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Conjugate Reciprocal Polynomials with All Roots on the Unit Circle

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Abstract. We study the geometry, topology and Lebesgue measure of the set of monic conjugate reciprocal polynomials of fixed degree with all roots on the unit circle. The set of such polynomials of degree N is naturally associated to a subset of $\mathbb{R}^{N-1}$. We calculate the volume of this set, prove the set is homeomorphic to the $N-1$ ball and that its isometry group is isomorphic to the dihedral group of order $2N$.

1 Introduction

Let N be a positive integer and suppose $f(x)$ is a polynomial in $\mathbb{C}[x]$ of degree N. If f satisfies the identity,

$$f(x) = x^N \overline{f(1/\overline{x})}, \tag{1.1}$$

then f is said to be *conjugate reciprocal*, or simply *CR*. Furthermore, if f is given by

$$f(x) = x^N + \sum_{n=1}^{N} c_n x^{N-n},$$

then (1.1) implies that $c_N = 1$, $c_{N-n} = \overline{c_n}$ for $1 \leq n \leq N-1$ and, if α is a zero of f, then so too is $1/\overline{\alpha}$. The purpose of this paper is to study the set of CR polynomials with all roots on the unit circle. The interplay between the symmetry condition on the coefficients and the symmetry of the roots allows for a number of interesting theorems about the geometry, topology and Lebesgue measure of this set.

CR polynomials have various names in the literature including reciprocal, self-reciprocal and self-inversive (though we reserve the term *reciprocal* for polynomials which satisfy an identity akin to (1.1) except without both instances of complex conjugation).

The condition on the coefficients of a conjugate reciprocal polynomial allows us to identify the set of CR polynomials with $\mathbb{R}^{N-1}$. To be explicit, let X_N be the $N-1 \times N-1$ matrix whose j,k entry is given by,

$$X_N[j,k] = \begin{cases} \frac{\sqrt{2}}{2}\left(\delta_{j,k} + \delta_{N-j,k}\right) & \text{if } 1 \leq j < N/2, \\ \delta_{j,k} & \text{if } j = N/2, \\ \frac{\sqrt{2}}{2}\left(i\delta_{N-j,k} - i\delta_{j,k}\right) & \text{if } N/2 < j < N, \end{cases} \tag{1.2}$$

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where $\delta_{j,k} = 1$ if $j = k$ and is zero otherwise. For instance,

$$X_5 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \end{bmatrix} \quad \text{and} \quad X_6 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & i \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -i & 0 \\ 1 & 0 & 0 & 0 & -i \end{bmatrix}.$$

The $\sqrt{2}/2$ factor is a normalization so that $|\det X_N| = 1$, a fact which is easily checked by induction on N (odd and even cases treated separately).

Given $\mathbf{a} \in \mathbb{R}^{N-1}$, $X_N\mathbf{a}$ is a vector in $\mathbb{C}^{N-1}$. Moreover if $\mathbf{c} = X_N\mathbf{a}$, then $c_{N-n} = \overline{c_n}$ for $1 \le n \le N-1$, and we may associate a CR polynomial to $\mathbf{a}$ by specifying that

$$\mathbf{a}(x) = (x^N + 1) + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

Let $\Delta \subset \mathbb{C}$ denote the open unit ball and let $\mathbb{T} \subset \mathbb{C}$ denote the unit circle. Equation (1.1) implies that there exist $\alpha_1, \alpha_2, \dots, \alpha_M \in \Delta$ and $\xi_1, \xi_2, \dots, \xi_L \in \mathbb{T}$ such that

$$\mathbf{a}(x) = \prod_{m=1}^{M}(x-\alpha)(x-1/\overline{\alpha})\prod_{l=1}^{L}(x-\xi_l), \tag{1.3}$$

where obviously $2M + L = N$. We define W_N to be the set

$$W_N = \{\mathbf{w} \in \mathbb{R}^{N-1} : \mathbf{w}(x) \text{ has all roots on } \mathbb{T}\},$$

so that W_N is in one-to-one correspondence with the set of CR polynomials of degree N with all roots on the unit circle. Figures 1 and 2 (see p. 1151 and p. 1156) show W_3 and W_4 respectively. Elements of W_N will be regarded as either CR polynomials or as vectors in $\mathbb{R}^{N-1}$ as is convenient.

From Figure 1 we see that ∂W_3 has a distinctive shape. In fact, it is a 3-cusped hypocycloid. (A hypocycloid is the curve produced by the image of a point on a circle as the circle rolls around the inside of a larger circle.) Similarly, the projection of W_4 onto the w_1, w_3 plane is bounded by a 4-cusped hypocycloid.

1.1 Statement of Results

The geometric properties of W_3 and W_4 (Figures 1 and 2) suggest many patterns in the structure of W_N in general.

Theorem 1.1 *W_N is homeomorphic to B^{N-1}, the closed $N-1$ dimensional ball.*

We will see that W_N is circumscribed by the sphere of radius $\sqrt{\binom{2N}{N} - 2}$. Moreover, the CR polynomials $(x+\zeta_N)^N$, where ζ_N is an N-th root of unity, correspond to the only points in W_N intersecting this sphere. Given $\mathbf{w} \in W_N$, there is a natural way to define a (cyclically ordered) partition of N, $\mathcal{P}(\mathbf{w})$, corresponding to the multiplicities of the cyclically ordered roots of $\mathbf{w}(x)$. In this manner, W_N has a finer structure imposed upon it.

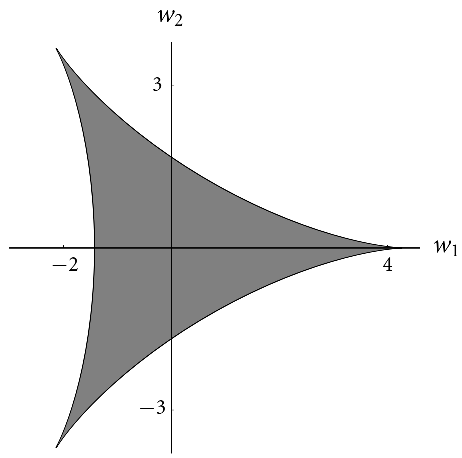

Figure 1: W_3, the set of (w_1, w_2) such that $x^3 + (w_1 + iw_2)x^2 + (w_1 - iw_2)x + 1$ has all roots on $\mathbb{T}$.

Corollary 1.2 *W_N has the structure of a coloured $N - 1$ simplex where the colouring of $\mathbf{w} \in W_N$ is $\mathcal{P}(\mathbf{w})$.*

The polynomials $(x + \zeta_N)^N$ are vertices in the above correspondence. This colouring seems to affect the geometry of W_N. For example, the edges of W_4 have different curvatures, depending on their colouring, and one can show that, despite appearances, W_4 is not star-shaped with respect to the origin. These geometric constraints are also evident in the isometry group of W_N, which is a proper subgroup of the group of isometries of an N-simplex.

Theorem 1.3 *The group of isometries of W_N is isomorphic to D_N. As such,* $\operatorname{Isom}(W_N)$ *is generated by R and C, where for any $\mathbf{w} \in W_N$, R corresponds to the action of multiplying each root of $\mathbf{w}(x)$ by ζ_N, and C corresponds to complex conjugation of the roots of $\mathbf{w}(x)$. That is, if $\mathbf{w}(x) = \prod_{n=1}^{N}(x - \xi_n)$, then*

$$R \cdot \mathbf{w}(x) = \prod_{n=1}^{N}(x - \zeta_N \xi_n) \quad \textit{and} \quad C \cdot \mathbf{w}(x) = \prod_{n=1}^{N}(x - \overline{\xi_n}).$$

We will also explicitly compute the volume of W_N by using techniques from random matrix theory. By the *volume* of W_N we simply mean its Lebesgue measure.

Theorem 1.4 *The volume of W_N is given by*

$$\operatorname{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$

That is, the volume of W_N is equal to the volume of the $N - 1$ dimensional ball of radius 2.

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Notice that $\mathrm{vol}(W_N)$ is always a rational number times an integer power of π, since when M is odd, $\Gamma(M)$ is a rational number times $\sqrt{\pi}$.

In Section 1.3 we will prove some preliminary results about W_N. Theorem 1.1 and Corollary 1.2 will be proved in Section 2. Theorem 1.3 will be proved in Section 3, and Theorem 1.4 will be proved in Section 4.

1.2 Motivation

Conjugate reciprocal polynomials (and more generally self-inversive polynomials) have appeared in the literature in the study of random polynomials, random matrix theory, and, most recently, speculative number theory. Self-inversive polynomials were first introduced by F. F. Bonsall and M. Marden [2]. (See Section 4.1 for the definition of self-inversive polynomials). Bonsall and Marden give a method for enumerating the zeros on the unit circle of a self-inversive polynomial. Other authors, in particular A. Schinzel, have been interested in necessary conditions for a self-inversive polynomial to have all its roots on the unit circle [7]. When finding such necessary conditions, one can without loss of generality restrict oneself to the case of CR polynomials. In this context, necessary conditions on the coefficients of a CR polynomial to have all roots on the unit circle translate to geometric information about W_N.

The statistical behavior of the roots of polynomials in W_N in the large N limit has been used in the study of quantum chaotic dynamics [1] and the distribution of zeros on the critical line of certain L-functions. In this vein, D. Farmer, F. Mezzadri and N. Snaith have studied the zeros of polynomials in W_N as a model for the distribution of zeros of L-functions which do not have an Euler product but nonetheless satisfy the Riemann hypothesis [6]. The second author was introduced to the problem of finding the volume of W_N by D. Farmer.

Likewise, reciprocal polynomials have been of interest in number theory to those studying Mahler measure [9] and those studying abelian varieties over finite fields. S. DiPippo and E. Howe use the volume of the set of monic real polynomials of degree N with all roots on the unit circle to give asymptotic estimates for the number of isogeny classes of N-dimensional abelian varieties over a finite field [4]. A monic polynomial with real coefficients and all roots on the unit circle is necessarily reciprocal and hence the set of all such polynomials of degree N is a set akin to W_N. The set of non-monic reciprocal polynomials with Mahler measure equal to one and degree at most N has been used [8] to give asymptotic estimates for the number of reciprocal polynomials in $\mathbb{Z}[x]$ of degree at most N with Mahler measure bounded by T as $T \to \infty$. The volume of the set of non-monic polynomials in $\mathbb{R}[x]$ of degree N with all roots in the closed unit disk has been used [3] to give asymptotic estimates for the number of polynomials in $\mathbb{Z}[x]$ of degree at most N with Mahler measure bounded by T as $T \to \infty$.

This paper will be largely concerned with the global properties of the coefficient vectors of CR polynomials in W_N. As we shall see, the CR condition together with the condition that all roots lie on the unit circle constrains the geometry and topology of W_N. Here we study the geometry, topology and volume of W_N, not for any particular application, but for their own sake.

1.3 Preliminary Results about W_N

The following proposition gives a useful characterization of elements of W_N based on their roots.

Proposition 1.5 *The vector* $\mathbf{w}$ *is in* W_N *if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^{N}(x - \xi_n),$$

where $\xi_1, \xi_2, \dots, \xi_N$ *are elements of* $\mathbb{T}$ *satisfying* $\xi_1\xi_2\cdots\xi_N = (-1)^N$.

Proof Suppose $\mathbf{w}$ is as in the statement of the proposition. Then

$$x^N\overline{\mathbf{w}(1/\overline{x})} = x^N \prod_{n=1}^{N}\Big(\frac{1 - \overline{\xi_n}x}{x}\Big) = (-1)^N \prod_{n=1}^{N}(\xi_n^{-1}x - 1) = \prod_{n=1}^{N} \xi_n(\xi_n^{-1}x - 1).$$

It follows that $\mathbf{w}(x) = \overline{\mathbf{w}(1/\overline{x})}$, and thus $\mathbf{w} \in W_N$. The converse is obvious since every element of W_N is a polynomial with all roots on the unit circle and constant coefficient 1. ■

In order to exploit both the symmetry of the coefficients of polynomials in W_N and the symmetry of the roots we introduce the map $E_N \colon \mathbb{C}^N \to \mathbb{C}^N$ given by $E_N(\alpha) = \mathbf{b}$ where

$$x^N + \sum_{n=1}^{N} b_{N-n}x^n = \prod_{n=1}^{N}(x - \alpha_n).$$

That is, the n-th coordinate function of $E_N(\alpha)$ is given by $(-1)^n e_n(\alpha_1, \alpha_2, \dots, \alpha_N)$. Clearly, since the coefficients of a monic polynomial are independent of the ordering of the roots, E_N induces a map (which we also call E_N) from $\mathbb{C}^N/S_N$ to $\mathbb{C}^N$, (where S_N is the symmetric group on N letters and $\mathbb{C}^N/S_N$ is the orbit space of $\mathbb{C}^N$ under the action of S_N on the coordinates of α.) The torus $\mathbb{T}^N$ sits in $\mathbb{C}^N$, and thus E_N gives a correspondence between the $\mathbb{T}^N/S_N$ and the set of polynomials of degree N in $\mathbb{C}[x]$ with all roots on the unit circle. We define Ω_N to be the subset of $\mathbb{T}^N/S_N$ given by

$$\Omega_N = t\{(\xi_1, \xi_2, \dots, \xi_N) : \xi_1\xi_2\cdots\xi_N = (-1)^N\}.$$

If $\mathbf{w} = E_N(\xi)$, then ξ will be referred to as a *root vector* of $\mathbf{w}(x)$. Clearly, by Proposition 1.5, the map E_N induces a homeomorphism between Ω_N and W_N.

We now turn to the structure of W_N viewed as a subset of $\mathbb{R}^{N-1}$.

Proposition 1.6 *W_N is a closed path connected set with positive volume. Moreover, the boundary of W_N is given by* $\partial W_N = \{\mathbf{w} \in W_N : \operatorname{disc}(\mathbf{w}) = 0\}$, *where* $\operatorname{disc}(\mathbf{w})$ *is the discriminant of* $\mathbf{w}(x)$.

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Proof Suppose $\mathbf{a} \in \mathbb{R}^{N-1}$, and $\mathbf{a} \notin W_N$. Then, by (1.3), there exists $\alpha \in \Delta$ such that $(x - \alpha)(x - 1/\overline{\alpha})$ is a factor of $\mathbf{a}(x)$. Since the root vector of a polynomial is a continuous function of the roots, there must exist some open neighborhood U of $\mathbf{a}$ such that $U \cap W_N = \varnothing$. It follows that W_N is closed.

Now suppose $\mathbf{w} \in W_N$ has a double root, that is, there exists $\xi_1, \xi_2, \dots, \xi_{N-1} \in \mathbb{T}$ such that $\mathbf{w}(x) = (x - \xi_1)^2(x - \xi_2) \cdots (x - \xi_{N-1})$. If U is an open neighborhood of $\mathbf{w}$, then, since the coefficients of a polynomial are continuous functions of the roots, there exists $\epsilon > 0$ such that $(x - \epsilon\xi_1)(x - \epsilon^{-1}\xi_1)(x - \xi_2) \cdots (x - \xi_{N-1})$ is a CR polynomial in $U \setminus W_N$. That is, $\mathbf{w} \in \partial W_N$. It follows that ∂W_N consists of polynomials with $\operatorname{disc} \mathbf{w} = 0$.

To see that W_N has positive volume, let $\mathbf{v}$ be in $\mathring{W}_N$, the interior of W_N. Then $\operatorname{disc}(\mathbf{v}) \neq 0$ and since the discriminant of a polynomial is a continuous function of the coefficients, there must exist an open neighborhood U of $\mathbf{v}$ such that all polynomials in U have non-zero discriminant. Thus $U \subset \mathring{W}_N$ and W_N has positive volume.

To see that W_N is path connected, let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two points in W_N. Marking $N - 1$ roots in each $\mathbf{w}_1$ and $\mathbf{w}_2$, it is clear that there is a continuous map between the $N - 1$ dimensional root vectors where all roots are in $\mathbb{T}$. By stipulating that the final roots are assigned to satisfy the CR condition, we see that this extends to a continuous map with image in W_N from $\mathbf{w}_1$ to $\mathbf{w}_2$. ∎

We now turn to the geometry of W_N. Our first result in this direction is identifying those points in W_N which are farthest from the origin. (The origin corresponds to the CR polynomial $x^N + 1$.)

Proposition 1.7 *If $\mathbf{w} \in W_N$, then $\|\mathbf{w}\|^2 \leq \binom{2N}{N} - 2$. Moreover, there is equality in this equality if and only if $\mathbf{w}(x) = (x + \zeta_N)^N$, where ζ_N is an N-th root of unity.*

Proof Suppose $\mathbf{w} \in W_N$ and that $\mathbf{w}(x) = \prod_{n=1}^{N}(x - \xi_n)$, for $\xi_1, \xi_2, \dots, \xi_N \in \mathbb{T}$. Let $\mathbf{b} = X_N\mathbf{w}$ and notice that

$$|b_n|^2 = \Big| \sum_{\mathbf{i} \in \mathcal{I}(N,n)} \Big\{ \prod_{m=1}^{n} \xi_{i_m} \Big\} \Big|^2,$$

where $\mathcal{I}(N, n) = \{\mathbf{i} \in \mathbb{Z}^n : 1 \leq i_1 < i_2 < \cdots < i_n \leq N\}$. Clearly then, $|b_n|^2 \leq \binom{N}{n}^2$ with equality when $\prod_{m=1}^{n} \xi_{i_m} = \prod_{m=1}^{n} \xi_{j_m}$ for every choice of $\mathbf{i}, \mathbf{j} \in \mathcal{I}(N, n)$. That is, there is equality exactly when $\xi_1 = \xi_2 = \cdots = \xi_N$. In this case, Proposition 1.5 implies that $\xi_1^N = (-1)^N$, which is only satisfied when $\xi_1 = \xi_2 = \cdots = \xi_N = \zeta_N$ for some N-th root of unity.

It follows that

$$\|\mathbf{w}\|^2 = \|\mathbf{b}\|^2 \leq \sum_{n=1}^{N-1} \binom{N}{n}^2 = \binom{2N}{N} - 2,$$

where the last equality comes from the well-known formula for the sum of the squares of binomial coefficients. By our previous remarks, equality is attained in the inequality only when $\mathbf{w}(x)$ is a polynomial of the form $(x + \zeta_N)^N$. ∎

This proposition gives us a hint of the geometric structure of W_N. Let $\zeta_N = e^{2\pi i/N}$, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N \in \mathbb{R}^{N-1}$ be determined by setting $\mathbf{v}_n(x) = (x + \zeta_N^n)^N$. In particular, $\mathbf{v}_N(x) = (x+1)^N$. For reasons which will become clear, we will call $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ the *vertices* of W_N. As we will see, every isometry of W_N must fix the origin. This together with Proposition 1.7 implies that the set of isometries of W_N must permute its vertices. We will also prove that the vertices span $\mathbb{R}^{N-1}$. It follows that if two isometries T_1 and T_2 induce the same permutation of the vertices, the isometry $T_1 T_2^{-1}$ fixes every vertex. Therefore, since $T_1 T_2^{-1}$ extends to an isometry that fixes a spanning set for R^{N-1}, it is the identity. Therefore, the group of isometries of W_N is isomorphic to a subgroup of S_N, as each isometry is uniquely determined by the permutation it induces on the set of vertices. In fact, Theorem 1.3 shows that this group is isomorphic to D_N, the group of isometries of a regular N-gon.

At present, for R and C defined as in Theorem 1.1, we will demonstrate the following.

Proposition 1.8 *R and C are isometries of W_N.*

We defer the proof that they generate the complete group of isometries until Section 3.2.

Proof We use $\|\cdot\|$ to denote the usual 2-norm on both $\mathbb{R}^{N-1}$ and $\mathbb{C}^{N-1}$. Given $\mathbf{w}_1, \mathbf{w}_2 \in W_N$, it is easily verified that $\|X_N \mathbf{w}_1\| = \|\mathbf{w}_1\|$. Thus, since $X_N\mathbf{w}_1 - X_N\mathbf{w}_2$ is the coefficient vector of a polynomial, Parseval's formula yields

$$\|\mathbf{w}_1 - \mathbf{w}_2\|^2 = \|X_N\mathbf{w}_1 - X_N\mathbf{w}_2\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\mathbf{w}_1(e^{i\theta}) - \mathbf{w}_2(e^{i\theta})\right|^2 \, d\theta. \tag{1.4}$$

Notice that $R \cdot \mathbf{w}_1(x) = \mathbf{w}_1(\zeta_N^{-1}x)$, and hence

$$\begin{aligned}\|R\cdot\mathbf{w}_1 - R\cdot\mathbf{w}_2\|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\left|R\cdot\mathbf{w}_1(e^{i\theta}) - R\cdot\mathbf{w}_2(e^{i\theta})\right|^2 \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\mathbf{w}_1(\zeta_N^{-1}e^{i\theta}) - \mathbf{w}_2(\zeta_N^{-1}e^{i\theta})\right|^2 \, d\theta = \|\mathbf{w}_1 - \mathbf{w}_2\|^2,\end{aligned}$$

where the last equation follows from an easy change of variables. Similarly, $C \cdot \mathbf{w}_1(x) = \overline{\mathbf{w}_1(\overline{x})}$ and thus,

$$\begin{aligned}\|C\cdot\mathbf{w}_1 - C\cdot\mathbf{w}_2\|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\left|C\cdot\mathbf{w}_1(e^{i\theta}) - C\cdot\mathbf{w}_2(e^{i\theta})\right|^2 \, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\mathbf{w}_1(e^{-i\theta}) - \mathbf{w}_2(e^{-i\theta})\right|^2 \, d\theta = \|\mathbf{w}_1 - \mathbf{w}_2\|^2.\end{aligned}$$

It follows that R and C are isometries of W_N. ■

1.4 The Colouring of W_4

As mentioned previously, every $\mathbf{w} \in W_N$ uniquely determines a partition $\mathcal{P}(\mathbf{w})$ of N up to cyclic ordering, corresponding to the multiplicities of the cyclically ordered roots of $\mathbf{w}(x)$. That is, we may decompose W_N into regions, *faces* if you will, determined by partitions of the integer N.

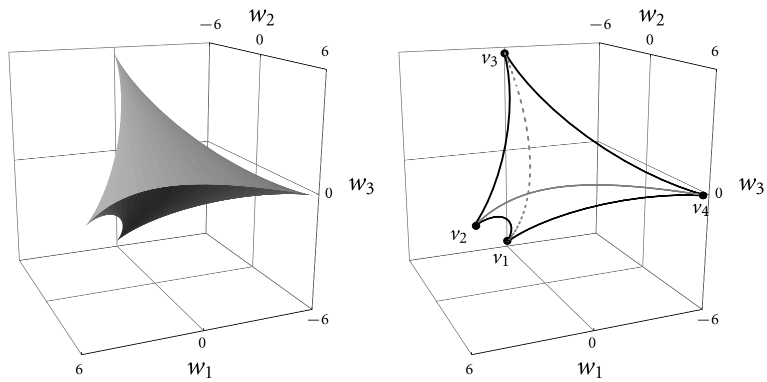

Figure 2: The faces of W_4

For example, Figure 2 demonstrates the decomposition of W_4 into a coloured simplex. The interior is associated with the partition $(1, 1, 1, 1)$. The four vertices, given by $\mathbf{v}_k(x) = (x - i^k)^4$ for $1 \le k \le 4$, each correspond to the partition (4). Likewise, there are four faces, each associated with the partition $(2, 1, 1)$. In contrast, there are two different partitions of 4 of length two: $(2, 2)$ and $(3, 1)$. Each of these partitions corresponds to faces of codimension 2, that is, *edges* of W_4. There are six edges, four of type $(3, 1)$ (black curves on the right in Figure 2) and two of type $(2, 2)$ (gray curves). One of the $(2, 2)$ edges joins $\mathbf{v}_2$ with $\mathbf{v}_4$ and the other joins $\mathbf{v}_1$ with $\mathbf{v}_3$.

To illustrate the difference between edges corresponding with the partition $(2, 2)$ and edges corresponding to $(3, 1)$, consider a path starting at $\mathbf{v}_4$ and traversing an edge to another vertex. The polynomial corresponding to $\mathbf{v}_4$ has a root of multiplicity four at $x = 1$. The edge associated with the partition $(2, 2)$ consists of polynomials with two double roots. If we imagine one of these double roots starting at $x = 1$ and traversing the unit circle in the counterclockwise direction, then the other double root must traverse the circle in the clockwise direction. Moreover, these double roots must traverse the circle at the same rate so that each intermediate polynomial on the $(2, 2)$ edge has constant coefficient equal to 1. A moment's thought reveals that the only other vertex which can be reached from $\mathbf{v}_4$ via a $(2, 2)$ edge corresponds to the polynomial with a root of multiplicity four at $x = -1$, *i.e.*, $\mathbf{v}_2$. The other $(2, 2)$ edge connects $\mathbf{v}_1$ with $\mathbf{v}_3$. On the other hand, if we look at an edge starting at $\mathbf{v}_4$ formed by dividing the root of multiplicity four into a root of multiplicity three traversing the circle in the counterclockwise direction and a single root traversing the circle in the clockwise direction, the single root must traverse the circle at a rate three times that of the triple root so that the intermediate polynomials have constant coefficient equal to 1. We see that when the triple root has reached $x = i$, then the single root, too, is at $x = i$ and thus a $(3, 1)$ edge connects $\mathbf{v}_4$ with $\mathbf{v}_1$. Similarly, when the triple root reaches $x = -1$, then so, too, has the single root, so that a $(3, 1)$ edge connects $\mathbf{v}_1$ with $\mathbf{v}_2$. Continuing in this manner, we see that any two vertices can be connected via an edge consisting of polynomials with a triple root and a single root.

The group generated by R acts transitively on both the set of $(2, 2)$ edges and the set of $(3, 1)$ edges. This, coupled with the action of C, demonstrates that there is an isometry sending any edge to itself which interchanges the vertex endpoints. Thus the curvature of each edge is symmetric. No isometry of W_N carries a $(2, 2)$ edge to a $(3, 1)$ edge. Indeed, the curvature of the $(2, 2)$ edges differs from the curvature of the $(3, 1)$ edges. This reflects the fact that $\operatorname{Isom}(W_4) \cong D_4$, not the full symmetric group.

2 The Topology of W_N

2.1 The Proof of Theorem 1.1

Recall that W_N is homeomorphic to Ω_N where Ω_N is the subset of $\mathbb{T}^N/S_N$ given by

$$\Omega_N = \{(\xi_1, \xi_2, \dots, \xi_N) : \xi_1\xi_2\cdots\xi_N = (-1)^N\}.$$

Let $c = 0$ if N is even and $c = 1/2$ if N is odd. After reparametrization by $\xi_n = e^{2\pi i\vartheta_n}$ we have

$$W_N \cong \Big\{ (\vartheta_1, \vartheta_2, \dots, \vartheta_N) : \Big\{ \sum_{n=1}^{N} \vartheta_n \Big\} = c \Big\}, \tag{2.1}$$

where $\vartheta_n \in \mathbb{R}/\mathbb{Z}$, for $1 \le n \le N$, and in this context $\{x\}$ denotes the fractional part of x. We will denote the set on the right-hand side of (2.1) by Θ_N, and the interior of this set by $\mathring{\Theta}_N$. Throughout this discussion, we will continue to refer to the ϑ_n as roots, and we will denote the reparameterized torus $\mathbb{R}/\mathbb{Z}$ as $\mathbb{T}_+$. (The subscript reflects the fact that in this context we are working with the additive torus.)

First, we will prove that the interior of W_N is homeomorphic to an open $N-1$ ball. Define $\Phi\colon \mathring{\Theta}_N \to \mathbb{T}_+^{N-1}/S_{N-1}$ as follows. Fix a basepoint $\psi = (\psi_1, \dots, \psi_N) \in \mathring{\Theta}_N$. Since ψ corresponds to a polynomial in the interior of W_N, by Proposition 1.6, all coordinates of ψ are unique. Fix some $n \in \{1, \dots, N\}$. We define $\Phi(\psi) = (\psi_1, \dots, \hat{\psi}_n, \dots \psi_N)$, where $\hat{\psi}_n$ means that ψ_n is omitted. We can extend Φ in a neighborhood of ψ since the roots vary continuously. Specifically, for $\vartheta = (\vartheta_1, \dots, \vartheta_N)$, $\Phi(\vartheta) = (\vartheta_1, \dots, \hat{\vartheta}_l, \dots, \vartheta_N)$ where ϑ_l is the closest root to ψ_n on $\mathbb{T}_+$. In fact, for any $\vartheta \in \mathring{\Theta}_N$, given a path in $\mathring{\Theta}_N$ from ψ to ϑ, we can extend Φ to ϑ due to the continuity of the roots and the fact that there are no multiple roots in the interior of W_N.

To define Φ on all of $\mathring{\Theta}_N$ it is enough to show that Φ does not depend on the choice of path. It suffices to show that if $\gamma\colon [0, 1] \to \mathring{\Theta}_N$ is a continuous loop based at ϑ, then $\Phi \circ \gamma(0) = \Phi \circ \gamma(1)$. Since there are no double roots in $\gamma(t)$, the continuous orbit of the roots of ϑ on $\mathbb{T}_+$ by γ is homotopic to a rotation. As such, there are continuous functions $\tau_n \colon [0,1] \to \mathbb{R}$ such that $\gamma(t) = (\vartheta_1 + \tau_1(t), \dots, \vartheta_N + \tau_N(t))$. The conjugate reciprocal condition translates to the condition

$$\Big\{ \sum_{n=1}^{N} (\vartheta_n + \tau_n(t)) \Big\} = c,$$

where c is as before. As $\{\sum_{n=1}^{N} \vartheta_n\} = c$, we conclude that $\{\sum_{n=1}^{N} \tau_n(t)\} = 0$.

Fixing representatives $\vartheta_1, \dots, \vartheta_N$ in $[0,1)$, we define $\sigma \colon [0,1] \to \mathbb{R}$ by

$$\sigma(t) = \sum_{n=1}^{N} (\vartheta_n + \tau_n(t)).$$

The continuity of σ follows from the continuity of the τ_n. Since γ is homotopic to a rotation, there is a unique $1 \le k' \le N$ such that $\gamma(\vartheta_k) = \vartheta_{k+k' \bmod N}$ for $1 \le k \le N$. It follows that $\sigma(1) = \sigma(0) + |k'|$. As $\{\sigma(t)\} = 0$ for all $t \in [0,1]$, this contradicts the continuity of σ, unless $k' \equiv 0 \bmod N$. Hence Φ is well defined on $\mathring{\Theta}_N$ and can be extended to all of Θ_N. Moreover, Φ is injective, since if $\Phi(\vartheta) = \Phi(\vartheta')$, then ϑ and ϑ' share $N-1$ roots and by the conjugate reciprocal condition, $\vartheta = \vartheta'$. The conjugate reciprocal condition allows us to define an inverse of Φ and, as Φ is continuous, we conclude that $\mathring{\Theta}_N$ is homeomorphic onto a subset of $\mathbb{T}_+^{N-1}/S_{N-1}$.

We now shift our attention to $\partial\Theta_N$; first we introduce some notation. Given positive integers M, m and n such that $1 \le n < m \le M$, define

$$L_m^M = \Big\{ (\vartheta_1, \dots, \vartheta_M) \in \mathbb{T}_+^M : \Big\{ 2\vartheta_n + \sum_{l=1, l\neq m}^{M} \vartheta_l \Big\} = c \Big\},$$

$$K_{n,m}^M = \{(\vartheta_1, \dots, \vartheta_M) \in \mathbb{T}_+^M : \vartheta_n = \vartheta_m\}.$$

We then set

$$L^M = \bigcup_{m=1}^{M} L_m^M, \quad K^M = \bigcup_{1 \le n < m \le M} K_{n,m}^M, \quad \text{and} \quad J^M = L^M \cup K^M.$$

Clearly L^M, K^M and J^M are stabilized by S_M, and thus (for instance) $(\mathbb{T}_+^M \setminus J^M)/S_M = (\mathbb{T}_+^M/S_M) \setminus (J^M/S_M)$. Returning to Φ, if a point in J^{M-1}/S_{N-1} is in the image of Φ, then its preimage necessarily has multiplicity at least two. Therefore $\Phi(\mathring{\Theta}_N)$ is a subset of $(\mathbb{T}_+^{N-1} \setminus J^{N-1})/S_{N-1}$ and Φ maps the boundary of Θ_N into J^{N-1}/S^{N-1}.

For any positive M, the removal of a hyperplane from $\mathbb{T}_+^M$ corresponds to the removal of a generator in the fundamental group. Since the hyperplanes $L_{m,n}^M$ are linearly independent, the fundamental group of $\mathbb{T}_+^M \setminus L^M$ is trivial. We conclude that the set $\mathbb{T}^M \setminus L^M$ is homeomorphic to a disjoint union of open M-balls. Therefore, the deletion from $\mathbb{T}_+^M \setminus L^M$ of the set of hyperplanes K^M is also homeomorphic to a disjoint union of open M-balls. Figure 3 shows one of these open balls in $\mathbb{T}_+^2 \setminus J^2$. The action by S_M on $\mathbb{T}_+^M$ is given by reflections through the $K_{m,n}^M$. As such, since K^M

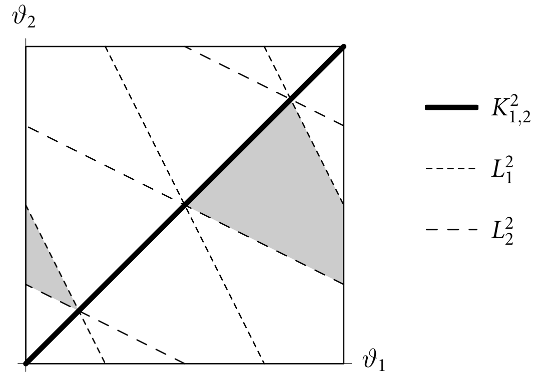

Figure 3: An open 1-ball in $\mathbb{T}^2_+ \setminus J^2$

and L^M are stabilized by this action, any reflection through $K^M_{m,n}$ must map a ball onto another ball. The image of $\mathring{\Theta}_N$ under Φ must therefore be one of these balls as $\partial\Theta_N$ maps into J^{N-1}, and we conclude that $\Phi(\Theta_N)$ is homeomorphic to a closed $(N-1)$-ball. Since Φ is a homeomorphism, we conclude that Θ_N and therefore W_N is homeomorphic to a closed $(N-1)$-ball.

2.2 The Proof of Corollary 1.2

Let $\mathcal{P}$ be a partition of N, *i.e.*, $\mathcal{P} = (n_1, \dots, n_M)$ is a vector of positive integers such that $n_1 + n_2 + \cdots + n_M = N$. Let $|\mathcal{P}| = M$ denote the length of the partition. By associating the cyclically ordered roots of $\mathbf{w}(x)$ with their multiplicities, $\mathbf{w} \in W_N$ determines a partition $\mathcal{P}(\mathbf{w})$ of N which is well defined up to cyclic ordering. (We will consider all partitions only up to cyclic ordering.) If $\mathcal{P} = (n_1, \dots, n_M)$ and $\mathcal{P}' = (n_1 + n_2, \dots, n_M)$, then we say that $\mathcal{P}'$ is obtained from $\mathcal{P}$ by reduction. Notice that reduction gives a partial ordering on the set of cyclically ordered partitions, and if the partition $\mathcal{P}''$ is obtained from $\mathcal{P}$ by a series of reductions we will write $\mathcal{P}'' \preceq \mathcal{P}$. We call each partition of N a colouring, and colour W_N according to the partition type of each $\mathbf{w} \in W_N$.

If $\mathbf{w} \in \partial W_N$ and $\mathcal{P}(\mathbf{w}) = (n_1, n_2, \dots, n_M)$, then there exist distinct $\vartheta_1, \vartheta_2, \dots, \vartheta_M \in \mathbb{T}_+$ such that $\mathbf{w}(x) = (x - e^{2\pi i\vartheta_1})^{n_1}(x - e^{2\pi i\vartheta_2})^{n_2} \cdots (x - e^{2\pi i\vartheta_M})^{n_M}$. Letting the ϑ_m vary, the continuity of the elementary symmetric functions implies that

$$\{(\vartheta_1, \vartheta_2, \dots, \vartheta_M) : \{n_1\vartheta_1 + n_2\vartheta_2 + \cdots + n_M\vartheta_M\} = c\}$$

locally parameterizes an $M-1$ dimensional ball in ∂W_N containing $\mathbf{w}$. Let $F(\mathbf{w})$ be the maximal connected subset of ∂W_N containing $\mathbf{w}$ such that if $\mathbf{x} \in F$, then $\mathcal{P}(\mathbf{x}) \preceq \mathcal{P}(\mathbf{w})$. We fix $1 \le n \le M$ and define $\Phi(\vartheta) = (\vartheta_1, \dots, \hat{\vartheta}_n, \dots, \vartheta_M)$. As in the proof of Theorem 1.1, Φ has a unique extension to F and we conclude that F is homeomorphic to a closed $M-1$ dimensional ball.

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The proof of Theorem 1.1 shows that $\Phi(\Theta_N)$ is homeomorphic to a simply connected subset of $\mathbb{T}_+^{N-1}$ bounded by hyperplanes. This is, in turn, isometric to a subset U of $\mathbb{R}^{N-1}$ bounded by hyperplanes, the boundary of which corresponds to $(\Phi(\Theta_N) \cap J^{N-1})/S_{N-1}$. It follows that U has the geometric structure of a polytope, and hence each $u \in U$ lies in the interior of a face of U. Moreover, the points in the boundary of U correspond to $\mathbf{w} \in W_N$ with $\operatorname{disc}(\mathbf{w}) = 0$. If $u \in U$ is in the intersection of M of these hyperplanes, *i.e.,* u is on a codimension M face, then u corresponds with a $\mathbf{w} \in W_N$ with $|\mathcal{P}(\mathbf{w})| = N - M$. In fact, this face is the image of $F(\mathbf{w})$ as defined above. Under this correspondence, we see that there are N vertices of the polytope, confirming that U has the structure of a simplex. It follows that W_N has the structure of a coloured simplex. Moreover, given any $\mathbf{w} \in W_N$ in the interior of a face F, and any $\mathbf{u} \in \partial F$, $\mathcal{P}(\mathbf{u}) \preceq \mathcal{P}(\mathbf{w})$.

3 The Geometry of W_N

Before proving Theorem 1.3 we need some results about the vertices of W_N.

3.1 The Vertices of W_N

Lemma 3.1 *The vertices of W_N span $\mathbb{R}^{N-1}$.*

Proof Let U be the $N-1 \times N-1$ matrix whose rows are given by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N-1}$. From the definition of $\mathbf{v}_n$ it is seen that the m, n entry of U is given by

$$U[m,n] = \begin{cases} \sqrt{2}\binom{N}{n}\cos\left(\frac{2\pi nm}{N}\right) & \text{if } 1 \le n < N/2, \\ \binom{N}{N/2}(-1)^{m-1} & \text{if } n = N/2, \\ [3pt]\sqrt{2}\binom{N}{n}\sin\left(\frac{2\pi(N-n)m}{N}\right) & \text{if } N/2 < n < N. \end{cases}$$

Hence,

$$\det U = \left(\prod_{n=1}^{\lfloor N/2 \rfloor}\binom{N}{n}\right)\left(\prod_{n=1}^{\lfloor (N-1)/2 \rfloor}\binom{N}{n}\right)\det U', \tag{3.1}$$

where U' is the $N-1 \times N-1$ matrix whose m, n entry is given by

$$U'[m,n] = \begin{cases} \frac{\sqrt{2}}{2}\left(\mathrm{e}\left(\frac{nm}{N}\right) + \mathrm{e}\left(\frac{(N-n)m}{N}\right)\right) & \text{if } 1 \le n < N/2, \\ \mathrm{e}\left(\frac{m}{2}\right) & \text{if } n = N/2, \\ [3pt]\frac{\sqrt{2}}{2}\left(i\,\mathrm{e}\left(\frac{nm}{N}\right) - i\,\mathrm{e}\left(\frac{(N-n)m}{N}\right)\right) & \text{if } N/2 < n < N, \end{cases}$$

and $\mathrm{e}(t) = e^{2\pi i t}$. That is, if we define the $N-1 \times N-1$ matrix V by $V[m,n] = \mathrm{e}(mn/N)$, then

$$U' = X_N^* V, \tag{3.2}$$

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where X_N^* is the conjugate transpose of X_N as defined in (1.2). This is convenient, since $|\det X_N| = 1$ and $|\det V| = |\det V'|$ where V' is the $N-1 \times N-1$ matrix given by

$$V'[m,n] = e^{\frac{2\pi imn}{N}-1}.$$

That is, V' is the Vandermonde matrix formed from the complex numbers

$$e(1/N), e(2/N), \dots, e((N-1)/N).$$

The well-known relationship between Vandermonde determinants and discriminants implies that $|\det V'| = |\operatorname{disc}((x^N-1)/(x-1))|^{1/2}$. This together with (3.1) and (3.2) yield

$$|\det U| = \Bigl(\prod_{n=1}^{\lfloor N/2\rfloor}\binom{N}{n}\Bigr)\Bigl(\prod_{n=1}^{\lfloor (N-1)/2\rfloor}\binom{N}{n}\Bigr)\Bigl|\operatorname{disc}\Bigl(\frac{x^N-1}{x-1}\Bigr)\Bigr|^{1/2}.$$

Since the discriminant of $(x^N-1)/(x-1)$ is nonzero, we conclude $\det U \neq 0$ and that the vertices of W_N span $\mathbb{R}^{N-1}$. ■

Lemma 3.2 $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_N = 0.$

Proof We use the fact that if $\zeta \neq 1$ is any N-th root of unity, then $\zeta + \zeta^2 + \dots + \zeta^N = (\zeta^N - 1)/(\zeta - 1) = 0$. If $0 < M < N$, then the coefficient of x^{N-M} in $\mathbf{v}_n(x)$ is given by $\binom{N}{M}\zeta_N^{nM}$. It follows that

$$\sum_{n=1}^{N}\binom{N}{M}\zeta_N^{nM} = \binom{N}{M}\sum_{n=1}^{N}(\zeta_N^M)^n = 0.$$

And since this is the coefficient of x^{N-M} in the polynomial $\mathbf{v}_1(x)+\mathbf{v}_2(x)+\dots+\mathbf{v}_N(x)$, we conclude that $\frac{1}{N}(\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_N)(x) = x^N + 1 = 0(x)$, which establishes the lemma. ■

Lemma 3.3 *Let* $1 < m, m' < N$. *Then* $\|\mathbf{v}_m - \mathbf{v}_{m'}\|^2 = \|\mathbf{v}_N - \mathbf{v}_k\|^2$ *where* $k = |m' - m|$. *Moreover,*

$$\|\mathbf{v}_N - \mathbf{v}_2\|^2 < \|\mathbf{v}_N - \mathbf{v}_4\|^2 < \dots < \|\mathbf{v}_N - \mathbf{v}_M\|^2 < 2\binom{2N}{N},$$

$$\|\mathbf{v}_N - \mathbf{v}_1\|^2 > \|\mathbf{v}_N - \mathbf{v}_3\|^2 > \dots > \|\mathbf{v}_N - \mathbf{v}_{M'}\|^2 > 2\binom{2N}{N},$$

where M *is the greatest even integer not exceeding* $N/2$ *and* M' *is the greatest odd integer not exceeding* $N/2$.

Proof First we notice that $\|\mathbf{v}_m - \mathbf{v}_{m'}\|^2 = \|\mathbf{v}_N - \mathbf{v}_k\|^2$ since the isometry R^{N-m} takes $\mathbf{v}_m$ to $\mathbf{v}_N$ and $\mathbf{v}'_m$ to $\mathbf{v}_k$. By (1.4),

$$\|\mathbf{v}_N - \mathbf{v}_k\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\bigl|(e^{i\theta}+1)^N - (e^{i\theta}+\zeta_N^k)^N\bigr|^2\,d\theta. \tag{3.3}$$

Now, if $-\pi < \theta < \pi$, then $\arg(e^{i\theta} + 1) = \theta/2$, and thus

$$\arg(e^{i\theta} + \zeta_N^k) = \arg(\zeta_N^k(1 + e^{i\theta}\zeta_N^{-k})) = \frac{\theta}{2} + \frac{\pi k}{N}.$$

It follows that $\arg\big((e^{i\theta} + \zeta_N^k)^N\big) = (-1)^k \arg((e^{i\theta} + 1)^N)$. That is, for fixed θ, $(e^{i\theta} + \zeta_N^k)^N$ and $(e^{i\theta} + 1)^N$ lie on a line passing through the origin, from which we conclude

$$\big|(e^{i\theta} + 1)^N - (e^{i\theta} + \zeta_N^k)^N\big|^2 = \big(|(e^{i\theta} + 1)^N| + (-1)^{k+1}|(e^{i\theta} + \zeta_N^k)^N|\big)^2. \tag{3.4}$$

Squaring out the right-hand side of (3.4) and substituting into (3.3), we find

$$\begin{aligned} \|\mathbf{v}_N - \mathbf{v}_k\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |(e^{i\theta} + 1)^N|^2\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} |(e^{i\theta} + \zeta_N^k)^N|^2\,d\theta \\ &\quad + (-1)^{k+1}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi} |(e^{i\theta} + 1)^N||(e^{i\theta} + \zeta_N^k)^N|\,d\theta\Big) \qquad (3.5)\\ &= 2\binom{2N}{N} + (-1)^{k+1}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi} |(e^{i\theta} + 1)^N||(e^{i\theta} + \zeta_N^k)^N|\,d\theta\Big), \qquad (3.6)\end{aligned}$$

where the binomial coefficient in (3.6) arises from an application of Parseval's formula to the first two integrals in (3.5) together with the familiar formula for the sum of the squares of the binomial coefficients.

Next we define $f(\theta) = |(e^{i\theta} + 1)^N|$, and we notice that $f(-\theta) = f(\theta)$ and $f(2\pi k/N - \theta) = |(e^{i\theta} + \zeta_N^k)|^N$. Consequently, we can write the integral in (3.6) as $f * f(2\pi k/N)$ where

$$f * f(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) f(t - \theta)\,d\theta.$$

The continuity of f implies that $f * f$ is itself continuous, and likewise the fact that f is even gives $f * f(-t) = f * f(t)$. The lemma will be proved by showing that $f * f(t)$ is increasing on $(-\pi, 0)$ and decreasing on $(0, \pi)$. In order to do this, we use the elementary fact that $f(\theta) = (2 + 2\cos\theta)^{N/2}$ and hence f is increasing on $(-\pi, 0)$ and decreasing on $(0, \pi)$. Moreover, f is differentiable on $(-\pi, \pi)$. Consequently, we may write

$$(f * f)'(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(\theta) f(t - \theta)\,d\theta = \frac{1}{\pi}\int_{0}^{\pi} f'(\theta)\{f(t - \theta) - f(t + \theta)\}\,d\theta,$$

where the latter equation holds since f is even.

Notice that for fixed t, $f(t - \theta) - f(t + \theta) = 0$ if and only if $\theta = 0$ or $\theta = \pi$, and $f'(\theta) < 0$ on $(0, \pi)$. It follows that, for fixed t, the integrand never changes sign. When $0 < t < \pi$ and $\theta = t/2$ the integrand is negative. From which we conclude that $(f * f)'(t) < 0$ on $(0, \pi)$. Similarly $(f * f)'(t) > 0$ on $(-\pi, 0)$, which establishes the lemma. ■

3.2 The Proof of Theorem 1.3

We begin by proving that every isometry of W_N fixes the origin.

Let D be the distance between 0 and a vertex of W_N. Let B denote the ball of radius D centred at 0 in $\mathbb{R}^{N-1}$, and let $S = \partial B$. Therefore S is the smallest sphere in $\mathbb{R}^{N-1}$ centred at 0 that circumscribes W_N. Notice that by Proposition 1.7, $\|\mathbf{w} - 0\| \leq D$ for all $\mathbf{w} \in W_N$, with equality precisely when $\mathbf{w}$ is a vertex. Let T be an isometry of W_N. If $T(0) = \mathbf{w}$ for some non-zero $\mathbf{w}$ in W_N, then $\|\mathbf{w} - \mathbf{w}'\| \leq D$ for all $\mathbf{w}' \in W_N$, and in particular, $\|\mathbf{w} - \mathbf{v}_n\| \leq D$ for $1 \leq n \leq N$. $T(B)$ is the ball of radius D centred at $\mathbf{w}$, and $W_N \subset B \cap T(B)$. Moreover, the vertices of W_N must lie in $S \cap T(B)$, and since $S \cap T(B)$ lies in the half space $\{\mathbf{u} \in \mathbb{R}^{N-1} : \mathbf{w} \cdot \mathbf{u} > 0\}$, we must have $\mathbf{w} \cdot \mathbf{v}_n > 0$ for $1 \leq n \leq N$. But this is a contradiction since, by Lemma 3.2,

$$0 = \mathbf{w} \cdot \Big(\sum_{n=1}^{N} \mathbf{v}_n \Big) = \sum_{n=1}^{N} \mathbf{w} \cdot \mathbf{v_n} > 0.$$

We conclude there is no such $\mathbf{w}$, and that every isometry of W_N fixes 0.

Since every isometry fixes the origin, Proposition 1.7 implies that any isometry must permute the vertices of W_N. Since R is transitive on the vertices, composing T with the appropriate power of R fixes $\mathbf{v}_N$. By Lemma 3.3, $\mathbf{v}_1$ and $\mathbf{v}_{n-1}$ are the unique vertices farthest from $\mathbf{v}_N$. Therefore, either $\mathbf{v}_1$ is fixed or it is exchanged with $\mathbf{v}_{N-1}$. Recall that $\mathbf{v}_m$ corresponds to $(x+\zeta_N^m)^N$, so conjugation exchanges $\mathbf{v}_1$ and $\mathbf{v}_{N-1}$. Composition with C, if necessary, results in an isometry S that fixes $\mathbf{v}_N$ and $\mathbf{v}_1$.

We will now show that any isometry that fixes both $\mathbf{v}_N$ and $\mathbf{v}_1$ is the identity. As R and C are in D_N, this will imply that $T \in D_N$, establishing the result. As the farthest vertices from $\mathbf{v}_1$ are $\mathbf{v}_N$ and $\mathbf{v}_2$, and S fixes $\mathbf{v}_1$, we conclude that $S(\mathbf{v}_2) = \mathbf{v}_2$ as S fixes $\mathbf{v}_N$. Continuing in this fashion, we see that S fixes all vertices. S fixes the origin, and therefore, if we let ℓ_n be the line segment connecting 0 and $\mathbf{v}_n$, S fixes ℓ_n pointwise. As a result, S fixes the span of the vertices, which is $\mathbb{R}^{N-1}$ by Lemma 3.1. We conclude that S is the identity.

4 The Volume of W_N

4.1 ω-Conjugate Reciprocal Polynomials

We will appeal to results and methods from random matrix theory in order to determine the volume of W_N. If λ_{N-1} is Lebesgue measure on $\mathbb{R}^{N-1}$, then

$$\text{vol}(W_N) = \lambda_{N-1}(W_N) = \int_{W_N} d\lambda_{N-1}(\mathbf{w}). \tag{4.1}$$

Our basic strategy will be to view elements of W_N as polynomials and employ a change of variables so that we are integrating over the roots of CR polynomials as opposed to the coefficients.

Along these lines, we must enlarge the set of polynomials under investigation. Given a fixed $\omega \in \mathbb{T}$, we say the degree N polynomial $f(x)$ is *ω-conjugate reciprocal* (or ω-CR) if $f(x) = \omega x^N \overline{f(1/\overline{x})}$. If a polynomial is ω-CR for some $\omega \in \mathbb{T}$, then it is

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also called *self-inversive*. In analogy with CR-polynomials, we define the $N-1\times N-1$ matrix $X_{N,\omega}$ by

$$X_{N,\omega}[j,k] = \begin{cases} \frac{\sqrt{2}}{2}(\delta_{j,k} + \omega\delta_{N-j,k}) & \text{if } 1 \le j < N/2, \\ \sqrt{\omega}\delta_{j,k} & \text{if } j = N/2, \\ \frac{\sqrt{2}}{2}(i\delta_{N-j,k} - i\omega\delta_{j,k}) & \text{if } N/2 < j < N, \end{cases} \tag{4.2}$$

for a fixed branch of the square root. It follows that if $\mathbf{a} \in \mathbb{R}^{N-1}$ and $\mathbf{c} = X_{N,\omega}\mathbf{a}$, then $(x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n}$ is ω-CR. We then define $W_{N,\omega}$ to be

$$W_{N,\omega} = \Big\{ \mathbf{a} \in \mathbb{R}^{N-1} : (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n} \text{ has all roots on } \mathbb{T}, \text{ where } \mathbf{c} = X_{N,\omega}\mathbf{a} \Big\}.$$

Now consider the map $E_{N,\omega} : \mathbb{T}^{N-1} \to W_N$ specified by $\mathbf{a} = E_{N,\omega}(\xi) = X_{N,\omega}^{-1}\mathbf{c}$ where $\mathbf{c}$ is obtained from ξ by

$$\Big(x - \frac{(-1)^N\omega}{\xi_1\xi_2\cdots\xi_{N-1}} \Big) \prod_{n=1}^{N-1}(x - \xi_n) = (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n}. \tag{4.3}$$

In order to determine the volume of W_N we need to compute the (absolute value of the) Jacobian of the map $E_{N,1}$. It shall be convenient, and no more difficult, to compute the Jacobian of $E_{N,\omega}$ for arbitrary ω.

Lemma 4.1 *Let $\omega \in \mathbb{T}$. The absolute value of the Jacobian of $E_{N,\omega}$ is given by*

$$|\operatorname{Jac} E_{N,\omega}(\xi_1, \xi_2, \dots, \xi_{N-1})| = \prod_{1 \le m < n \le N} |\xi_n - \xi_m|, \tag{4.4}$$

where

$$\xi_N = \frac{(-1)^N\omega}{\xi_1\xi_2\cdots\xi_{N-1}}.$$

Moreover,

$$|\operatorname{Jac} E_{N,\omega}(\xi_1, \xi_2, \dots, \xi_{N-1})| = |\operatorname{Jac} E_{N,1}(\xi_1, \xi_2, \dots, \xi_{N-1})|. \tag{4.5}$$

Proof By (4.3),

$$\begin{aligned} c_n &= (-1)^n e_n(\xi_1, \xi_2, \dots, \xi_N) \\ &= (-1)^n \big(e_n(\xi_1, \xi_2, \dots, \xi_{N-1}) + \xi_N e_{n-1}(\xi_1, \xi_2, \dots, \xi_{N-1}) \big), \end{aligned} \tag{4.6}$$

where e_n is the n-th elementary symmetric function. The Jacobian of $E_{N,\omega}$ is given by

$$\operatorname{Jac} E_{N,\omega} = \det \Big[\frac{\partial a_n}{\partial \xi_m} \Big]_{n,m=1}^{N-1} = \det X_{N,\omega}^{-1} \det \Big[\frac{\partial c_n}{\partial \xi_m} \Big]_{n,m=1}^{N-1}.$$

From (4.6) an easy calculation reveals

$$\frac{\partial c_n}{\partial \xi_m} = (-1)^n \Big(1 - \frac{\xi_N}{\xi_m}\Big) e_{n-1,m},$$

where $e_{n-1,m} = e_{n-1,m}(\xi_1, \xi_2, \ldots, \xi_{N-1})$ is the $(n-1)$-st elementary symmetric function in all variables except ξ_m, and we use the convention that $e_{0,m} = 1$. We conclude that

$$\text{(4.7)} \qquad \operatorname{Jac} E_{N,\omega} = \det X_{N,\omega}^{-1} \Big(\prod_{\ell=1}^{N-1} \xi_\ell^{-1}(\xi_N - \xi_\ell)\Big) \det\big[(-1)^{n-1} e_{n-1,m}\big]_{n,m=1}^{N-1}.$$

Let U be the $N-1 \times N-1$ matrix $[(-1)^{n-1}e_{n-1,m}]_{n,m=1}^{N-1}$ and notice that the m-th column of U is comprised of the coefficients of the polynomial

$$f_m(x) = \prod_{\substack{\ell=1\\ \ell\neq m}}^{N-1}(x - \xi_\ell) = \sum_{n=1}^{N-1}(-1)^{n-1} e_{n-1,m} x^{N-1-n},$$

and clearly

$$f_m(\xi_k) = \delta_{m,k} \prod_{\substack{\ell=1\\ \ell\neq m}}^{N-1}(\xi_k - \xi_\ell).$$

Thus, if we set $V = [\xi_m^j]_{m,j=1}^{N-1}$ (that is, V is the Vandermonde matrix in the variables $\xi_1, \xi_2, \ldots, \xi_{N-1}$), then VU is a diagonal matrix, and

$$\det VU = (-1)^{\binom{N-1}{2}} \prod_{1\leq m<n<N} (\xi_n - \xi_m)^2.$$

Using the familiar formula for the Vandermonde determinant, we see

$$\det U = (-1)^{\binom{N-1}{2}} \prod_{1\leq m<n<N} (\xi_n - \xi_m).$$

It is easy to verify from (1.2) and (4.2) that $\det X_{N,\omega} = (\sqrt{\omega})^N \det X_N$ and hence (4.7) leads us to

$$|\operatorname{Jac} E_{N,\omega}| = \prod_{1\leq m<n\leq N} |\xi_n - \xi_m|.$$

To prove (4.5) we note that

$$(-1)^N \prod_{n=1}^{N} \omega^{-1/N}\xi_n = \omega^{-1}\Big((-1)^N \prod_{n=1}^{N} \xi_n\Big) = \omega^{-1}\omega = 1.$$

Thus we may transform an ω-CR polynomial into a CR polynomial by multiplying each of the roots by $\omega^{-1/N}$ (for any fixed branch of the N-th root). But this corresponds to multiplying $\operatorname{Jac} E_{N,\omega}$ by a complex number of modulus 1. ■

4.2 The Proof of Theorem 1.4

Given $\mathbf{w}(x) = \prod_{n=1}^{N}(x-\xi_n) \in \mathring{W}_N$, there are $N!$ different choices of root vectors $\xi \in \mathbb{T}^{N-1}$ associated with $\mathbf{w}$. That is, since one of the roots is determined by the others, there are $\binom{N}{N-1}$ ways of choosing $N-1$ independent roots, and then $(N-1)!$ ways of ordering them. Setting $\xi_n = e^{i\theta_n}$, it follows from (4.1) that

$$\begin{aligned}\operatorname{vol}(W_N) &= \frac{1}{N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\operatorname{Jac}E_{N,1}(e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_{N-1}})\right|d\theta_1\cdots d\theta_{N-1}\\ &= \frac{1}{N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}\left|\operatorname{Jac}E_{N,1}(e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_{N-1}})\right|d\vartheta\Big)\,d\theta_1\cdots d\theta_{N-1}.\end{aligned}$$

By (4.5) we may replace $|\operatorname{Jac}E_{N,1}|$ with $\left|\operatorname{Jac}E_{N,e^{i\vartheta}}\right|$

$$\operatorname{vol}(W_N) = \\ \frac{1}{N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}\left|\operatorname{Jac}E_{N,e^{i\vartheta}}(e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_{N-1}})\right|d\vartheta\Big)\,d\theta_1\cdots d\theta_{N-1}.$$

By the change of variables $\theta_N = N\pi + \vartheta - (\theta_1+\cdots+\theta_{N-1})$ and (4.4), we discover that

$$\operatorname{vol}(W_N) = \frac{1}{2\pi N!}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le m<n\le N}|e^{i\theta_n}-e^{i\theta_m}|\,d\theta_1\,d\theta_2\cdots d\theta_N.$$

The value of this integral has been calculated by F. Dyson in the context of random matrix theory [5]. Using Dyson's value for this integral, we have

$$\operatorname{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})},$$

which is the volume of the $N-1$ dimensional ball of radius 2.

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