FOUNDATIONS OF A THEORY OF THE MOTION OF THE ORBIT PLANE OF HYPERION

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Abstract. The principles are set out for the construction of a theory of the motion of the orbit plane of Hyperion, using the mixed set of angle parameters, using different reference planes for different angles, which it has proved convenient to use. It is found that this leads to additional terms, which have not been shown in previous published theories. The theory is developed in general principles exactly, and in detail as far as is needed to enable comparison to be made with the observational data at present available, and, from parameters which have been derived from opposition means from the period 1875 to 1922, the co-efficients of some of the larger long-period terms are computed.

1. Introduction

It has become usual, when developing theories of the motion of Hyperion *in* its orbit plane, including the effects of the very close resonance of orbital period with Titan, to use longitudes, including the longitude of the apse, referred to the Ecliptic and Equinox (*i.e.* First Point of Aries), of course of some specified date. However, when dealing with the motion *of* the orbit plane, the governing equations are very much simplified by using parameters which refer the orientation of the plane to the equator plane, or ring plane, of Saturn, since the orbit planes of Hyperion and Titan (and in fact of all the satellites except Iapetus) are inclined at quite small angles to that plane, and the differential equations for the rectangular-type orbital plane parameters may be treated as linear, for any precision of the theory which has been so far required.

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2. The parameters employed

Let us now examine the effects of using this mixed set of parameters, referred to two different reference planes, in work on the motion of Hyperion.

Let us use the following notation:

i for the inclination of the orbit plane to Saturn's equator plane,

h for the longitude of the ascending node of the orbit, on Saturn's equator plane, measured from the ascending node of Saturn's equator on the Ecliptic,

I for the inclination of the orbit plane to the Ecliptic,

Ω for the longitude of the ascending node of the orbit, on the ecliptic, measured from the Equinox,

I_e for the inclination of the equator plane of Saturn to the Ecliptic,

Ω_e for the longitude of the ascending node of Saturn's equator on the Ecliptic, also measured from the Equinox.

A consistent canonical set of orbital parameters may be constructed by using Saturn's equator plane as the reference plane, which has the advantage of being effectively fixed in orientation, since the gravitational couple on Saturn is so small. One way to remove one source of complication from the task of comparing results with those obtained by the use of more usual reference systems, would be to measure all longitudes from the equinox, along the Ecliptic to the to the ascending node of Saturn's equator on the Ecliptic, and then along Saturn's equator to the ascending node of the orbit on Saturn's equator, and then (except for the longitude of the node itself) along the orbit. (Then longitudes so defined would be subject to precession, almost entirely due to the precessional motion of the equinox along the ecliptic.) A canonical set of orbital parameters may be set up in which all longitudes are defined in this way, thus avoiding any complication arising from the use of different reference planes in the construction of a perturbation theory (including the use of a Lie series transformation to separate the long-period effects from those of short period.) Let us denote by $\hat{\psi}$ a longitude defined on this basis (*i.e.* using the Ecliptic, equator plane of Saturn, and orbit plane).

Since, however, most reduction of observational data proceeds on the basis of longitudes measured in the more conventional way, i.e. from the Equinox along the Ecliptic to the ascending node of the orbit on the Ecliptic, it is necessary, in interpreting longitudes predicted by such a canonically consistent theory as is considered above, to relate the two systems of longitudes. Let us denote by ψ a longitude defined in the conventional way (*i.e.* using only the Ecliptic and orbit planes). Usually the theory will involve differences of longitudes of the two satellites, of the type $\hat{\psi}_H - \hat{\psi}_T$ (or corresponding differences of their apse longitudes, etc.). Such a difference will differ from the corresponding difference $\psi_H - \psi_T$ by quantities of the second order in the inclinations i_H and i_T of the satellites' orbits to Saturn's equator plane, and that will, in most aspects of the motion, lead to negligible consequences in the predictions to any precision which has so far been required. But, as observational data of finer precision are acquired, it will become necessary to take these differences into account.

3. The equations for the perturbations

As mentioned above, the differential equations for the motion of the orbit plane will be approximately linear if expressed in terms of the rectangular-type parameters

$$\begin{aligned} p &= \sin\, i\, \sin\, h = \sin\, I\, \sin\,(\Omega - \Omega_e) \\ q &= \sin\, i\, \cos\, h = \sin\, I\, \cos\, I_e\, \cos\,(\Omega - \Omega_e) - \cos\, I\, \sin\, I_e \end{aligned} \qquad (1)$$

The Lagrange equations for the mixed set of orbital parameters $(\lambda, \varpi, a, e, q, p)$, with the disturbing function, R, expressed in terms of this same set, are found to be, without approximation,

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial \lambda}, \\ \frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{Y}{na^2}\frac{\partial R}{\partial \lambda} - \frac{X}{na^2}\frac{\partial R}{\partial \varpi}, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{Y}{na^2}\frac{\partial R}{\partial e} - \frac{Z}{nab}\mathcal{P}, \\ \frac{\mathrm{d}\varpi}{\mathrm{d}t} &= \frac{X}{na^2}\frac{\partial R}{\partial e} - \frac{Z}{nab}\mathcal{P}, \\ \frac{\mathrm{d}q}{\mathrm{d}t} &= -\frac{\cos\, i}{nab}\frac{\partial R}{\partial p} - \frac{Z}{nab}(q\cos I + \sin I_e)\left\{\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right\}, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= +\frac{\cos\, i}{nab}\frac{\partial R}{\partial q} - \frac{Z}{nab}p\cos\, I\left\{\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right\}, \end{aligned}$$

where

$$\begin{aligned} X &= \frac{\sqrt{1-e^2}}{e}, \\ Y &= X - \frac{1-e^2}{e}, \\ Z &= \frac{1}{1+\cos\, I}, \end{aligned}$$

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$$\mathcal{P} = p\cos I \frac{\partial R}{\partial p} - (q\cos I + \sin I_e)\frac{\partial R}{\partial q},$$

and we note that

$$\cos I = \cos i \cos I_e - q\sin I_e$$

and

$$\cos i = \sqrt{1-(q^2+p^2)}.$$

4. The terms from the perturbations by Titan

The most important part of the disturbing function, R, is that corresponding to the effect of Titan:

$$R_T = Gm_T\left\{\frac{1}{\Delta} - \frac{r_H}{r_T}\cos \mathcal{S}\right\},$$

in which

G is the constant of gravitation,
m_T is the mass of Titan,
Δ is the distance between Hyperion and Titan,
r_H is the distance between Hyperion and Saturn,
r_T is the distance between Titan and Saturn,
$\mathcal{S}$ is the angle subtended at the centre of Saturn by Hyperion and Titan,

so that

$$\Delta^2 = r_H^2 + r_T^2 - 2r_H r_T\cos \mathcal{S}.$$

Now let us put

$$R_T = R_0 + \delta R,$$

in which R_0 is R_T as evaluated with $\mathcal{S}$ replaced by $\psi_H - \psi_T$, the difference between the true longitudes of Hyperion and Titan, and with Δ replaced by Δ_0, which is Δ also evaluated with $\mathcal{S}$ replaced by $\psi_H - \psi_T$. Thus R_0 is that part of R_T giving the main part of the perturbations in the orbit plane, and δR contains all of the terms in R_T involving the parameters of the orbit plane. Then we find, taking proper account of the use of the different reference planes used in the definitions of the various angles, that, to second order in q_H and p_H (the values of q and p, respectively, for Hyperion), we obtain

$$\delta R = Gm_T\left\{r_H r_T\left\{\frac{1}{\Delta_0^3}-\frac{1}{r_T^3}\right\}\mathcal{F} + \frac{3}{4}\cdot\frac{r_H^2 r_T^2}{\Delta_0^5}\tan^2\left(\frac{I_e}{2}\right)(p_H-p_T)^2\{1-\cos 2(\psi_H-\psi_T)\}\right\},$$

where

$$\begin{aligned}\mathcal{F} = \frac{1}{4}\Bigg\{&-\left\{(q_H-q_T)^2+\left\{1+2\tan^2\left(\frac{I_e}{2}\right)\right\}(p_H-p_T)^2\right\}\cos(\psi_H-\psi_T)\\ &+\{(q_H-q_T)^2-(p_H-p_T)^2\}\cos(\psi_H+\psi_T-2\Omega_e)\\ &+2(q_H-q_T)(p_H-p_T)\sin(\psi_H+\psi_T-2\Omega_e)\\ &+2\Bigg\{(p_Hq_T-q_Hp_T)+2\tan\left(\frac{I_e}{2}\right)(p_H-p_T)\\ &\quad+\tan^2\left(\frac{I_e}{2}\right)(p_Hq_H-p_Tq_T)\Bigg\}\sin(\psi_H-\psi_T)\Bigg\}.\end{aligned}$$

Substituting these terms into the Lagrange equations for the rates of change of q_H and p_H, we find some cancellation of terms, leading to some simplification in the terms of lowest order (as the comments at the end of section 2 lead us to expect), and that $\frac{\mathrm{d}q_H}{\mathrm{d}t}$ has to first order, in fact the terms

$$n_H m'\mathcal{K}\Big\{(p_H-p_T)\{\cos(\psi_H-\psi_T)+\cos(\psi_H+\psi_T-2\Omega_e)\} +(q_H-q_T)\{\sin(\psi_H-\psi_T)-\sin(\psi_H+\psi_T-2\Omega_e)\}\Big\},$$

and that $\frac{\mathrm{d}p_H}{\mathrm{d}t}$ has, also to first order,

$$n_H m'\mathcal{K}\Big\{(q_H-q_T)\{-\cos(\psi_H-\psi_T)+\cos(\psi_H+\psi_T-2\Omega_e)\} +(p_H-p_T)\{\sin(\psi_H-\psi_T)+\sin(\psi_H+\psi_T-2\Omega_e)\}\Big\},$$

where

$$\mathcal{K} = \frac{1}{2}a_H r_H r_T\left\{\frac{1}{\Delta_0^3}-\frac{1}{r_T^3}\right\},$$

and

$$m' = \frac{m_T}{m_S},$$

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where m_S is the mass of Saturn. The extra terms arising from the use of mixed reference planes do not cancel out in the expressions for $\frac{\mathrm{d}\lambda_H}{\mathrm{d}t}$ and $\frac{\mathrm{d}\varpi_H}{\mathrm{d}t}$, even to first order, and, to this order, both have the terms

$$n_H m' \mathcal{K} \tan\left(\frac{I_e}{2}\right)\Big\{(q_H - q_T)\Big\{-\cos(\psi_H - \psi_T) + \cos(\psi_H + \psi_T - 2\Omega_e)\Big\}$$

$$+(p_H - p_T)\sin(\psi_H + \psi_T - 2\Omega_e)$$

$$+\Big\{p_T + \Big\{\cos\ I_e \sec^2\left(\frac{I_e}{2}\right) - 2\tan\left(\frac{I_e}{2}\right)\Big\}p_H\Big\}\sin(\psi_H - \psi_T)\Big\}$$

$$+\frac{3}{4}\frac{r_H r_T}{\Delta_0^5}\tan^2\left(\frac{I_e}{2}\right)\cos\ I_e p_H\{1 - \cos\ 2(\psi_H - \psi_T)\}.$$

Now from the results of the theory of the motion in the orbit plane (Message, 1989, 1993), we find the expressions, in which we indicate by "$< \mathcal{F} >$" the result of averaging a quantity "$\mathcal{F}$" over $\lambda_H - \lambda_T$, to isolate the long-period and critical terms,

$$< K\cos(\psi_H - \psi_T) > = \sum_i \sum_j \mathcal{A}_{i,j}\ \cos(i\tau - j\zeta),$$

$$< K\sin(\psi_H - \psi_T) > = \sum_i \sum_j \mathcal{B}_{i,j}\ \sin(i\tau - j\zeta),$$

$$< K\exp\ \{\iota\,(\psi_H + \psi_T - 2\Omega_e)\} > = \sum_i \sum_j \mathcal{C}_{i,j}\ \exp\ \{\iota\,(i\tau - j\zeta)\},$$

where τ is the argument of the free libration (of about 21 month period), and ζ is the linear part of the argument $\varpi_H - \varpi_T$ (of about $18\frac{3}{4}$ year period), and ι is $\sqrt{-1}$.

5. The terms from other perturbations

From the solar perturbations, the main term in R is

$$\frac{1}{2}n_0^2 r_H^2\,(3\cos^2\ \mathcal{S}_0 - 1),$$

where n_0 is the mean motion in the relative motion of the Sun and Saturn, and $\mathcal{S}_0$ is the angle subtended at the centre of Saturn by the Sun and Hyperion. The largest long-period parts of this term are

$$\frac{3}{8}n_0^2 r_H^2\Big\{1+\cos^2 I_e - q_H\sin 2I_e - q_H^2\cos 2I_e + p_H^2\cos^2 I_e$$
$$+\Big\{\sin^2 I_e + q_H\sin 2I_e + q_H^2\cos 2I_e + p_H^2(\cos^2 I_e - 2)\Big\}\cos 2(\lambda_0-\Omega_e)$$
$$+2\Big\{p_H\sin I_e + q_H p_H\cos I_e\Big\}\sin 2(\lambda_0-\Omega_e)\Big\},$$

from which the largest long-period solar terms in $\dfrac{dq_H}{dt}$ are

$$\frac{3n_0^2}{4n_H}\Big\{1+\frac{1}{2}e^2\Big\}\Big\{\Big\{\cos^2 I_e + \{\cos^2 I_e - 2\}\cos 2(\lambda_0-\Omega_e)\Big\}p_H$$
$$+\Big\{\sin I_e + q_H\cos I_e\Big\}\sin 2(\lambda_0-\Omega_e)\Big\},$$

and the largest long-period solar terms in $\dfrac{dp_H}{dt}$ are

$$\frac{3n_0^2}{4n_H}\Big\{1+\frac{1}{2}e^2\Big\}\Big\{-\frac{1}{2}\sin 2I_e - q_H\cos 2I_e + p_H\cos I_e\sin 2(\lambda_0-\Omega_e)$$
$$+\frac{1}{2}\{\sin 2I_e + 2q_H\cos 2I_e\}\cos 2(\lambda_0-\Omega_e)\Big\}.$$

The largest long-period term from the effect of the figure of Saturn in $\dfrac{dq_H}{dt}$ is

$$\frac{3}{2}n_H^2 R_e^2 J_2 p_H,$$

and the corresponding term in $\dfrac{dp_H}{dt}$ is

$$-\frac{3}{2}n_H^2 R_e^2 J_2 q_H.$$

Here R_e is the radius of Saturn's equator, and J_2 is the co-efficient of the second zonal harmonic in the external gravitational field of Saturn.

6. The solution of the equations.

To proceed to a solution of the equations for q_H and p_H, introduce the complex variable

$$\mathcal{Z} = \kappa(q_H - q_T) + \frac{\iota}{\kappa}(p_H - p_T),$$

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where κ is a constant to be chosen. The equations may then be written, to the precision to which we have been working,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}t} &= \iota n_H \Big\{-\alpha\mathcal{Z} - \alpha'\bar{\mathcal{Z}} - \delta_s + \sum_j \beta_j \exp(\iota u_j)\,\bar{\mathcal{Z}} \\ &\quad - \sum_j \gamma_j \exp(\iota w_j)\,\mathcal{Z} \\ &\quad + \sum_j \rho_j \exp(\iota v_j)\Big\},\end{aligned}$$

where α, α', δ_s, β_j, γ_j, and ρ_j are constants, and u_j, w_j, and v_j are linear functions of the time (corresponding to the various terms in the equations for $\frac{\mathrm{d}q_H}{\mathrm{d}t}$ and $\frac{\mathrm{d}p_H}{\mathrm{d}t}$). The constant κ is chosen so that α' takes the value zero. This is found to require that, approximately,

$$\kappa^4 = 1 - \frac{3\,n_0^2\,\sin^2 I_e}{4\,n_H^2\,m'\,\mathcal{A}_{0,0}},$$

which gives κ the value 0.999565..... The constant α is given by

$$\alpha = m'\mathcal{A}_{0,0} + \frac{3}{2}R_e^2 J_2 + ... \approx 1.403m',$$

and

$$\delta_s = \frac{3n_0^2}{8n_H^2}\,\sin 2I_e \approx 0.00000118.....$$

We note that, if the β_j were all zero (as would be true in the presence only of those terms which Woltjer (1928) took), then the linear equation for $\mathcal{Z}$ would be solvable exactly. However, some of the β_j are in fact significant. The solution may be written in the form

$$\begin{aligned}\mathcal{Z} &= c\,\exp(-\iota v) - \frac{\delta_s}{\alpha} \\ &\quad + \sum_j a_j \exp(\iota v_j) \\ &\quad + \sum_j b_j \exp\{\iota(u_j + v)\} \\ &\quad + \sum_j b'_j \exp\{\iota(w_j - v)\} \\ &\quad + \sum_j\sum_k c_{j,k} \exp\{\iota(u_j - v_k)\}\end{aligned}$$

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$$+\sum_j \sum_k c'_{j,k} \exp \{\iota(w_j + v_k)\}$$
$$+\sum_j \sum_k d_{j,k} \exp \{\iota(u_j - u_k - v)\}$$
$$+\sum_j \sum_k \sum_\ell e_{j,k,\ell} \exp \{\iota(u_j - u_k + v_\ell)\}$$
$$+ \text{etc} \ldots$$

in which the constants of integration are the amplitude, c, of the free oscillation, and the phase of the linear argument, v, of the free oscillation. Substituting this solution into the differential equation, and equating the co-efficients of each of the (infinite number of) periodic terms, leads to an array of algebraic equations which may be solved by iteration to give the values of the amplitudes a_j, b_j, b'_j, $c_{j,k}$, $c'_{j,k}$, $d_{j,k}$, $e_{j,k,\ell}$, etc,..., of the forced terms, and the rate of change, χ, say, of the argument v of the free oscillation.

7. Identification of some of the major terms.

Let us now identify some of the main forced terms in the motion of the orbit plane, beginning with those of type $a_j \exp(\iota v_j)$.

Corresponding to $j = 1$ let us set the term arising from the precession of the orbit plane of Titan. This has a period of about 690 years and the relevant argument is $v_1 = 41.4° - 0.5213°(t - 1880.25)$, with t in years. Then the solution gives $a_1 = 0.041°$ and the main contribution to q_H is $0.333° \cos v_1$ and to p_H is $0.333° \sin v_1$.

Corresponding to $j = 2$ let us set a term with argument $v_2 = \Omega_0$, the node of the orbit of the relative motion of the Sun and Saturn. Since the mean motion of this is about 6 seconds of arc per century, it is effectively constant in this context. The contribution to q_H is $-0.745°$ and that to p_H is $-0.037°$.

Corresponding to $j = 3$ let us set a term with argument $v_3 = 2\lambda_0 - \Omega_0 + \pi$. For this we find that $a_3 = -0.018°$; the contribution to q_H is $-0.018° \cos(2\lambda_0 - \Omega_0)$ and that to p_H is $-0.018° \sin(2\lambda_0 - \Omega_0)$.

A significant term of the type $b_j \exp\{\iota(u_j + v)\}$ is associated with that term in the disturbing function which has argument $8\lambda_H - 6\lambda_T - 2\Omega_H$ and appears in the present theory with $u_1 = 2(\zeta + \varpi_T - \Omega_e) + \pi$, which has period 10.3 years. The theory gives $b_1 = -0.013°$ and the contribution to q_H is

$$0.013° \sin\{2(\zeta + \varpi_T - \Omega_e) - v\},$$

and that to p_H is

$$0.013° \cos\{2(\zeta + \varpi_T - \Omega_e) - v\}.$$

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It remains, to the precision to which we are at present working, to consider the free oscillation. A fresh analysis of the values of the opposition means of orbital parameters which derive from the observational data from the time interval 1875 to 1922 gives for the amplitude $c = 0.521° \pm 0.012°$ and, for the argument v, $94.91° \pm 1.41° - (2.651° \pm 0.097°) \cdot (t - 1900.0)$. This rate gives an estimate for the mass of Titan, in terms of that of Saturn, of $(2.73 \pm 0.11) \cdot 10^{-4}$, but since the data span only a fraction of the period of this free term, about 136 years, this estimate cannot be considered to have very high weight.

Work is in hand to make an analysis of all the available observational data in one solution in comparison with this theory, which it is hoped will improve the estimates of the parameters. To make comparison with more precise observational data, as may become available for example from the "Cassini" mission, would require the retaining in the theory of more terms than we have considered in section 7, and perhaps also of terms of higher than second order in q_H and p_H in the expression for δR in section 4 above. Also it might possibly be necessary to include, in the theory of the motion *in* the orbit plane, those terms of second order in the mass of Titan resulting from the effect of terms in δR on λ_H and ϖ_H (and perhaps also a and e), though such terms will be very small indeed, however, having as factors the square of Titan's mass and also the very small angle of inclination of the orbit plane to Saturn's equator plane.

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