure $\nu \in \mathcal{P}(E)$, where $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ is the corresponding line graph. However, in order to show that the algorithm terminates ν -almost everywhere, we need to assume additionally that ν is \mathcal{G} -bounded; see Section 5.

Fix \mathcal{G} and ν as above, and a Radon–Nikodym cocycle ρ_ν of ν with respect to \mathcal{E} .

Definition 4.1. We say that a partial coloring $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ does not admit an improvement of weight $L \in \mathbb{N}$ if

$$\nu \left(\left\{ e \in U_c : \exists \text{ 3-step Vizing chain } W_c(e) \text{ s. t. } \sum_{f \in W_c(e)} \rho_\nu(e, f) \leq L \right\} \right) = 0.$$

If this condition is not satisfied, then we say that c admits an improvement of weight L .

Theorem 4.2. *Let $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ be a partial coloring and $L \in \mathbb{N}$. Then there is a partial coloring $c'; E \rightarrow [\Delta(\mathcal{G}) + 1]$ that does not admit improvement of weight L with the property that $\nu(\text{dom}(c) \setminus \text{dom}(c')) = 0$ and*

$$\nu(\{e \in E : c(e) \neq c'(e)\}) \leq L\nu(U_c),$$

where $c(e) \neq c'(e)$ also includes the situation when $e \in \text{dom}(c') \setminus \text{dom}(c)$.

Proof. The strategy of the proof follows closely [GP20, Proof of Proposition 5.4]. For a partial coloring $d; E \rightarrow [\Delta(\mathcal{G}) + 1]$, define A_d to be the set of those $e \in U_d$ for which there exists a 3-step Vizing chain $W_d(e)$ such that

$$\sum_{f \in W_d(e)} \rho_\nu(e, f) \leq L.$$

Clearly, $\nu(A_d) = 0$ if and only if d does not admit improvement of weight L . Set $c_0 = c$. We use induction to build a transfinite sequence of partial colorings $(c_\alpha)_{\alpha < \aleph_1}$ that satisfy the following:

1. for every $\alpha < \beta < \aleph_1$, we have $\nu(\text{dom}(c_\alpha) \setminus \text{dom}(c_\beta)) = 0$,
2. for every $\alpha < \aleph_1$, if $\nu(A_{c_\alpha}) \neq 0$, then $\nu(U_{c_{\alpha+1}}) < \nu(U_{c_\alpha})$,
3. for every $\alpha < \beta < \aleph_1$, if $c_\alpha \neq c_\beta$, then $\nu(U_{c_\beta}) < \nu(U_{c_\alpha})$,
4. if $c_\alpha = c_{\alpha+1}$ for some $\alpha < \aleph_1$, then $c_\alpha = c_\beta$ for every $\alpha \leq \beta < \aleph_1$,
5. for every $\alpha < \beta < \aleph_1$,

$$\nu(\{e \in E : c_\alpha(e) \neq c_\beta(e)\}) \leq L \sum_{\alpha \leq \alpha' < \beta} \nu(S_{\alpha'}) = L\nu \left(\bigcup_{\alpha \leq \alpha' < \beta} S_{\alpha'} \right) \leq L\nu(U_c),$$

where $S_{\alpha'} = U_{c_{\alpha'}} \setminus U_{c_{\alpha'+1}}$.

Once we build such a sequence, then we are done. Indeed, conditions (3) and (4) guarantee the existence of $\alpha < \aleph_1$ such that $c_\alpha = c_\beta$ for every $\alpha \leq \beta < \aleph_1$ as there are no strictly decreasing sequences of real numbers of length \aleph_1 . Define $c' = c_{\alpha_0}$, where α_0 is the minimal ordinal with this property. Then (1) (with the choice $0 < \alpha_0$) implies $\nu(\text{dom}(c) \setminus \text{dom}(c')) = 0$, (2) implies that $c' = c_{\alpha_0}$ does not admit improvement of weight L and (5) (with the choice $0 < \alpha_0$) implies that $\nu(\{e \in E : c(e) \neq c'(e)\}) \leq L\nu(U_c)$.

Successor stage $\alpha \mapsto \alpha + 1$. Suppose that we have constructed a sequence of partial colorings $(c_\beta)_{\beta \leq \alpha}$ such that the property (1)–(5) hold for every (pair of) ordinal(s) less or equal than α . If $\nu(A_{c_\alpha}) = 0$, then setting $c_{\alpha+1} = c_\alpha$ clearly works. Suppose that $\nu(A_{c_\alpha}) > 0$, and pick any Borel assignment $e \in A_{c_\alpha} \mapsto W_{c_\alpha}(e)$ with the property $W_{c_\alpha}(e)$ is a 3-step Vizing chain and $\sum_{f \in W_{c_\alpha}(e)} \rho_\nu(e, f) \leq L$ for every $e \in A_{c_\alpha}$.

Case I. There is $k \in \mathbb{N}$ such that

$$\nu(\{e \in A_{c_\alpha} : |W_{c_\alpha}(e)| = k\}) > 0.$$

By [KST99, Proposition 4.6] (applied on the $2k + 2$ power graph of \mathcal{E} that is of bounded degree), there is a set $S_\alpha \subseteq A_{c_\alpha}$ with the property that

- (a) $\nu(S_\alpha) > 0$,
- (b) $e \neq e' \in S_\alpha$ are at least $2k + 2$ far apart in the graph distance of \mathcal{E} ,
- (c) $|W_{c_\alpha}(e)| = k$ for every $e \in S_\alpha$.

Set $T_\alpha = \bigcup_{e \in S_\alpha} W_{c_\alpha}(e)$, and observe that

$$\nu(T_\alpha) = \int_{e \in S_\alpha} \sum_{f \in W_{c_\alpha}(e)} \rho_\nu(e, f) \, d\nu \leq L\nu(S_\alpha) \tag{*}$$

by item (2) in Proposition 2.2. As $\{W_{c_\alpha}(e)\}_{e \in S_\alpha}$ are pairwise of positive distance from each other by (b) and (c) and each $W_{c_\alpha}(e)$ is augmenting, there is a partial coloring $c_{\alpha+1}$ with the property that

- (i) $T_\alpha \subseteq \text{dom}(c_{\alpha+1})$,
- (ii) $c_\alpha \upharpoonright (\text{dom}(c_\alpha) \setminus T_\alpha) = c_{\alpha+1} \upharpoonright (\text{dom}(c_\alpha) \setminus T_\alpha)$.

We claim that $c_{\alpha+1}$ works as required, namely, let $\alpha' < \alpha + 1$, then

1. $\nu(\text{dom}(c_{\alpha'}) \setminus \text{dom}(c_{\alpha+1})) = 0$ as $\nu(\text{dom}(c_{\alpha'}) \setminus \text{dom}(c_\alpha)) = 0$ and $\text{dom}(c_{\alpha+1}) = \text{dom}(c_\alpha) \cup S_\alpha$ by (i) and (ii),
2. follows from $\text{dom}(c_{\alpha+1}) = \text{dom}(c_\alpha) \cup S_\alpha$ together with (a),
3. follows from (2) combined with inductive assumption (3),
4. if $c_{\alpha'} = c_{\alpha'+1}$ for some $\alpha' < \alpha$, then $\nu(A_{c_\alpha}) = 0$ by the inductive assumption (4) and (2),
5. observe that $\nu(S_\alpha \cap S_\beta) = 0$ for every $\beta < \alpha$ by the inductive assumption (1); consequently, when combined with the inductive assumption (5) and (*), we have

$$\begin{aligned} \nu(\{e \in E : c_{\alpha'}(e) \neq c_{\alpha+1}(e)\}) &\leq \nu(\{e \in E : c_{\alpha'}(e) \neq c_\alpha(e)\}) \\ &\quad + \nu(\{e \in E : c_\alpha(e) \neq c_{\alpha+1}(e)\}) \\ &\leq L \sum_{\alpha' \leq \beta < \alpha} \nu(S_\beta) + \nu(T_\alpha) \\ &\leq L \sum_{\alpha' \leq \beta < \alpha+1} \nu(S_\beta) \\ &= L\nu\left(\bigcup_{\alpha' \leq \beta < \alpha+1} S_{\alpha'}\right) \leq L\nu(U_c). \end{aligned}$$

Case II. There is no $k \in \mathbb{N}$ such that

$$\nu(\{e \in A_{c_\alpha} : |W_{c_\alpha}(e)| = k\}) > 0.$$

In another words, ν -almost every 3-step Vizing chain is infinite. Observe that this can happen if and only if there is an assignment (defined for ν -almost every $e \in A_{c_\alpha}$) $e \in A_{c_\alpha} \mapsto (x_e, \alpha_e, \beta_e)$, where $x_e \in V$ and $\alpha_e, \beta_e \in [\Delta + 1]$ such that $W_{c_\alpha}(e) = M(e) \frown P_{c_\alpha}(x_e, \alpha_e/\beta_e)$. Using finite additivity of ν and [KST99, Proposition 4.6], we find $10 < k \in \mathbb{N}$, $\alpha, \beta \in [\Delta + 1]$ and $R_\alpha \subseteq A_{c_\alpha}$ such that

- (a) $\nu(R_\alpha) > 0$,
- (b) $|M(e)| = k$, $\alpha_e = \alpha$ and $\beta_e = \beta$ for every $e \in R_\alpha$,
- (c) $e \neq e' \in R_\alpha$ are at least $5k$ far apart in the graph distance of \mathcal{E} .

Note that this implies that if $e \neq e' \in R_\alpha$, then

- $x_e \neq x_{e'}$ and consequently $P_{c_\alpha}(x_e, \alpha/\beta)$ and $P_{c_\alpha}(x_{e'}, \alpha/\beta)$ are vertex disjoint,
- $M(e)$ and $M(e')$ are at least $2k$ apart in the graph distance of \mathcal{E} .

However, it can happen that $M(e) \cap P_{c_\alpha}(x_{e'}, \alpha/\beta) \neq \emptyset$.

We address this issue as follows. Define an auxiliary directed graph \mathcal{H} on R_α as follows. For $e \neq e' \in R_\alpha$, let (e, e') be an oriented edge if $P_{c_\alpha}(x_{e'}, \alpha/\beta)$ intersect $B_{\mathcal{E}}(e, 2k)$, the ball of radius $2k$ around e . Note that as $|B_{\mathcal{E}}(e, 2k)| \leq (2\Delta)^{2k}$ for every $e \in R_\alpha$, the graph \mathcal{H} has uniformly bounded outdegree. By [KST99, Proposition 4.5], we can write $R_\alpha = \bigcup_{n \in \mathbb{N}} R_{\alpha, n}$, where each $R_{\alpha, n}$ is \mathcal{H} -independent. By σ -additivity of ν , we find $n \in \mathbb{N}$ such that $\nu(R_{\alpha, n}) > 0$ and set $S_\alpha = R_{\alpha, n}$.

Let $e \neq e' \in S_\alpha$. By the definition, we have that $W_{c_\alpha}(e)$ and $W_{c_\alpha}(e')$ are vertex disjoint. Moreover, $(c_\alpha)_e$ extends c_α as $e \in \text{dom}((c_\alpha)_e)$ and $\text{dom}(c_\alpha) \subseteq \text{dom}((c_\alpha)_e)$. Set $T_\alpha = \bigcup_{e \in S_\alpha} W_{c_\alpha}(e)$, and define

$$c_{\alpha+1}(f) = \begin{cases} c_\alpha(f) & \text{if } f \notin T_\alpha \\ (c_\alpha)_e(f) & \text{if } f \in W_{c_\alpha}(e), \text{ where } e \in S_\alpha \text{ is the unique such edge.} \end{cases}$$

It follows immediately that

- (i) $T_\alpha \subseteq \text{dom}(c_{\alpha+1})$,
- (ii) $c_\alpha \upharpoonright (\text{dom}(c_\alpha) \setminus T_\alpha) = c_{\alpha+1} \upharpoonright (\text{dom}(c_\alpha) \setminus T_\alpha)$.

Observe that $c_{\alpha+1}$ is a partial coloring. Indeed, if $x \in V$ is not covered by any edge of distance $k + 2$ to some $e \in S_\alpha$, then $c_{\alpha+1} \upharpoonright N_G(x) = c_\alpha \upharpoonright N_G(x)$ (as the only modification is a shift of some of the infinite α/β -paths). On the other hand, if $x \in V$ is of distance at most $k + 2$ to some $e \in S_\alpha$, then $c_{\alpha+1} \upharpoonright N_G(x) = (c_\alpha)_e \upharpoonright N_G(x)$, hence $c_{\alpha+1}$ is a partial coloring. The same reasoning as above shows that

$$\nu(T_\alpha) = \int_{e \in S_\alpha} \sum_{f \in W_{c_\alpha}(e)} \rho_\nu(e, f) \, d\nu \leq L\nu(S_\alpha) \tag{**}$$

by item (2) in Proposition 2.2. Verifying the conditions (1)–(5) can be done mutatis mutandis as in the Case (I).

Limit stage $\beta \nearrow \alpha$. Suppose that $\alpha < \aleph_1$ is a limit ordinal and we have constructed $(c_\beta)_{\beta < \alpha}$ such that the property (1)–(5) hold for every (pair of) ordinal(s) strictly less than α . We claim that $c_\alpha(e) := \lim_{\beta \rightarrow \alpha} c_\beta(e)$ is defined ν -almost everywhere, that is, the sequence of colors $(c_\beta(e))_{\beta < \alpha}$ eventually stabilizes (in fact the colors in the sequence are changed only finitely many times) for ν -almost every $e \in E$. This follows from the Borel–Cantelli lemma as, by the previous paragraph, the sequence $(T_\beta)_{\beta < \alpha}$, where T_β is the set of edges that changed their color in the β th step, satisfies

$$\sum_{\beta < \alpha} \nu(T_\beta) \leq \sum_{\beta < \alpha} L\nu(S_\beta) \leq L\nu(U_c) < \infty.$$

Hence, the set of edges $e \in E$ for which $\{\beta < \alpha : e \in T_\beta\}$ is cofinal in α has to be ν -null.

It remains to show that (1)–(5) continuous to hold with α . Let $\alpha' < \alpha$, then

1. by the fact that $\alpha < \aleph_1$, the inductive assumption and construction of c_α , we have

$$\nu(\{e \in \text{dom}(c_{\alpha'}) : \exists \alpha' < \beta' < \alpha \ e \notin \text{dom}(c_{\beta'}) \text{ or } \lim_{\beta \rightarrow \alpha} c_\beta(e) \text{ not defined}\}) = 0,$$

consequently, $\nu(\text{dom}(c_{\alpha'}) \setminus \text{dom}(c_\alpha)) = 0$,

2. is not relevant in limit stages,
3. if $c_{\alpha'} \neq c_\alpha$, then $c_{\alpha'} \neq c_{\alpha'+1}$ (see (4)); this implies that

$$\nu(U_{c_{\alpha'}}) > \nu(U_{c_{\alpha'+1}}) \geq \nu(U_{c_\alpha})$$

as $\nu(\text{dom}(c_{\alpha'+1}) \setminus \text{dom}(c_\alpha)) = 0$ by (1) (where the strict inequality follows from the inductive assumption (3)),

4. if $c_{\beta'} = c_{\beta''}$ for some $\beta' < \alpha$ and every $\beta' \leq \beta'' < \alpha$, then $c_\alpha(e) = \lim_{\beta \rightarrow \alpha} c_\beta(e) = c_{\beta'}(e)$, hence $c_\alpha = c_{\beta'}$,
5. if $c_{\alpha'}(e) \neq c_\alpha(e)$, then there must be $\alpha' \leq \beta < \alpha$ such that $e \in T_\beta$ as otherwise $c_\alpha(e) = \lim_{\beta \rightarrow \alpha} c_\beta(e) = c_{\alpha'}(e)$, consequently, we have

$$\begin{aligned} \nu(\{e \in E : c_{\alpha'}(e) \neq c_\alpha(e)\}) &\leq \sum_{\alpha' \leq \beta < \alpha} \nu(T_\beta) \leq \sum_{\alpha' \leq \beta < \alpha} L\nu(S_\beta) \\ &= L \sum_{\alpha' \leq \beta < \alpha} \nu(S_\beta) = L\nu\left(\bigcup_{\alpha' \leq \beta < \alpha} S_\beta\right) \\ &\leq L\nu\left(\bigcup_{\beta < \alpha} S_\beta\right) \leq L\nu(U_c) \end{aligned}$$

by the definition of T_β and S_β combined with the fact that $\nu(S_\beta \cap S_{\beta'}) = 0$ for every $\beta < \beta' < \alpha$ which follows by the inductive assumption (1).

This finishes the proof. □

5. Cocycle bounded on edges

Recall that two Borel probability measures $\mu, \nu \in \mathcal{P}(V)$ are equivalent if $\mu(A) = 0$ if and only if $\nu(A) = 0$ for every $A \in \mathcal{B}$. We restate Theorem 1.2 in a compact form for the convenience of the reader.

Theorem 5.1. *Let $\Delta \in \mathbb{N}$, $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$ and $\mu \in \mathcal{P}(V)$ be a \mathcal{G} -quasi-invariant. Then there is an equivalent \mathcal{G} -bounded Borel probability measure $\nu \in \mathcal{P}(V)$.*

Proof. Let $\mathcal{G}^{[k]}$ denote the Borel graph on V , where (x, y) form an edge if and only if $\text{dist}_{\mathcal{G}}(x, y) = k$. Then $\mathcal{G}^{[1]} = \mathcal{G}$ and $\mathcal{G}^{[k]}$ has degree bounded by Δ^k . Use repeatedly [KST99, Proposition 4.6] to find a Borel proper edge coloring $\{A_i^k\}_{i=1}^{2\Delta^k}$ of $\mathcal{G}^{[k]}$ for each $k \in \mathbb{N} \setminus \{0\}$. Using any Borel linear order on V , vertices covered by A_i^k can be split into two disjoint Borel sets $A_{i,0}^k \subseteq V$ and $A_{i,1}^k \subseteq V$ together with Borel isomorphisms $f_{i,0}^k : A_{i,0}^k \rightarrow A_{i,1}^k$ and $f_{i,1}^k : A_{i,1}^k \rightarrow A_{i,0}^k$ such that $\{x, f_{i,0}^k(x)\}, \{y, f_{i,1}^k(y)\} \in A_i^k$ for every $x \in A_{i,0}^k$ and $y \in A_{i,1}^k$. We also set $A_{0,0}^0 = V$ and $f_{0,0}^0 = \text{id}_V$. Observe that for every $(x, y) \in F_{\mathcal{G}}$ there is exactly one triplet (k, i, j) , where $k = \text{dist}_{\mathcal{G}}(x, y)$, $i \in 2\Delta^{\text{dist}_{\mathcal{G}}(x, y)}$ and $j \in \{0, 1\}$ such that $f_{i,j}^k(x) = y$.

Denote as $\mu_{i,j}^k$ the push-forward of $\mu \upharpoonright A_{i,1-j}^k$ via $(f_{i,j}^k)^{-1}$, where $j \in \{0, 1\}$. We have

$$\mu_{i,j}^k(B) = \mu(f_{i,j}^k(B \cap A_{i,j}^k)) = \int_B \mathbf{1}_{A_{i,j}^k}(x) \rho_\mu(x, f_{i,j}^k(x)) \, d\mu \tag{5.1}$$

for every $B \in \mathcal{B}$, where ρ_μ is the Radon–Nikodym cocycle with respect to μ . In particular, $\mu_{i,j}^k(V) \leq 1$. Define

$$\tilde{\nu} = \mu + \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,0}^k + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,1}^k \right). \tag{5.2}$$

Claim 5.2. *$\tilde{\nu}$ is a finite Borel measure on V that is equivalent with μ . The Radon–Nikodym derivative $\Omega = \frac{d\tilde{\nu}}{d\mu}$ can be explicitly written as*

$$\Omega(x) = 1 + \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,0}^k}(x) \rho_\mu(x, f_{i,0}^k(x)) + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,1}^k}(x) \rho_\mu(x, f_{i,1}^k(x)) \right)$$

for μ (and $\tilde{\nu}$) almost every $x \in V$. In particular, $\frac{1}{\Omega} = \frac{d\mu}{d\tilde{\nu}}$.

Proof. Let $n \in \mathbb{N}$, and define

$$\tilde{\nu}_n = \mu + \sum_{k=1}^n \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,0}^k + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,1}^k \right).$$

As $\mu_{i,j}^k(V) \leq 1$, we see that $\tilde{\nu}_n$ is a finite Borel measure on V that is equivalent with μ . Indeed, if $\mu(A) = 0$, then by the definition of \mathcal{G} -quasi-invariance we have that $\tilde{\nu}_n(A) = 0$, and if $\tilde{\nu}_n(A) = 0$, then $\mu(A) = 0$ as $\mu(A) \leq \tilde{\nu}_n(A)$ for every $A \in \mathcal{B}$. Moreover, it is easy to see that the Radon–Nikodym derivative $\Omega_n = \frac{d\tilde{\nu}_n}{d\mu}$ satisfies

$$\Omega_n(x) = 1 + \sum_{k=1}^n \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,0}^k}(x) \rho_\mu(x, f_{i,0}^k(x)) + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,1}^k}(x) \rho_\mu(x, f_{i,1}^k(x)) \right)$$

by Equation (5.1).

Observe that the limit $\Omega(x) = \lim_{n \rightarrow \infty} \Omega_n(x)$ is defined for every $x \in V$ as $\{\Omega_n(x)\}_{n \in \mathbb{N}}$ is increasing, and we have

$$\lim_{n \rightarrow \infty} \int_A \Omega_n(x) \, d\mu = \int_A \Omega(x) \, d\mu$$

for every $A \in \mathcal{B}$ by the monotone convergence theorem. Note that by the definition of Ω_n we have

$$\lim_{n \rightarrow \infty} \tilde{\nu}_n(A) = \int_A \Omega(x) \, d\mu \tag{5.3}$$

for every $A \in \mathcal{B}$.

The sequence $\{\tilde{\nu}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the total variation distance $\|\cdot\|_{TV}$. Indeed, we have

$$\|\tilde{\nu}_m - \tilde{\nu}_n\|_{TV} = (\tilde{\nu}_m - \tilde{\nu}_n)(V) = \left(\sum_{k=n}^m \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,0}^k + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mu_{i,1}^k \right) \right)(V) \leq \frac{1}{2^{n-1}}$$

as $\tilde{\nu}_m - \tilde{\nu}_n$ is a finite Borel (positive) measure for every $m > n > 1$. Consequently, there is a finite Borel measure $\tilde{\nu}$ on (V, \mathcal{B}) such that $\tilde{\nu}_n \xrightarrow{TV} \tilde{\nu}$. In particular, we have $\tilde{\nu}_n(A) \rightarrow \tilde{\nu}(A) < \infty$ for every $A \in \mathcal{B}$. Combined with Equation (5.3), we see that $\Omega = \frac{d\tilde{\nu}}{d\mu}$ as desired.

It remains to show that μ and $\tilde{\nu}$ are equivalent. If $\mu(A) = 0$, then $\tilde{\nu}(A) = \lim_{n \rightarrow \infty} \tilde{\nu}_n(A) = 0$ as in that case $\tilde{\nu}_n(A) = 0$ for every $n \in \mathbb{N}$. On the other hand, if $\tilde{\nu}(A) = 0$, then $\int_A \Omega(x) d\mu = 0$ as $\Omega = \frac{d\tilde{\nu}}{d\mu}$. This can only happen if $\mu(A) = 0$ as $\Omega \geq 1$. \square

Let ν be a normalization of $\tilde{\nu}$, that is, $\nu = K\tilde{\nu}$ for some $0 < K < \infty$. It is easy to see that ν and $\tilde{\nu}$, hence also ν and μ , are equivalent. Consequently, ν is \mathcal{G} -quasi-invariant. Write ρ_ν for the Radon–Nikodym cocycle of ν with respect to \mathcal{G} .

Claim 5.3. *There is a μ -conull \mathcal{G} -invariant set $A \subseteq V$ such that*

$$\rho_\nu(x, y) = \rho_\mu(x, y) \frac{\Omega(y)}{\Omega(x)} \tag{5.4}$$

for every $x, y \in A$ such that $(x, y) \in F_{\mathcal{G}}$.

Proof. Let $g : C \rightarrow V$ be a Borel injection such that $(x, g(x)) \in F_{\mathcal{G}}$ for every $x \in C$. Set $g(C) = D$, then $g : C \rightarrow D$ is a Borel bijection. As μ and ν are equivalent, $\frac{\Omega}{K} = \frac{\nu}{\mu}$ and $\rho_\mu(x, g(x)) \frac{\Omega(g(x))}{\Omega(x)}$ are nonnegative. We have

$$\begin{aligned} \int_C \rho_\mu(x, g(x)) \frac{\Omega(g(x))}{\Omega(x)} d\nu(x) &= \int_C \rho_\mu(x, g(x)) \frac{\Omega(g(x))}{\Omega(x)} \frac{\Omega(x)}{K} d\mu(x) \\ &= \int_C \rho_\mu(x, g(x)) \frac{1}{K} \Omega(g(x)) d\mu(x). \end{aligned}$$

By (2) Proposition 2.2 applied to $f(x, y)$ that is defined as $\frac{1}{K}\Omega(y)$ whenever $g(x) = y$ and 0 otherwise, we have

$$\begin{aligned} \int_C \rho_\mu(x, g(x)) \frac{1}{K} \Omega(g(x)) d\mu(x) &= \int_D \frac{1}{K} \Omega(y) d\mu(y) \\ &= \nu(D) = \nu(g(C)) \\ &= \int_C \rho_\nu(x, g(x)) d\nu(x) \end{aligned}$$

and the claim follows from the uniqueness of the Radon–Nikodym cocycle. \square

It remains to show that $\rho_\mu(x, y)\Omega(y) \leq 4\Delta\Omega(x)$ for every $(x, y) \in E$, as this clearly implies $\rho_\nu(x, y) = \rho_\mu(x, y) \frac{\Omega(y)}{\Omega(x)} \leq 4\Delta$. Let $(x, y) \in E$. Suppose that $y \in A_{i,j}^k$, that is, $z = f_{i,j}^k(y)$ is well defined and satisfies $\text{dist}_{\mathcal{G}}(y, z) = k$. We already observed that the triplet (k, i, j) is unique. As $(x, y) \in F_{\mathcal{G}}$, there is exactly one triplet (k', i', j') such that $x \in A_{i',j'}^{k'}$ and $f_{i',j'}^{k'}(x) = z$. Moreover, as $(x, y) \in E$ we have that $k' \leq k + 1$. Indeed, we have $\text{dist}_{\mathcal{G}}(x, z) = k' \leq \text{dist}_{\mathcal{G}}(x, y) + \text{dist}_{\mathcal{G}}(y, z) = k + 1$. The same reasoning with the roles of y and x interchanged implies that the assignment $(k, i, j) \mapsto (k', i', j')$ is a bijection. Observe that

$$\frac{1}{2^{k+1}(2\Delta^{k+1})} \mathbf{1}_{A_{i,j}^k}(y) \rho_\mu(x, f_{i,j}^k(y)) \leq \frac{1}{2^{k'}(2\Delta^{k'})} \mathbf{1}_{A_{i',j'}^{k'}}(x) \rho_\mu(x, f_{i',j'}^{k'}(x)) \tag{5.5}$$

holds whenever $y \in A_{i,j}^k$. We have

$$\begin{aligned}
 \rho_\mu(x, y)\Omega(y) &= \rho_\mu(x, y) \left(1 + \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,0}^k}(y) \rho_\mu(y, f_{i,0}^k(y)) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,1}^k}(y) \rho_\mu(y, f_{i,1}^k(y)) \right) \right) \\
 &= \rho_\mu(x, y) + \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{2^k} \left(\frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,0}^k}(y) \rho_\mu(x, f_{i,0}^k(y)) \right. \\
 &\quad \left. + \frac{1}{2\Delta^k} \sum_{i=1}^{2\Delta^k} \mathbf{1}_{A_{i,1}^k}(y) \rho_\mu(x, f_{i,1}^k(y)) \right) \\
 &\leq 4\Delta \left(\frac{1}{4\Delta} \rho_\mu(x, f_{0,0}^0(y)) + \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{2^{k+1}} \left(\frac{1}{2\Delta^{k+1}} \sum_{i=1}^{2\Delta^{k+1}} \mathbf{1}_{A_{i,0}^k}(y) \rho_\mu(x, f_{i,0}^k(y)) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2\Delta^{k+1}} \sum_{i=1}^{2\Delta^{k+1}} \mathbf{1}_{A_{i,1}^k}(y) \rho_\mu(x, f_{i,1}^k(y)) \right) \right) \\
 &\leq 4\Delta \left(1 + \sum_{k' \in \mathbb{N} \setminus \{0\}} \frac{1}{2^{k'}} \left(\frac{1}{2\Delta^{k'}} \sum_{i=1}^{2\Delta^{k'}} \mathbf{1}_{A_{i',0}^{k'}}(x) \rho_\mu(x, f_{i',0}^{k'}(x)) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2\Delta^{k'}} \sum_{i=1}^{2\Delta^{k'}} \mathbf{1}_{A_{i',1}^{k'}}(x) \rho_\mu(x, f_{i',1}^{k'}(x)) \right) \right) \\
 &= 4\Delta\Omega(x),
 \end{aligned}$$

where we used (1) Proposition 2.2 to get the second equality and Equation (5.5) together with the fact that $(k, i, j) \mapsto (k', i', j')$ is a bijection to get the second inequality. \square

6. Double counting argument

In this section, we show that if a partial edge coloring c does not admit improvement of weight L , then the measure of uncolored edges has to be $O\left(\frac{1}{\log^2(L)L}\right)$ under the assumption that ν is \mathcal{G} -bounded.

Theorem 6.1. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ be the corresponding line graph, $\nu \in \mathcal{P}(E)$ be \mathcal{E} -bounded, that is, ν satisfy $\rho_\nu(e, f) \leq 8\Delta$ for every $e, f \in E$ such that $(e, f) \in I_{\mathcal{G}}$, and $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ be a partial coloring that does not admit improvement of weight $L \in \mathbb{N}$, where $\log_{8\Delta}(L) \geq (8\Delta)^{20}$. Then*

$$\nu(U_c) \leq \frac{64(4\Delta)^7(\Delta!)^{14}}{\log_{8\Delta}^2(L)L}.$$

Proposition 6.2. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ be the corresponding line graph and $\nu \in \mathcal{P}(E)$ be \mathcal{E} -bounded. Suppose that $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ is a partial coloring that does not admit improvement of weight $L \in \mathbb{N}$, where $\frac{\log_{8\Delta}(L)}{2} - 1 \geq 2\Delta$. Then ν -almost every $e \in U_c$ is $\frac{\log_{8\Delta}(L)}{4}$ -bad for c .*

Proof. Let $W_c(e)$ be a 3-step Vizing chain at e . We have

$$L \leq \sum_{f \in W_c(e)} \rho_v(e, f) \leq \sum_{k \leq |W_c(e)|} (8\Delta)^k \leq (8\Delta)^{|W_c(e)|+1}.$$

Consequently, $|W_c(e)| \geq \log_{8\Delta}(L) - 1 \geq \frac{\log_{8\Delta}(L)}{2} + 2\Delta$. □

We get an immediate corollary of Theorem 3.17.

Proposition 6.3. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ be the corresponding line graph and $\nu \in \mathcal{P}(E)$ be \mathcal{E} -bounded. Suppose that $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ is a partial coloring that does not admit improvement of weight $L \in \mathbb{N}$, where $\log_{8\Delta}(L) \geq (8\Delta)^{20}$. Then we have*

$$|\mathcal{V}_c(e)| \geq \frac{1}{(4\Delta)^7} \log_{8\Delta}^2(L)$$

for ν -almost every $e \in U_c$.

Proposition 6.4. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$, $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ be the corresponding line graph and $\nu \in \mathcal{P}(E)$ be \mathcal{E} -bounded. Suppose that $c; E \rightarrow [\Delta(\mathcal{G}) + 1]$ is a partial coloring that does not admit improvement of weight $L \in \mathbb{N}$, where $\log_{8\Delta}(L) \geq (8\Delta)^{20}$. Then for ν -almost every $e \in U_c$ and every $(f^1, f^2) \in \mathcal{V}_c(e)$, we have*

$$\sum_{f \in P_c^3} \rho_v(e, f) \geq \frac{L}{2},$$

where $W_c(e, f^1, f^2) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3)$.

Proof. Set $P = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3)$. It follows from the definition of $\mathcal{V}_c(e)$ that $s = l(P) \leq 3\Delta + \frac{\log_{8\Delta}(L)}{2}$. We have

$$\begin{aligned} L &\leq \sum_{f \in W_c(e, f^1, f^2)} \rho_v(e, f) = \sum_{f \in P} \rho_v(e, f) + \sum_{f \in P_c^3} \rho_v(e, f) \\ &\leq (8\Delta)^{3\Delta + \frac{\log_{8\Delta}(L)}{2} + 2} + \sum_{f \in P_c^3} \rho_v(e, f). \end{aligned}$$

Consequently, $\frac{L}{2} \leq L - (8\Delta)^{3\Delta + 2} L^{\frac{1}{2}} \leq \sum_{f \in P_c^3} \rho_v(e, f)$ as needed. □

Define an auxiliary Borel oriented bipartite multigraph \mathcal{H}_c with vertex set $U_c \sqcup \text{dom}(c)$ such that (e, f) is an edge if $f \in P_c^3$ for some

$$W_c(e, f^1, f^2) = (F_c^1) \frown (P_c^1) \frown (F_c^2) \frown (P_c^2) \frown (F_c^3) \frown (P_c^3),$$

where $(f^1, f^2) \in \mathcal{V}_c(e)$. Note that \mathcal{H}_c is in general a multigraph as there might be different 3-step Vizing chains at the same edge e for which $(e, f) \in \mathcal{H}_c$ for the same $f \in \text{dom}(c)$. The following is the crucial observation for the double counting argument; note that the right-hand side of the inequality does not depend on L .

Proposition 6.5. $\text{deg}_{\mathcal{H}_c}(f) \leq 32(\Delta!)^{14}$ for every $f \in \text{dom}(c)$.

Proof. There are at most 2Δ choices for α_3, β_3 and z_3 such that $f \in P_c^3 = P_c(z_3, \alpha_3/\beta_3)$. There are at most Δ choices for y_3 , and at most $\Delta! \Delta$ choices for a fan F_c^3 at x_3 . For P_c^2 , we deduce that there are

at most $4(\Delta + 1)^2$ choices for z_2 and α_2, β_2 as the last edge of P_c^2 has to intersect the first edge of F_c^3 . Similar estimates hold for F_c^2, P_c^1 and F_c^1 . Altogether, we get that

$$\text{deg}_{\mathcal{H}_c}(f) \leq 2\Delta(\Delta!\Delta^2)^3(4(\Delta + 1)^2)^2 \leq 32(\Delta!)^{14},$$

as desired. □

Proof of Theorem 6.1. Define a function $\mathbf{F}(e, f)$ that counts the number of oriented edges from e to f in \mathcal{H}_c . By (2) Proposition 2.2, we have

$$\int_E \sum_{f \in [e]_\varepsilon} \mathbf{F}(e, f) \rho_\nu(e, f) \, d\nu(e) = \int_E \sum_{e \in [f]_\varepsilon} \mathbf{F}(e, f) \, d\nu(f). \tag{DC}$$

Using Proposition 6.5, we get an upper bound for the right-hand side of Equation (DC) as

$$\int_E \sum_{e \in [f]_\varepsilon} \mathbf{F}(e, f) \, d\nu(f) \leq \int_{\text{dom}(c)} \text{deg}_{\mathcal{H}_c}(f) \, d\nu(f) \leq 32(\Delta!)^{14}.$$

Using the definition of $\mathcal{V}_c(e)$ for $e \in U_c$, Proposition 6.4 and Proposition 6.3, we get an lower bound for the left-hand side of Equation (DC) as

$$\begin{aligned} \int_E \sum_{f \in [e]_\varepsilon} \mathbf{F}(e, f) \rho_\nu(e, f) \, d\nu(e) &= \int_{U_c} \sum_{(f^1, f^2) \in \mathcal{V}_c(e)} \sum_{f \in P_c^3} \rho_\nu(e, f) \, d\nu(e) \\ &\geq \int_{U_c} \sum_{(f^1, f^2) \in \mathcal{V}_c(e)} \frac{L}{2} \, d\nu(e) \\ &\geq \int_{U_c} \frac{1}{(4\Delta)^7} \log_{8\Delta}^2(L) \frac{L}{2} \, d\nu(e) = \frac{1}{2(4\Delta)^7} \log_{8\Delta}^2(L) L \nu(U_c). \end{aligned}$$

Altogether, we infer that

$$\nu(U_c) \leq \frac{64(4\Delta)^7(\Delta!)^{14}}{\log_{8\Delta}^2(L)L}$$

as desired. □

7. Proof of the main result

We restate Theorem 1.1, in a compact form, for the convenience of the reader.

Theorem 7.1. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < \infty$ and $\mu \in \mathcal{P}(V)$. Then there is a Borel map $c : E \rightarrow [\Delta(\mathcal{G}) + 1]$ that is a proper edge coloring μ -almost everywhere.*

Proof. Apply Proposition 2.1 to \mathcal{G} to get $\hat{\mu}$, then apply Theorem 5.1 to $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ and $\hat{\mu}$ to get $\nu \in \mathcal{P}(E)$ with the property that $\rho_\nu(e, f) \leq 8\Delta$ for every $e, f \in E$ such that $(e, f) \in I_{\mathcal{G}}$, and $\mu(\{x \in V : \exists e \in A \, x \in e\}) = 0$ for every $A \in \mathcal{C}$ such that $\nu(A) = 0$.

Set $L_n = A \cdot (8\Delta)^n$, where $A \in \mathbb{N}$ is such that $\log_{8\Delta}(L_n) \geq (8\Delta)^{20}$ for every $n \in \mathbb{N}$, and define inductively, using Theorem 4.2, a sequence $\{c_n\}_{n \in \mathbb{N}}$ such that c_n does not admit improvement of weight L_n for every $n \in \mathbb{N}$ and

$$\nu(\{e \in E : c_n(e) \neq c_{n+1}(e)\}) \leq L_{n+1} \nu(U_c)$$

for every $n \in \mathbb{N}$.

The Borel–Cantelli lemma implies that $c(e) = \lim_{n \rightarrow \infty} c_n(e)$ is defined ν -almost everywhere as

$$\begin{aligned} \sum_{n \in \mathbb{N}} L_{n+1} \nu(U_{c_n}) &\leq \sum_{n \in \mathbb{N}} \frac{64(4\Delta)^7 (\Delta!)^{14} L_{n+1}}{\log_{8\Delta}^2(L_n) L_n} \\ &= \sum_{n \in \mathbb{N}} \frac{64(4\Delta)^7 (\Delta!)^{14} (8\Delta)^{n+1}}{n^2 (8\Delta)^n} = \sum_{n \in \mathbb{N}} \frac{128(4\Delta)^8 (\Delta!)^{14}}{n^2} < \infty \end{aligned}$$

by Theorem 6.1. Altogether, this shows that c is correct at μ -almost every vertex by the definition of ν . \square

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