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Second Order Dehn Functions of Asynchronously Automatic Groups

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Abstract. Upper bounds of second order Dehn functions of asynchronously automatic groups are obtained.

In [1], [2], [6], [7], and [9], the author and others have studied the theory of second order Dehn functions of groups. We also have carried out calculations of second order Dehn functions for some classes of groups and some group extensions in [1], [10], [11], and [12]. The main purpose of this article is to obtain the upper bounds of second order Dehn functions of asynchronously automatic groups.

For the first order Dehn functions of asynchronously combable groups we have the following: if a group G admits an asynchronously bounded combing σ, and if the length of σ is bounded by a function f for all positive integer n, then the first order Dehn function $\delta_G^{(1)}(n)$ of G is bounded above by $nf(n)$ [5, Lemma 4.1]. Thus, if G is an asynchronously automatic group then G admits an asynchronously bounded combing σ with a simple exponential length [6, Theorem 7.3.4], and so the first order Dehn function of G is bounded above by an simple exponential function. We show (Corollary 2.4) that this is also true for the second order Dehn function of G.

1 Definitions and Notations

Throughout this paper, we denote by G a group finitely presented by $\mathcal{P} = \langle \boldsymbol{x} \,;\, \boldsymbol{r} \rangle$ and identify G with the group defined by $\mathcal{P}$. Alternatively, we regard $\mathcal{P}$ as a two-complex where the underlying graph $\mathcal{P}^{(1)}$ consists of a *single vertex o*, the set $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ of *edges* and the set $\boldsymbol{r}$ of *defining paths*.

We refer the reader to [3] (also see [8]) for the concept of *pictures* over $\mathcal{P}$. Applying certain operations (bridge moves, insertions or deletions of folding pairs, insertions or deletions of floating circles) on (*spherical*) pictures (see [3]) we then define an equivalence relation among spherical pictures over $\mathcal{P}$. Let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$. We denote by $A(\mathbb{P})$ the number of discs in $\mathbb{P}$, and write $\langle \mathbb{P} \rangle_{\mathcal{P}}$ (or simply $\langle \mathbb{P} \rangle$) for the equivalence class represented by the spherical picture $\mathbb{P}$. All these equivalence classes then form a left $\mathbb{Z}G$-module (the *second homotopy module* of $\mathcal{P}$) denoted by $\pi_2(\mathcal{P})$ (see [3] for details).

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Let $\mathbf{X}$ be a set of generators of the module $\pi_2(\mathcal{P})$. Then if $\mathbb{P}$ is a spherical picture we define the *volume* $V_{\mathbf{X}}(\langle\mathbb{P}\rangle)$ of $\langle\mathbb{P}\rangle$ with respect to $\mathbf{X}$ to be the least value of m over all expressions

$$\sum_{i=1}^{m} \varepsilon_i g_i \zeta_i$$

($\varepsilon_i = \pm 1$, $g_i \in G$, $\zeta_i \in \mathbf{X}$, $i = 1, 2, \dots, m$) equal to $\langle\mathbb{P}\rangle$. We then define the *second order Dehn function* $\delta^{(2)}_{\mathcal{P},\mathbf{X}}$ of $\boldsymbol{\mathcal{P}}$ with respect to $\mathbf{X}$ by

$$\delta^{(2)}_{\mathcal{P},\mathbf{X}}(n) = \max\{V_{\mathbf{X}}(\langle\mathbb{P}\rangle) : A(\mathbb{P}) \le n\} \quad (n = 1, 2, \dots).$$

For a group G of type F_3 (that is, a group given by a finite presentation for which π_2 is finitely generated) this is a group invariant up to $\sim$-equivalence [2] (see also [9], [13]), which we write as $\delta^{(2)}_G$, the second order Dehn function of G, where the $\sim$-equivalence is defined as follows. For two increasing functions $f, g\colon \mathbb{N} \longrightarrow \mathbb{R}^+$ write $f \preceq g$ if there is a constant $a > 0$ such that $f(n) \le ag(an) + an$ ($n \in \mathbb{N}$). Then $f \sim g$ if $f \preceq g$ and $g \preceq f$.

Recall that the *Cayley graph* $\Gamma_{\boldsymbol{x}}(G)$ of group G with respect to $\boldsymbol{x}$ is a graph with a vertex for each element of G (the vertex is also denoted by this element), and with a directed edge $e = (g, x)$ from g to $g\overline{x}$ for each $x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$ (where for each word W on $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$, $\overline{W}$ is the element of G represented by W) and each $g \in G$.

Let $g, g' \in G$ and let γ be a path in $\Gamma_{\boldsymbol{x}}(G)$ from g to g', say

$$\gamma = (g, x_1^{\varepsilon_1})(g\bar{x}_1^{\varepsilon_1}, x_2^{\varepsilon_2}) \cdots (g\overline{x_1^{\varepsilon_1} \cdots x_{n-1}^{\varepsilon_{n-1}}}, x_n^{\varepsilon_n})$$

$\varepsilon_i = \pm 1$ ($1 \le i \le n$). We denote $\iota(\gamma) = g$ the *initial* of γ, $\tau(\gamma) = g'$ the *terminal* of γ, and $L(\gamma) = n$ the *length* of γ. Reading off the second coordinates of the edges of γ gives a unique word $w_\gamma = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ on $\boldsymbol{x}$. We then can define the *projection* map $p_o\colon \Gamma_{\boldsymbol{x}}(G) \longrightarrow \boldsymbol{\mathcal{P}}^{(1)}$ given by

$$p_o(\gamma) = w_\gamma, \quad p_o(g) = o, \quad \text{for any path } \gamma \text{ and any } g \in G.$$

Thus, for any word $w = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ on $\boldsymbol{x}$ and for any $g \in G$ we have a unique lift of w in $\boldsymbol{\mathcal{P}}^{(1)}$ at g

$$t_g(w) = (g, x_1^{\varepsilon_1})(g\bar{x}_1^{\varepsilon_1}, x_2^{\varepsilon_2}) \cdots (g\overline{x_1^{\varepsilon_1} \cdots x_{n-1}^{\varepsilon_{n-1}}}, x_n^{\varepsilon_n}).$$

We then call t_g the *lift map* from the set of all words on $\boldsymbol{x}$ to the set of all paths at g of $\Gamma_{\boldsymbol{x}}(G)$.

Let $\tilde{\boldsymbol{r}} = \{t_g(R) : R \in \boldsymbol{r}, g \in G\}$. We then obtain a two-complex, the *universal covering* $\tilde{\boldsymbol{\mathcal{P}}} = \langle \Gamma_{\boldsymbol{x}}(G) \,;\, \tilde{\boldsymbol{r}} \rangle$ of $\boldsymbol{\mathcal{P}}$, with vertex set G, edge set

$$\{(g, x^\varepsilon) : g \in G, x \in \boldsymbol{x}, \varepsilon = \pm 1\},$$

and defining path set $\tilde{\boldsymbol{r}}$.

We also have the definition of pictures over two-complex $\tilde{\boldsymbol{\mathcal{P}}}$ (see [9]). Moreover, given a picture $\mathbb{P}$ over $\boldsymbol{\mathcal{P}}$, then for any $g \in G$ we have a unique picture, the lift of $\mathbb{P}$ at g, $t_g(\mathbb{P}) = \tilde{\mathbb{P}}_g$ over $\tilde{\boldsymbol{\mathcal{P}}}$ of $\mathbb{P}$ at g, satisfying $p_o(\tilde{\mathbb{P}}_g) = \mathbb{P}$ (also see [9]).

Let $\mu\colon \hat{F}(\boldsymbol{x},\boldsymbol{x}^{-1}) \longrightarrow G$ be the monoid homomorphism given by $x^{\varepsilon} \longmapsto \bar{x}^{\varepsilon}$ ($x \in \boldsymbol{x}$, $\varepsilon = \pm 1$), where $\hat{F}(\boldsymbol{x},\boldsymbol{x}^{-1})$ is the free monoid generated by $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$.

By assigning unit length to each edge of $\Gamma_{\boldsymbol{x}}(G)$, $\Gamma_{\boldsymbol{x}}(G)$ is a geodesic metric space. The *distance* $d_{\boldsymbol{x}}(1,g)$ from the identity to any element g in $\Gamma_{\boldsymbol{x}}(G)$ is the word length of a shortest word w with $\mu(w) = g$ denoted by $|g|_{\boldsymbol{x}}$, or simply $|g|$. The distance $d_{\boldsymbol{x}}(g_1,g_2)$, or simply $d(g_1,g_2)$ of any two elements g_1,g_2 of G in $\Gamma_{\boldsymbol{x}}(G)$ is then $|g_1^{-1}g_2|_{\boldsymbol{x}}$.

A *combing* $\sigma\colon G \longrightarrow \hat{F}(\boldsymbol{x},\boldsymbol{x}^{-1})$ is a section of μ, or equivalently, σ is a set of chosen paths in $\Gamma_{\boldsymbol{x}}(G)$ such that for any $g \in G$, $\sigma(g)$ is a choice of a path from the identity to g. We write σ_g for this chosen path and call this path a *combing line*. If $\sigma_g = e_1e_2\cdots e_n$, then for any non-negative integer time t, if $t \leq n$ we write $\sigma_g(t) = \tau(e_t)$ and if $t > n$ we write $\sigma_g(t) = \tau(e_n) = g$.

A combing σ of $\Gamma_{\boldsymbol{x}}(G)$ is said to have the (*synchronous*) *K-fellow traveller property* if there exists a non-negative constant K such that the combing paths to any vertices g_1,g_2 with $d(g_1,g_2) \leq 1$ in $\Gamma_{\boldsymbol{x}}(G)$ are within a distance K of each other at any integer time $t \geq 0$, *i.e.*,

$$d\big(\sigma_{g_1}(t),\sigma_{g_2}(t)\big) \leq K$$

and we will say that σ_{g_1} and σ_{g_2} are (*synchronous*) *K-fellow travellers* in $\Gamma_{\boldsymbol{x}}(G)$.

We say that G is (*synchronously*) *combable* if it admits a combing having the K-fellow traveller property. Any *synchronously automatic group* (for definition see [6, Section 2.3]) is a synchronously combable group [6].

Let

$$\Omega = \{\psi\colon \mathbb{N} \longrightarrow \mathbb{N}\ ;\ \psi(0) = 0, \psi(n+1) = \psi(n) \text{ or } \psi(n)+1, n \in \mathbb{N}\}$$

where all ψ are unbounded. Given a combing σ of G and for any $g,h \in G$, we set

$$E_{\sigma}(g,h) = \min_{\psi,\psi' \in \Omega}\left\{\max_{t \in \mathbb{N}}\left\{d\Big(\sigma_g\big(\psi(t)\big),\sigma_h\big(\psi'(t)\big)\Big)\right\}\right\}.$$

Then the *asynchronous width* of σ is defined to be

$$\Phi(n) = \max\{E_{\sigma}(g,h) : d(1,g), d(1,h) \leq n, d(g,h) = 1, g,h \in G\}$$

for all $n \in \mathbb{N}$. If Φ is bounded by a constant K then we say that σ has the *asynchronous K-fellow traveller property*.

We say that G is *asynchronously combable* if it admits a combing having the asynchronous K-fellow traveller property. Any *asynchronously automatic group* (for definition see [6, Section 7.2]) is asynchronously combable [6, Theorem 7.3.6].

The *length* $L_{\sigma}(n)$ of a given combing σ is defined by:

$$L_{\sigma}(n) = \max_{g \in G}\{L(\sigma_g) : d(1,g) \leq n\}.$$

If there is a increasing function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ so that $L_{\sigma}(n) \leq f(n)$ for all n, then we say that the length of σ is bounded by f.

Let σ be a combing of G. Let $D\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a function. If for any integer $r > 0$, $g \in G$ and for all integers s, t with $0 \leq s, t \leq L(\sigma_g)$ one has that $|s-t| > D(r)$ implies $d\big(\sigma_g(s), \sigma_g(t)\big) > r$, we then say that D is a *departure function* for the combing σ. It is well known that all asynchronously automatic groups are asynchronously combable with departure function [6, Theorems 7.2.4, 7.2.8].

2 Main Results

Lemma 2.1 **[5, Lemma 4.1]** *Suppose that G admits a combing σ having the asynchronous K-fellow traveller property. Let $\mathcal{P}_1 = \langle \boldsymbol{x}\,; \boldsymbol{r}_1 \rangle$ where $\boldsymbol{r}_1$ is the set of all words $w = x_1 \cdots x_n$ on $\boldsymbol{x}$ such that $n \leq 2(K+1)$ and $\overline{x_1 \cdots x_n} = 1$ in G. Then $\mathcal{P}_1$ is a (finite) presentation for G.*

Proof Consider an edge $e = (g, x^{\varepsilon})$ of $\Gamma_{\boldsymbol{x}}(G)$ $(x \in \boldsymbol{x}, g \in G, \varepsilon = \pm 1)$. We have a pair of monotone unbounded functions ψ_g and $\psi_{g\bar{x}^{\varepsilon}}$ such that

$$d\Big(\sigma_g\big(\psi_g(t)\big), \sigma_{g\bar{x}^{\varepsilon}}\big(\psi_{g\bar{x}^{\varepsilon}}(t)\big) \Big) \leq K$$

for all $t \in \mathbb{N}$. Thus, if we choose a geodesic (called *space-like* segments) in $\Gamma_{\boldsymbol{x}}(G)$ from $\sigma_g\big(\psi_g(t)\big)$ to $\sigma_{g\overline{x}^{\varepsilon}}\big(\psi_{g\bar{x}^{\varepsilon}}(t)\big)$ then we have a subgraph of $\Gamma_{\boldsymbol{x}}(G)$ of the form as depicted in Figure 1 which consists of some triangles and/or trapezoids. We will fix such a subgraph for e denoted $\tilde{\Omega}_e$. Let ρ be the boundary path of a triangle or trapezoid in $\tilde{\Omega}_e$. Then ρ has a length at most $2K + 2$. Thus, if the projection of this path to $\mathcal{P}_1$ is U, *i.e.*, $p_o(\rho) = U$ then $L(U) \leq 2(K+1)$.

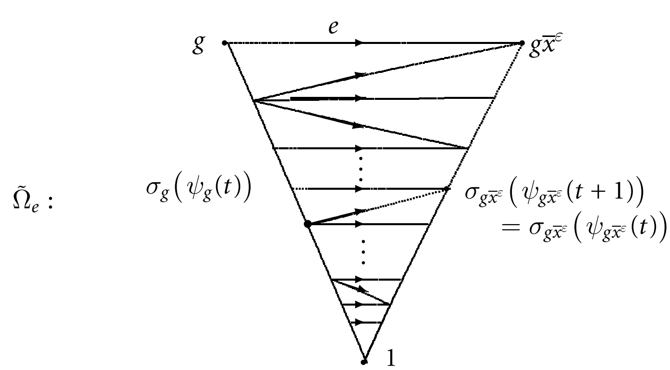

Figure 1

Now, for any given word $w = x_1 \cdots x_n$ on $\boldsymbol{x}$ with $\overline{x_1 x_2 \cdots x_n} = 1$ in G and for any $g \in G$, we lift w to a closed path $t_g(w)$ in $\Gamma_{\boldsymbol{x}}(G)$. We can fill $t_g(w)$ in with some $\tilde{\Omega}_e$ for each edge e of $t_g(w)$ to obtain a planar subgraph of $\Gamma_{\boldsymbol{x}}(G)$ denoted $\tilde{\Lambda}_{w,g}$ as demonstrated in Figure 2. By projecting $\tilde{\Lambda}_{w,g}$ to $\mathcal{P}_1$ we obtain a van Kampen diagram Λ_w (the images of space-like segments in $\tilde{\Lambda}_{w,g}$ under p_o will also be called the *space-like words* of Λ_w) for $p_o\big(t_g(w)\big) = w$ over $\mathcal{P}_1$ and this completes our proof.

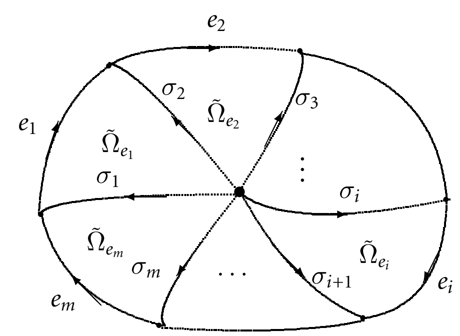

Figure 2

We now further suppose that L_σ is bounded by an increasing function f and σ admits a departure function D. Our next task is to construct an explicit set of spherical pictures that will represent the second homotopy generators for the finite presentation $\mathcal{P}_1$ found in Lemma 2.1.

For each $R \in \boldsymbol{r}_1$ and each $g \in G$, by lifting R at g in $\Gamma_{\boldsymbol{x}}(G)$ we obtain the universal covering $\tilde{\mathcal{P}}_1$ of $\mathcal{P}_1$. Let $\tilde{R}_g = e_1e_2\cdots e_m$ be a defining path of $\tilde{\mathcal{P}}_1$, $R \in \boldsymbol{r}_1$, $g \in G$. Since $\iota(e_1) = g$, $m \le 2(K+2)$, it follows that $L(e_i) \le |g| + K + 1$ and so $L(\sigma_{\iota(e_i)}) \le f(|g| + K + 1)$, $1 \le i \le m$. By padding terms which are trivial paths to the ends of $\sigma_{\iota(e_i)}$'s if necessary, we may extend them to paths $\sigma_i = e_{i1}e_{i2}\cdots e_{ih}$ $(1 \le i \le m)$ with e_{ij}'s empty paths for $j > L(\sigma_{\iota(e_i)})$ and where $h = \max\{L(\sigma_{\iota(e_i)}) : 1 \le i \le m\} \le f(|g| + K + 1)$.

Consider $\tilde{\Lambda}_{R,g}$ (defined in the proof of Lemma 2.1). For each $1 \le j < h$, starting at $\tau(e_{1j})$ on σ_1 we travel along m space-like segments back to σ_1 at some $\tau(e_{1T_j})$ as shown in Figure 3. Denote this path by γ'_j. Then the length of γ'_j is at most mK. Obviously we have $T_{j-1} \le T_j$ by the monotonicity of each element of $\tilde{\Omega}$. We note that the diagrams $\tilde{\Lambda}_{R,g}$ $(R \in \boldsymbol{r}_1, g \in G)$ are spherical.

Since $|\sigma_1(j) - \sigma_1(T_j)| \le Km$, $|T_j - j| \le D(Km)$, $0 \le j \le h$. Let γ_j be the path obtained by adding to γ'_j the segment denoted ζ_j of σ_1 from $\tau(e_{1T_j})$ to $\tau(e_{1j})$. Then $L(\gamma_j) \le mK + D(mK)$. In addition, we require that γ_0 is the empty path at 1 and γ_h is $\tilde{R}_g$. Between γ_{j-1} and γ_j we have a *drum* of $\tilde{\Lambda}_{R,g}$ (by cutting it along e_{1j} as shown in Figure 4) consisting of at most $2mK \cdot D(2mK+1)$ of those basic triangles and trapezoids. We then have a picture (the dualization of this drum) $\tilde{\mathbb{C}}^{(j)}_{R,g}$ over $\tilde{\mathcal{P}}$ containing at most $2mK \cdot D(2mK+1)$ discs and at most $4mK(K+1) \cdot D(2mK+1)$ arcs.

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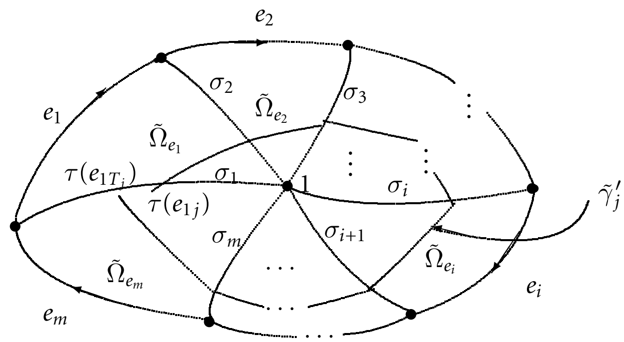

Figure 3

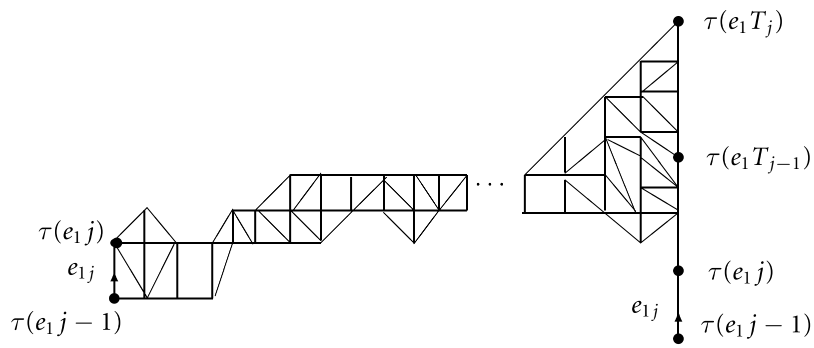

Figure 4

Since $L(\gamma_j) \leq mK + D(mK)$, we can choose a picture $\mathbb{A}_R^{(j)}$ over $\mathcal{P}_1$ with boundary label $p_0(\gamma_j)$ and $A(\mathbb{A}_R^{(j)}) \leq \delta_{\mathcal{P}_1}^{(1)}\big(mK + D(mK)\big)$. We also can assume that the total number of arcs in $\mathbb{A}_R^{(j)}$ is at most $2(K+1)\delta_{\mathcal{P}_1}^{(1)}\big(mK + D(mK)\big)$.

Let $\tilde{\mathbb{A}}_{R,g}^{(j)}$ be the lift of $\mathbb{A}_R^{(j)}$ at $\tau(e_{1j})$. Then $\tilde{\mathbb{A}}_{R,g}^{(j)}$ is a picture over $\tilde{\mathcal{P}}_1$ with boundary label γ_j. We then obtain a spherical picture $\tilde{\mathbb{B}}_{R,g}^{(j)}$ over $\tilde{\mathcal{P}}$ at $\iota(e_{1j})$ of the form shown in Figure 5 with

$$A(\tilde{\mathbb{B}}_{R,g}^{(j)}) \leq 2\delta_{\mathcal{P}_1}^{(1)}\big(mK + D(mK)\big) + 2mK \cdot D(2mK+1),$$

$$\#\text{arcs of } \tilde{\mathbb{B}}_{R,g}^{(j)} \leq 4mK(K+1) \cdot D(2mK+1) + 4(K+1)\delta_{\mathcal{P}_1}^{(1)}\big(mK + D(mK)\big).$$

Corresponding to $\tilde{\Lambda}_{R,g}$ we also have a spherical picture $\tilde{\mathbb{P}}_{R,g}$ which is of the form demonstrated in Figure 6.

The following lemma now is true.

Lemma 2.2 *Let G and $\mathcal{P}_1$ be defined as in Lemma 2.1 where L_σ is bounded by an*

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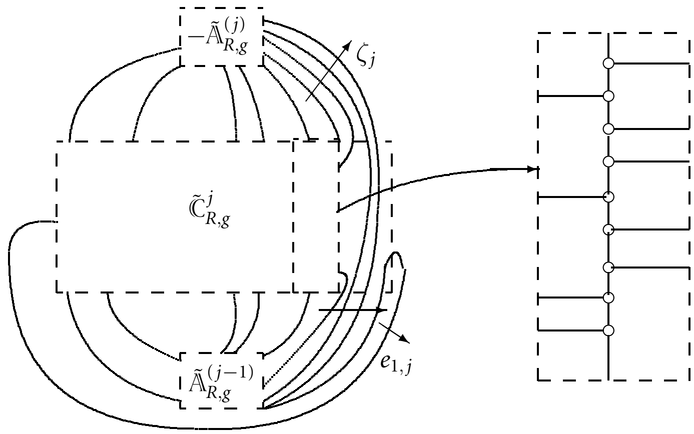

Figure 5

increasing function f and σ admits a departure function D. Then

$$\tilde{\mathbb{P}}_{R,g} \sim \sum_{j=1}^{h} (\tilde{\mathbb{B}}^{(j)}_{R,g})^{e_{1,j+1}\cdots e_{1h}}.$$

Since the disc numbers of all spherical pictures $\tilde{\mathbb{B}}^{(j)}_{R,g}$ are bounded by $2\delta^{(1)}_{\mathcal{P}_1} \cdot \big(mK + D(mK)\big) + 2mK \cdot D(2mK+1)$ and the arc numbers of these pictures are bounded by $4mK(K+1) \cdot D(2mK+1) + 4(K+1)\delta^{(1)}_{\mathcal{P}_1}\big(mK + D(mK)\big)$, if we let $\mathbf{X}$ be the set of all images of these pictures under the projection p_o then $\mathbf{X}$ is finite.

Theorem 2.3 *Let G and $\mathcal{P}_1$ be defined as in Lemma 2.1 where L_σ is bounded by an increasing function f and σ admits a departure function D, and let $\mathbf{X}$ be defined as above. Then*

(i) $\mathbf{X}$ *generates* $\pi_2(\mathcal{P}_1)$*;*
(ii) $\delta^{(2)}_G \preceq nf(n)$.

Proof Let n be any positive integer, and let $\mathbb{P}$ be a minimal connected spherical picture over $\mathcal{P}_1$ with n discs $\Delta_1, \Delta_2, \ldots, \Delta_n$, labeled $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \ldots, R_n^{\varepsilon_n}$. Let the word $x_{i1}\cdots x_{i,j_i}$ $(1 \le i \le n)$ on $\boldsymbol{x}$ be the label of a minimal transverse path in $\mathbb{P}$ from the basepoint of $\mathbb{P}$ to the basepoint of Δ_i. Then $j_i \le 2(K+1)n$ by elementary graph theory. Let $\tilde{\mathbb{P}}_1$ be the lift of $\mathbb{P}$ at 1 in $\tilde{\mathcal{P}}$. Then the discs $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n$ of $\tilde{\mathbb{P}}_1$ are labeled by $\tilde{R}^{\varepsilon_1}_{1,g_1}, \ldots, \tilde{R}^{\varepsilon_n}_{n,g_n}$ where $g_i = \overline{x_{i1}\cdots x_{i,j_i}}$, $1 \le i \le n$. We convert $\tilde{\mathbb{P}}_1$ to a picture $\tilde{\mathbb{P}}'_1$ by replacing each $\tilde{\Delta}_i$ by the complement of the disc labeled $\tilde{R}^{-\varepsilon_i}_{i,g_i}$ in $\varepsilon_i\tilde{\mathbb{P}}_{R_i,g_i}$. Suppose that there is an arc labeled e_j connecting $\tilde{\Delta}_i$ and $\tilde{\Delta}_{i+1}$. Then in $\tilde{\mathbb{P}}'_1$ we see that all subpictures $\tilde{\mathbb{C}}^{(q)}_j$ $(1 \le q \le h)$ as shown in Figure 6 will be canceled as in $\tilde{\mathbb{P}}^{\varepsilon_{i+1}}_{R_{i+1},g_{i+1}}$ we have the same subpictures with opposite symbols since $\tilde{\Omega}_{e_j}$ is fixed. Thus, $\tilde{\mathbb{P}}'$ can

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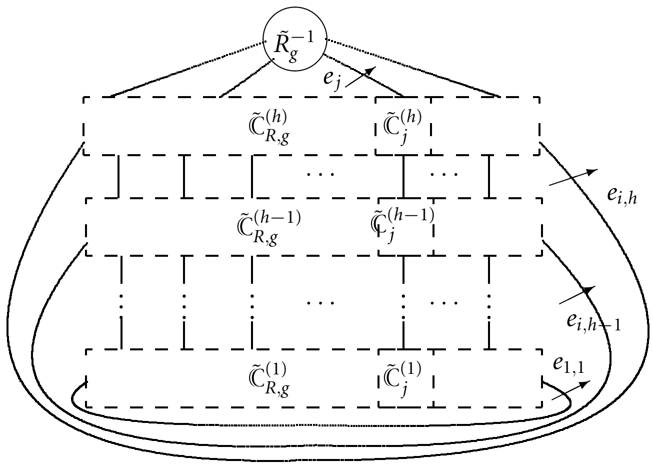

Figure 6

be transformed to the empty picture by bridge moves and eliminations of canceling pairs. Thus, by Lemma 1.5 of [3], for certain paths λ_i $(i = 1, \dots, n)$,

$$\langle \tilde{\mathbb{P}}_1 \rangle = \sum_{i=1}^{n} \varepsilon_i \langle \tilde{\mathbb{P}}_{R_i,g_i}^{\lambda_i} \rangle.$$

By Lemma 2.2, applying the projection p_o then gives an expression for $\langle \mathbb{P} \rangle$ involving at most $nf\big(2(K+1)n + K + 1\big)$ terms of $\mathbf{X}$.

Now let $\mathbb{P}$ be an arbitrary spherical picture $\mathbb{P}$ over $\mathcal{P}$ with n discs having nontrivial components $\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_q$ with $n_1, n_2, \dots, n_q$ discs respectively, where $n_1 + n_2 + \cdots + n_q = n$. Then there are words $U_1, U_2, \dots, U_q$ on $\boldsymbol{x}$ such that

$$\langle \mathbb{P} \rangle = \sum_{i=1}^{q} \overline{U}_i \cdot \langle \mathbb{P}_i \rangle.$$

Thus, using the previous paragraph we get

$$\begin{aligned} V_{\mathcal{P}_1,\mathbf{X}}(\mathbb{P}) &\le \sum_{i=1}^{q} n_i f\big(2(K+1)n_i + K + 1\big) \\ &\le nf\big(2(K+1)n + K + 1\big) \end{aligned}$$

as required.

If G is asynchronously automatic, then by the proof of Theorem 7.3.4 of [6] the function f in the above theorem then can be taken as a simple exponential one. Furthermore, if G is automatic, then f can be taken as a linear one. Thus, we have:

Corollary 2.4 *If G is asynchronously automatic then $\delta_G^{(2)} \preceq e^n$; and if G is synchronously automatic then $\delta_G^{(2)} \preceq n^2$.*

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Remark Let F be a free group of rank 3 on a, b and c. Let ϕ be an automorphism of F defined by

$$a \longrightarrow c, \quad b \longrightarrow ac, \quad c \longrightarrow bc.$$

We then form the semi-direct products $K = F \times_{\phi} \langle s, \bar{s} \rangle$ with $s, \bar{s}$ all acting via ϕ, and furthermore, form another semidirect product $G = K \times_{\hat{\phi}} \langle u, t \rangle$, where u acts via the automorphism $\hat{\phi}$ of K defined by

$$F \xrightarrow{\phi} F, \quad s \mapsto s, \quad \bar{s} \mapsto \bar{s},$$

and t acts via the identity automorphism. Thus, G is asynchronously automatic [4], and so by the above corollary, $\delta_G^{(2)}(n) \preceq e^n$. In Example 2 of [12], we also have this upper bound. In fact, in [12] we proved that $e^{\sqrt{n}} \preceq \delta_G^{(2)}(n) \preceq e^n$. We then can guess that the upper bounds in Corollary 3.4 are optimal.

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