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ON A CONJECTURE OF WOOD

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Abstract. We show that there exists a locally compact separable metrizable space L such that $C_0(L)$, the Banach space of all continuous complex-valued functions vanishing at infinity with the supremum norm, is almost transitive. Due to a result of Greim and Rajagopalan [**3**], this implies the existence of a locally compact Hausdorff space $\tilde{L}$ such that $C_0(\tilde{L})$ is transitive, disproving a conjecture of Wood [**9**]. We totally owe our construction to a topological characterization due to Sánches [**8**].

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1. Introduction, main theorem and preliminaries. For a locally compact Hausdorff space L, $C_0(L)$ denotes the Banach space of all continuous complex-valued functions on L vanishing at infinity, equipped with the supremum norm. A Banach space X is said to be *transitive* (resp. *almost transitive*) if the isometry group $G(X)$ acts transitively on the unit sphere $S(X) = \{x \in X \,|\, \|x\| = 1\}$ (resp. the orbit $G(X) \cdot x$ is dense in $S(X)$ for each $x \in S(X)$). In [**9**], Wood conjectured that $C_0(L)$ is not transitive for any locally compact Hausdorff space L unless L is a singleton. In [**3**], the conjecture was verified for the Banach space $C_0(L:\mathbf{R})$ of all *real-valued* continuous functions on L vanishing at infinity. Greim and Rajagopalan proved in [**3**] that the existence of a locally compact Hausdorff space L with $C_0(L)$ being almost transitive implies the existence of a locally compact Hausdorff space $\tilde{L}$ such that $C_0(\tilde{L})$ is transitive. In [**7**], the verification of the conjecture was reduced to the case in which L has the metrizable one-point compactification αL. Furthermore, Sánches in [**8**] gave an explicit topological characterization of the space L with the almost transitive $C_0(L)$. Theorem 2 and 3 of [**8**] are restated below. Following [**8**], we say that a locally compact Hausdorff space L is *an Wood space* (resp. *an almost Wood space*) if $C_0(L)$ is transitive (resp. almost transitive).

THEOREM 1.1 ([**8**, Theorem 2,3]). *Let L be a locally compact metrizable space such that the one-point compactification $\alpha L = L \cup \{\infty\}$ is metrizable and* $\dim \alpha L = 1$. *Then L is an almost Wood space if and only if L satisfies the following condition:*

for each pair of sequences $\{E_i | i = 0, \ldots, n\}$ and $\{F_i | i = 0, \ldots, n\}$ of compact subsets of αL satisfying

(1) $\alpha L = \cup_{i=0}^{n} E_i = \cup_{i=0}^{n} F_i$,
(2) *the point of infinity $\infty \in E_0 \cap F_0$, and*
(3) $E_i \cap E_j = F_i \cap F_j = \emptyset$ *if* $|i - j| > 1$,

there exists a homeomorphism $\varphi : \alpha L \to \alpha L$ such that $\varphi(\infty) = \infty$ and $\varphi(E_i) \subset F_{i-1} \cup F_i \cup F_{i+1}$ for each $i = 0, \ldots, n$, where $F_{-1} = F_{n+1} = \emptyset$.

The clear characterization above led us to find out a non-trivial almost Wood space, and hence, by [**3**], a non-trivial Wood space.

The almost Wood space L given in this note is (the pseudo-arc)$\backslash$\{a singleton\}. The *pseudo-arc* P is the topologically unique compact connected metric space satisfying the following two conditions (a) and (b). A *chain* of a metric space X is an open cover $C = \{C(0), \dots, C(n)\}$ of X such that $C(i) \cap C(j) \neq \emptyset$ if and only if $|i - j| \leq 1$. Each member $C(i)$ of a chain $C = \{C(0), \dots, C(n)\}$ is called a *link*. Also we fix a metric on X and let $\operatorname{mesh} C = \max_{i=0,\dots,n} \operatorname{diam} C(i)$, where $\operatorname{diam} C(i)$ is the diameter of $C(i)$ with respect to the metric.

(a) For each $\epsilon > 0$, there exists a chain C such that $\operatorname{mesh} C < \epsilon$.

(b) For each pair of points $x, y \in P$, there exists a homeomorphism $\varphi : P \to P$ such that $\varphi(x) = y$.

The space was constructed by R. H. Bing [**1**] and E. E. Moise [**6**]. It has been playing an important role in continuum theory (in topology) and our result is another simple application of the topology of the pseudo-arc.

MAIN THEOREM. *Let P be the pseudo-arc, $p \in P$ and let $L = P \backslash \{p\}$. Then $C_0(L)$ is almost transitive.*

In the rest of this section, we recall some properties of the pseudo-arc. [**5**] is an excellent survey article on the space, from which we quote all results below. First observe that $\dim P = 1$ by the above condition (a) and the connectedness of P.

DEFINITION 1.2. (1) A function $f : \{0, 1, \dots, m\} \to \{0, \dots, n\}$ is called a *pattern* if $|f(i) - f(i+1)| \leq 1$ for each $i = 0, \dots, m-1$.

(2) Let $C = \{C(0), \dots, C(n)\}$ and $D = \{D(0), \dots, D(m)\}$ be chains of the pseudo-arc. The chain D is said to *follow a pattern f in the chain C* if $\operatorname{cl} D(i) \subset C(f(i))$ for each $i = 0, \dots, m$. Here $\operatorname{cl} D(i)$ denotes the closure of $D(i)$.

THEOREM 1.3 ([**5**, p. 108]). *For each point $p \in P$ and for each $\epsilon > 0$, there exists a chain $C = \{C(0), \dots, C(n)\}$ such that* $\operatorname{mesh} C < \epsilon$ *and* $p \in C(0) \backslash \cup_{i=1}^{n} C(i)$.

The following is an immediate consequence of [**5**, p. 108, Theorem[60]], being applied to $j = i_0 = 0$, for which the hypothesis of the theorem is automatically satisfied.

THEOREM 1.4. *Let p be a point of the pseudo-arc P and let $C = \{C(0), \dots, C(n)\}$ be a chain of P such that $p \in C(0)$. For each pattern $f : \{0, \dots, m\} \to \{0, \dots, n\}$ with $f(0) = 0$, there exists a chain $D = \{D(0), \dots, D(m)\}$ which follows f and $p \in D(0)$.*

The following theorem, here stated for the pseudo-arc only, provides us with the standard method to construct homeomorphisms of the pseudo-arc. The proof of main theorem given in the next section is a simple application of Theorem 1.5 below to a specific situation.

For a chain $C = \{C(0), \dots C(n)\}$ of the pseudo-arc P and for a point $p \in P$, $\operatorname{st}(p, C)$ denotes the collection of links containing p. Note that $\operatorname{st}(p, C)$ consists of at most two links.

THEOREM 1.5 ([**5**, p. 109]). *Let $\{C_i | i = 0, 1, \dots\}$ and $\{D_i | i = 0, 1, \dots\}$ be two sequences of chains of the pseudo-arc P such that,*

(1) *for each $i = 0, 1, \dots,$ C_i and D_i has the same number of links, so they are written as $C_i = \{C_i(0), \dots C_i(n_i)\}$ and $D_i = \{D_i(0), \dots D_i(n_i)\}$,*

(2) $\lim_{i\to\infty} \operatorname{mesh} C_i = \lim_{i\to\infty} \operatorname{mesh} D_i = 0$*, and*

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(3) *for each i, there exists a pattern $f_i : \{0, 1, \ldots, n_{i+1}\} \to \{0, \ldots n_i\}$ such that C_{i+1} follows f_i in C_i and D_{i+1} follows f_i in D_i.*
Then there exists a homeomorphism $h : P \to P$ such that, if $\mathrm{st}(p, C_i) \subset \{C_i(j), C_i(j+1)\}$, then $h(x) \in D_i(j) \cup D_i(j+1)$, for each $j = 0, \ldots, n_i$ and for each $i = 0, 1, \ldots$.

REMARK. In [**5**], a stronger condition
(2′) $\mathrm{mesh}\, C_i < 1/i$ and $\mathrm{mesh}\, D_i < 1/i$
is required instead of the condition (2) above. The proof of the theorem easily shows that the same conclusion holds under the condition (2) above.

COROLLARY 1.6. *Under the above notation, assume further that a point $p \in P$ satisfies $p \in C_i(0) \cap D_i(0)$ for each i. Then the above homeomorphism $h : P \to P$ satisfies $h(p) = p$.*

Proof. Since $h(p) \in D_i(0) \cup D_i(1)$, we have $d(p, h(p)) \leq \mathrm{mesh}\, D_i \to 0$ as $i \to \infty$. The conclusion follows immediately.

2. Proof of Main theorem. Recall [**8**] that $C_0(L)$ is *almost positive transitive* if, for each pair of non-negative norm-one functions $f, g \in C_0(L)$ and for each $\epsilon > 0$, there exists an isometry $T : C_0(L) \to C_0(L)$ such that $\|Tf - g\| < \epsilon$. Also we say that $C_0(L)$ allows *nearly polar decompositions* if, for each $f \in C_0(L)$ and for each $\epsilon > 0$, there exists an isometry $T : C_0(L) \to C_0(L)$ such that $\|f - T(|f|)\| < \epsilon$. It is easy to see that $C_0(L)$ is almost transitive if and only if it is almost positive transitive and allows nearly polar decompositions (cf. [**8**, p. 315]).

The following Lemma is implicitly stated in [**8**, Theorem 2], in which the proof is omitted. We give a proof here for completeness.

LEMMA 2.1. *Let L be a locally compact metrizable space such that the one-point compactification αL is metrizable with $\dim \alpha L = 1$. Then $C_0(L)$ allows nearly polar decompositions.*

Proof. Take an arbitrary $f \in C_0(L)$ and extend it canonically to the function on αL by defining $f(\infty) = 0$. Since $\dim \alpha L = 1$, one can show that, for each $\epsilon > 0$, there exists a continuous function $f_\epsilon : \alpha L \to \mathbf{C}$ such that $\|f - f_\epsilon\| < \epsilon$ and $f_\epsilon(x) \neq 0$ for each $x \in \alpha L$ (see [**4**, Theorem 18], or [**2**, p. 76, 1.9.B]). Define $T : C_0(L) \to C_0(L)$ by

$$(Tg)(x) = \frac{f_\epsilon(x)}{|f_\epsilon(x)|} \cdot g(x), \quad g \in C_0(L) \text{ and } x \in L.$$

Clearly T is an isometry. For an arbitrary $x \in L$, we estimate $|(T|f|)(x) - f(x)|$ as follows. Here the complex conjugate of $f(x)$ is denoted by $\overline{f(x)}$.

$$\begin{aligned}
|(T|f|)(x) - f(x)|^2 &= \left| \frac{f_\epsilon(x)}{|f_\epsilon(x)|} \cdot |f(x)| - f(x) \right|^2 \\
&= 2|f(x)|^2 - |f(x)| \frac{f_\epsilon(x)\overline{f(x)} + \overline{f_\epsilon(x)} f(x)}{|f_\epsilon(x)|} \\
&= 2|f(x)| \left(|f(x)| - \mathrm{Re} \frac{f_\epsilon(x)\overline{f(x)}}{|f_\epsilon(x)|} \right) \\
&\leq 2\|f\| \left(\frac{(|f_\epsilon(x)| + \epsilon)|f_\epsilon(x)| - \mathrm{Re} f_\epsilon(x)\overline{f(x)}}{|f_\epsilon(x)|} \right) \\
&= 2\|f\| \left(\epsilon + \frac{|f_\epsilon(x)|^2 - \mathrm{Re} f_\epsilon(x)\overline{f(x)}}{|f_\epsilon(x)|} \right).
\end{aligned}$$

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Making use of the inequality $\|f - f_\epsilon\| < \epsilon$, it is easy to see

$$\frac{|f_\epsilon(x)|^2 - \mathrm{Re} f_\epsilon(x)\overline{f(x)}}{|f_\epsilon(x)|} < 2\epsilon.$$

Thus we have $\|T|f| - f\| = \sup_{x\in L} |(T|f|)(x) - f(x)| \le \sqrt{6\|f\|\epsilon}$, completing the proof.

Proof of Main Theorem. Let P be the pseudo-arc and take a point $p \in P$. We show that $L = P\backslash\{p\}$ is an almost Wood space. Clearly $\alpha L \cong P$, a compact connected metrizable space and $\dim P = 1$. Thus $C_0(L)$ allows nearly polar decompositions by Lemma 2.1. Hence it suffices to verify the conditions required in Theorem 1.1, yet it is more convenient to prove directly the almost positive transitivity of $C_0(L)$, along with exactly the same idea as that of [**8**, Theorem 3].

Take arbitrary pair of non-negative, norm-one functions $f, g \in C_0(L)$ and extend them to functions on $\alpha L = P$ by defining $f(p) = g(p) = 0$. Then $f(\alpha L) = g(\alpha L) = [0, 1]$. Fix a positive integer n and let $I_i = (\frac{i}{n} - \frac{1}{4n}, \frac{i+1}{n} + \frac{1}{4n}) \cap [0, 1]$ for $i = 0, \ldots, n-1$. Let $C_0(i) = f^{-1}(I_i)$ and $D_0(i) = g^{-1}(I_i)$ for $i = 0, \ldots, n-1$. $C_0 = \{C_0(0), \ldots, C_0(n-1)\}$ and $D_0 = \{D_0(0), \ldots, D_0(n-1)\}$ are chains of P. As $f(p) = 0$, we have $p \in C_0(0)$ and similarly $p \in D_0(0)$.

Take a small $0 < \epsilon_1 < 1/2$ so that every subset E of P with $\operatorname{diam} E < \epsilon_1$ is contained in a link of D_0. By Theorem 1.3, there exists a chain $D_1 = \{D_1(0), \ldots, D_0(n_1)\}$ with $\operatorname{mesh} D_1 < \epsilon_1$ such that $p \in D_1(0)$. Note that, for each link $D_1(i)$, there exists a link of D_0 containing $D_1(i)$. Hence there exists a pattern which D_1 follows in D_0. Let $f_1 : \{0, \ldots, n_1\} \to \{0, \ldots, n\}$ be a such pattern. Apply Theorem 1.4 to take a chain $C_1 = \{C_1(0), \ldots, C_1(n_1)\}$ which follows f_1 in C_0 such that $p \in C_1(0)$.

Take a small $0 < \epsilon_2 < 1/3$ so that every subset F of P with $\operatorname{diam} F < \epsilon_2$ is contained in a link of C_1. Again by Theorem 1.3, there exists a chain $C_2 = \{C_2(0), \ldots, C_2(n_2)\}$ with $\operatorname{mesh} C_2 < \epsilon_2$ such that $p \in C_2(0)$. Let $f_2 : \{0, \ldots, n_2\} \to \{0, \ldots, n_1\}$ be a pattern which C_2 follows in C_1. We make another application of Theorem 1.4 to obtain a chain $D_2 = \{D_2(0), \ldots, D_2(n_2)\}$ which follows f_2 in D_1 such that $p \in D_2(0)$. Note that $\operatorname{mesh} D_2 \le \operatorname{mesh} D_1 < 1/2$.

Continuing this process, we obtain sequences of chains $\{C_i | i = 0, 1, \ldots\}$ and $\{D_i | i = 0, 1, \ldots\}$ together with a sequence $\{f_i : \{0, \ldots, n_i\} \to \{0, \ldots, n_{i-1}\} | i = 1, 2, \ldots\}$ of patterns ($n_0 = n$) which satisfy the hypothesis of Theorem 1.5. Moreover the sequences have an additional property: $p \in \cap_{i=0}^{\infty} C_i(0) \cap \cap_{i=0}^{\infty} D_i(0)$. By Theorem 1.5 and Corollary 1.6, there exists a homeomorphism $\varphi : P \to P$ with $\varphi(p) = p$ such that, if $\operatorname{st}(x, C_0) \subset \{C_0(j), C_0(j+1)\}$, then $\varphi(x) \in D_0(j) \cup D_0(j+1)$.

Take a point $x \in C_0(j)$. Then by the definition of C_0, $f(x) \in (\frac{j}{n} - \frac{1}{4n}, \frac{j+1}{n} + \frac{1}{4n})$. Since $\varphi(x) \in D_0(j) \cup D_0(j+1)$, we have similarly $g(\varphi(x)) \in (\frac{j}{n} - \frac{1}{4n}, \frac{j+2}{n} + \frac{1}{4n})$. Hence $|f(x) - g(\varphi(x))| \le \frac{2}{n} + \frac{1}{2n} = \frac{5}{2n}$. Since x is an arbitrary point of L, we have $\|f - Tg\| \le \frac{5}{2n}$. For a given $\epsilon > 0$, we may take n so that $\frac{5}{2n} < \epsilon$ to complete the proof of the almost positive transitivity of $C_0(L)$.

This completes the proof of Main Theorem.

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