PROJECTIONS IN CERTAIN CONTINUOUS FUNCTION SPACES C(H) AND SUBSPACES OF C(H) ISOMORPHIC WITH C(H)

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Notation. If A and B are sets then $A - B = \{x \mid x \in A,\ x \notin B\}$. This notation is also used if A and B are linear spaces. If X and Y are Banach spaces an *embedding* of X into Y is a continuous linear mapping u of X onto a closed subspace of Y which is $1 - 1$. In this case X is said to be *embedded* in Y. If $|ux| = |x|$ for every $x \in X$ ($|\ldots|$ stands for norm), then u *embeds* X *isometrically* into Y. If u is onto then X and Y are *isomorphic* and if, in addition, $|ux| = |x|$ for every $x \in X$, then X and Y are *isometric.* Then an embedding u has a continuous inverse u^{-1} (**4**, p. 36) defined on uX and this fact is used below without further reference. The conjugate space of X is denoted by X'. Unless otherwise noted, all topological spaces considered are Hausdorff spaces.

1. Introduction. We consider Banach spaces over the real numbers R only. Let B be a Banach space with the following property: If X is a subspace of a Banach space Y and if u is a bounded linear map from X to B, then u has a bounded extension u_1 from Y to B. Such a B is said to have *property P*, or the *extension property*, and we write (B,P). If u_1 can always be taken so that $|u_1| \leqslant t|u|$, then B is said to have *property* P_t and we write (B,P_t) (**4**, pp. 94–95). If B has the above property subject to the restriction that Y be separable, B is said to have the *separable extension property* and we write (B,S) and (B,S_t) in place of (B,P) and (B,P_t) respectively. Clearly (B,P) implies (B,S) and (B,P_t) implies (B,S_t).

With the above terminology the Hahn–Banach theorem asserts (R,P_1). Phillips (**13**, p. 538) noted that for any set H one has $(m(H),P_1)$ where $m(H)$ is the set of bounded real-valued functions on H with supremum norm. Goodner (**6**) and Nachbin (**12**) characterized (B,P_1) spaces as spaces isometric to a $C(H)$ space with H compact and extremally disconnected, provided the unit ball of B has an extreme point.* Kelley (**10**) removed the extreme point assumption. Implicit in the proofs of these characterizations was the theorem:

If $Y \supset B$ and $Y/B = R$ implies there is a projection with norm one from Y to B, then (B,P_1). In § 3 a different proof of this is given. In § 4 another representation of a P_1 space is given provided it is well situated in a $C(H)$ space

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*$C(H)$ is the space of bounded continuous functions on H with $|f| = \sup\{|f(h)| \mid h \in H\}$.

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where H is compact and extremally disconnected. The condition is that there exists a projection p such that if $u = 2p - I$, then $|u| = 1$.

The following (**4**, p. 94) *are equivalent for a Banach space B:*
(a) (B,P),
(b) If $X \supset B$ there is a continuous projection p of X onto B.
(c) If $Y \supset B_1$ and B_1 is isomorphic to B there is a continuous projection p of Y onto B_1.
(d) If B_1 is an embedding of B in some $m(H)$, there is a continuous projection of $m(H)$ onto B_1.

The proofs are similar to calculations in Theorem 7 below. From these conditions it is seen that if (B,P), then (B,P_t) for some t since, letting u be an isometric embedding of B in some $m(H)$ and p a projection of $m(H)$ onto $u(B)$, then if $X \supset Y$ and v is a map of Y to B, the map uv of Y to $m(H)$ has an extension v_1, $|v_1| = |uv|$. The map $u^{-1}pv_1$ is an extension of v from Y to B, and $|u^{-1}pv_1| \leqslant |u^{-1}||p||v_1|$ so that $|p|$ provides a t.

Akilov observed that if B is a complete Banach lattice whose unit ball has a least upper bound y, then $(B,P_{|y|})$. If $Y \supset X$, and u is a map from X to B, substituting the function p, $p(x) = |u||x|y$, for the subadditive linear functional in the proof of the Hahn–Banach theorem, one shows there exists an extension u_1 of u and $|u_1| \leqslant |u|\ |y|$ (**6**, p. 94). If H is compact and extremally disconnected, then $C(H)$ is a complete Banach lattice whose unit sphere has a least upper bound y, $y(h) = 1$ for all h, and $|y| = 1$ (**6**, p. 103). Hence Kelley's result and Akilov's result with $|y| = 1$ provide complete characterizations of spaces (B,P_1).

Goodner (**6**, p. 102) proved that *if B is a sublattice of $C(H)$ and p is a projection of $C(H)$ onto B with $|p| = 1$, then $U(B)$ has a least upper bound y and* $|y| = 1$. Hence (**4**, p. 101) there is $C(K)$ for which B is isometric to $C(K)$. The important step is to show that p is a positive map, that is, if $f \geqslant 0$, then $pf \geqslant 0$.

THEOREM 1. *If $B \subset C(H)$ and p is a positive projection of $C(H)$ onto B, then B is a Banach lattice whose unit sphere has an upper bound. Hence* (4, p. 101) *B is isomorphic to $C(K)$ for some K. If H is compact and extremally disconnected then B is a complete lattice and B is isomorphic to a space with property P_1.*

Proof. Define an order in B by saying b is non-negative in B if and only if there is an $f \geqslant 0$ in $C(H)$ for which $pf = b$. Let F be the set of such b. Then F is a closed cone (**4**, p. 97) and so orders B. If b_1 and $b_1 - b$ are in F, then b_1 and $b_1 - b$ are non-negative in $C(H)$. Hence $b_1 \geqslant b \vee 0$, where $b \vee 0$ stands for sup $(b,0)$ in the lattice (and $b \wedge 0 = \inf\ (b,0)$), so $pb_1 = b_1 \geqslant p(b \vee 0)$. Hence $p(b \vee 0)$ provides a supremum in B for b and 0. Also $p(b \wedge 0)$ then provides an infimum so that B is a lattice. If $p(b \vee 0) - p(b \wedge 0) - (p(b_1 \vee 0) - p(b_1 \wedge 0))$ is non-negative in B, it is non-negative in $C(H)$, and since $p(b_1 \vee 0) - p(b_1 \wedge 0) \geqslant 0$,

$$|p(b \vee 0) - p(b \wedge 0)| \geqslant |p(b_1 \vee 0) - p(b_1 \wedge 0)|$$

so that B is a Banach lattice. Also $U(B)$ has an upper bound $p(i)$ where $i(h) = 1$ for all h. If H is compact and extremally disconnected then $C(H)$ is a complete lattice. Let A have an upper bound in B, say b. Then b is an upper bound for A in $C(H)$ and if x is the supremum of A in $C(H)$, $b \geqslant x$, and $pb = b \geqslant px$ so px is a supremum for A in B.

To prove the last part we have that B is a complete Banach lattice whose unit ball has a least upper bound $e = p(i)$. Hence $(B, P_{|e|})$.

Define a new norm on B by letting $||b||$ be the greatest lower bound of all t for which $-te \leqslant b \leqslant te$. If $|b| \leqslant 1$, then $-i \leqslant b \leqslant i$ so $||b|| \leqslant 1$. Hence B is isomorphic to B with its new norm (**4**, p. 37). If $b \vee 0 + (-b) \vee 0 \geqslant c \vee 0 + (-c) \vee 0$ and if $te \geqslant b \vee 0 + (-b) \vee 0$, then $te \geqslant c \vee 0 + (-c) \vee 0$ so that $||b \vee 0 + (-b) \vee 0|| \geqslant ||c \vee 0 + (-c) \vee 0||$. Hence with the new norm B is a complete Banach lattice whose unit sphere has a least upper bound e, and $||e|| = 1$, and so B with its new norm is a P_1 space.

Substituting a Banach lattice Y for $C(H)$, then the above proof shows that B can be given an order in which it is a Banach lattice having a unit if Y has and complete if Y is.

The Banach spaces m, c, c_0 are the spaces of bounded sequences, convergent sequences, and sequences convergent to 0 respectively. In each case

$$|x| = \sup_n |x_n|,$$

for $n \in N$. N stands for the positive integers. Clearly $c_0 \subset c \subset m$. Phillips (**13**, p. 539; **8**) showed there is no continuous projection of m onto c_0. His main step was (**4**, p. 32) to show that if u is a map of m to c_0, then u^2 is a compact map. Grothendieck (**7**, p. 169) proved that if B is a separable subspace of $C(H)$ with H compact and extremally disconnected then there is no continuous projection of $C(H)$ to B unless B is finite dimensional.

Goodner (**6**, p. 98; **1**) showed that no L space whose dimension is greater than two has property P_1. In (**3**) it is shown that a map p from a $C(H)$ space to a weakly complete subspace is weakly compact and that p^2 is then compact. Hence *an infinite dimensional weakly complete space cannot have property P*. In particular, no infinite dimensional reflexive space or L space can have property P.

Sobczyk (**15**) proved that if $X \supset c_0$ and if X is separable, then there is a projection p of X to c_0 and $|p| \leqslant 2$ (see § 3 below). Hence (c_0, S_2).

These results answer affirmatively Banach's conjecture (**2**, pp. 192–193) that $\dim_1 (X) = \dim_1 (Y)$ is not sufficient to prove X is isomorphic to Y. Form $X = m \oplus c_0$, where if $f = f_1 + f_2$, $f_1 \in m$, $f_2 \in c_0$, then $|f| = \max (|f_1|, |f_2|)$. Then $\dim_1 (X) = \dim_1 (m)$; but if u is an isomorphism of m onto X, then $u^{-1}pu$ is a projection of m onto uc_0, where p is the projection $pf = f_2$. Hence no such u exists and X and m are not isomorphic. Similarly, $\dim_1 (C([0, 1]))$ $\dim_1 C([0, 1]) \oplus 1_2)$, but $C([0, 1])$ is not isomorphic to $C([0, 1]) \oplus 1_2$. In

§ 2 it is shown that if (X, P_s) and if $\dim_1(X) = \dim_1(m)$, then X is isomorphic to m. In this section we examine a class of subspaces of certain $C(H)$ spaces and show they are isomorphic to the given $C(H)$ space.

In § 3 we consider separable spaces with property S and give new proofs of Sobczyk's result and a recent result of McWilliams **(11)**.

2. A class of subspaces of $C(H)$ isomorphic to $C(H)$. A class of spaces was examined in Theorem 1 which includes, up to isomorphism, finite dimensional spaces and finite direct sums of P_1 spaces. An element of the class was found to be isomorphic to a P_1 space. In this section the remaining known P_t spaces are shown to be isomorphic to P_1 spaces.

DEFINITION. *A Banach space of sequences X is a Banach space whose elements are sequences $x = \{x_n\}$ of real numbers and if d_n is defined on X by $d_n(x) = x_n$, then $\{d_n\}$ is a uniformly bounded sequence in X'.*

Notation. If u is a continuous linear mapping from the Banach space X to the Banach space Y, denote by u' the conjugate mapping

$$u':Y' \to X' \quad (u'y'(x) = y'(ux) \text{ for every } x \in X, y' \in Y').$$

THEOREM 2. *Let X be a Banach space of sequences and let u be an isomorphism of X into the Banach space B. Suppose p is a continuous projection from B onto uX. Define $d_i \in X'$ as above, and let $e_i = (u^{-1}p)'d_i$. Let $\{n_i\}$ be a subsequence of N. If $X_1 = \{x \in X \mid x_{n_i} = 0, i = 1, 2, \ldots,\}$ is isomorphic to X, then $B_1 = \{b \in B \mid e_{n_i}(b) = 0, i = 1, 2, \ldots,\}$ is isomorphic to B.*

Proof. Let $v(X_1) = X$ be the promised isomorphism of X_1 onto X and let $q = 1 - p + uv^{-1}u^{-1}p$.

$q:B \to B_1$: We have $u^{-1}pq = u^{-1}puv^{-1}u^{-1}p = v^{-1}u^{-1}p$ since $u^{-1}p(1-p) = 0$ and $pu = u$. Hence $e_{n_i}(qb) = d_{n_i}(v^{-1}u^{-1}pb) = 0$ and so $qb \in B_1$ for every $b \in B$.

$qB = B_1$: Let $b_1 \in B_1$. Then $u^{-1}pb_1 = x_1 \in X_1$ since $d_{n_i}(x_1) = (u^{-1}p)'d_{n_i}(b_1) = e_{n_i}(b_1) = 0$ for each i. Let $b = uvu^{-1}pb_1$. Then

$$\begin{aligned} q((1-p)b_1 + b) &= (1-p)b_1 + uv^{-1}u^{-1}p(1-p)b_1 + (1-p)b + uv^{-1}u^{-1}pb \\ &= (1-p)b_1 + 0 + 0 + uv^{-1}u^{-1}puvu^{-1}pb_1 = (1-p)b_1 + pb_1 = b_1. \end{aligned}$$

q is $1-1$: If $qb = 0$, then $(1-p)b = 0 = uv^{-1}u^{-1}pb$, since $(1-p)b \in (1-p)B$ and $uv^{-1}u^{-1}pb \in pB$. Thus $pb = b$ so that $0 = uv^{-1}u^{-1}pb = uv^{-1}u^{-1}b$. Since $uv^{-1}u^{-1}$ is $1-1$, $b = 0$. Q.E.D.

If H is compact and if there is a sequence $\{h_n\} \subset H$ of distinct elements and if

$$h_n \xrightarrow{n\to\infty} h_0 \notin \{h_n\}$$

one constructs an image of c_0 in $C(H)$ as follows. About each h_n choose an open neighbourhood U_n such that $U_j \cap U_n \neq \phi$ implies $j = n$. Select $b_n \in C(H)$ such that $|b_n| = 1 = b_n(h_n)$ and $b_n(h) = 0$ if $h \notin U_n$. If $x \in c_0$ the functions $\sum_1^k x_n b_n = f_k$ form a Cauchy sequence in $C(H)$ so that

$$g_k \xrightarrow{k\to\infty} g$$

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for some $g \in C(H)$ and $g(h_n) = x_n$ for each $n \in N$. Hence c_0 can be embedded in $C(H)$ by letting $ux = g$.

COROLLARY 2.1. *If $\{h_n\}$ is a sequence of distinct elements in a compact space H such that $h_n \to h_0 \notin \{h_n\}$, and if $\{h_{n_i}\}$ is a subsequence of $\{h_n\}$ such that $\{h_n\} - \{h_{n_i}\}$ is infinite, then $B_1 = \{b \in C(H) | b(h_{n_i}) = 0\}$ is isomorphic to $C(H)$.*

Proof. With the above notation define w from $C(H)$ to c_0 by $(wb)_n = b(h_n) - b(h_0)$. Then $p = uw$ is a projection from $C(H)$ onto $u(c_0)$. It is easily seen that if $X_1 = \{x \in c_0 | x_{n_i} = 0\}$, then c_0 is isomorphic to X_1. Using the notation of Theorem 2 one has that $b_1 \in B_1$ if and only if $e_{n_i}(b) = 0$ since $b_1 \in B_1$ implies

$$e_{n_i}(b_1) = d_{n_i}(u^{-1}pb_1) = d_{n_i}(wb_1) = b_1(h_{n_i}) = 0$$

and if $e_{n_i}(b_1) = 0$, then $b_1(h_{n_i}) = 0$ so that $b_1 \in B_1$.

By Theorem 2 $C(H)$ is isomorphic to B_1.

Remarks. With the hypothesis of Corollary 2.1 one can project from $C(H)$ onto B_1. If q on c_0 is defined by

$$(qx)_j = \begin{cases} 0 \text{ if } j \in \{n_i\} \\ x_j \text{ otherwise,} \end{cases}$$

define p_1 by $(p_1 b)(h) = (uqw - uw + 1)b(h) - b(h_0)$ for every $b \in B$, $h \in H$.

If $b_1 \in B_1$, then $wb_1 \in X_1$ so that $qwb_1 = wb_1$ and $b(h_0) = 0$ so $p_1 b_1 = uqwb_1 - uwb_1 + b_1 = uwb_1 - uwb_1 + b_1 = b_1$. If $b \in C(H)$, then

$$\begin{aligned} (uqwb - uwb + b)h_{n_i} &= (qwb)_{n_i} - (uwb)h_{n_i} + b(h_{n_i}) \\ &= b(h_{n_i}) - (uwb)h_{n_i} \\ &= b(h_{n_i}) - (wb)_{n_i} \\ &= b(h_{n_i}) - b(h_{n_i}) + b(h_0) \\ &= b(h_0) \\ \text{so} \quad (p_1 b)h_{n_i} &= 0 \end{aligned}$$

and thus $p_1 b \in B_1$.

Notation. If $W \subset H$ denote by $C_W(H)$ the set of $b \in C(H)$ such that $b(h) = 0$ if $h \in W$. Thus with the conditions of Corollary 2.1 we have $C_{\{h_{n_i}\}}(H)$ is isomorphic to $C(H)$.

THEOREM 3. *In any infinite topological H, if $C(H) = B \oplus Y$, where B and Y are closed and Y is finite dimensional, there are points $h_1, \ldots, h_n$ such that B is isomorphic to $C_{\{h_1, \ldots, h_n\}}(H)$.*

Proof. We use induction on n, the dimension of Y. If $Y = (y)$*, define $pf = f - f(h_1)x$ where h_1 is such that $y(h_1) \neq 0$ and $x = (1/y(h_1))y$. Then $pf(h_1) = f(h_1) - f(h_1)x(h_1) = 0$ and $pf = f$ if $f(h_1) = 0$. Hence p is a projection of $C(H)$ onto $C_{\{h_1\}}(H)$. $(py)h = x(h) - x(h_1)x(h) = x(h) - x(h)$ so

*$(y_1, \ldots, y_n)$ denotes the subspace of Y generated by $y_1, \ldots, y_n$.

$px = 0$. Hence $pB = C_{\{h_1\}}(H)$. If $pb = 0$, then $b(h) - b(h_1)x(h) = 0$ for all h so $b = b(h_1)x$. Since b and x are in complementary subspaces, $b = 0$ and p is an isomorphism of B with $C_{\{h_1\}}(H)$.

Assume the theorem true if dim $(Y) = n - 1$ and let $C(H) = B \oplus Y$ where dim $(Y) = n$, say $Y = (y_1, \ldots, y_n)$. Then $C(H) = B \oplus (y_1) \oplus (y_2, \ldots, y_n)$ and, by the induction hypothesis, there are points $h_2, \ldots, h_n$ such that $B \oplus (y_1)$ is isomorphic to $C_{\{h_2, \ldots, h_n\}}(H)$. Let v be the isomorphism. Let $vy_1 = x$ and h_1 a point at which $x(h_1) \neq 0$.

Let $f_1 = (1/x(h_1))x$. Let p be the projection of

$$C_{\{h_2, \ldots, h_n\}}(H) \text{ onto } C_{\{h_1, \ldots, h_n\}}(H)$$

defined by $pf = f - f(h_1)f_1$. Consider the map pv of B onto

$$C_{\{h_1, \ldots, h_n\}}(H), \ (pvy = px = x - x(h_1)f_1 = 0).$$

If $pvb = 0$, then $vb = (vb)(h_1)f_1$ or

$$vb = \frac{(vb)(h_1)}{x(h_1)}x = \frac{(vb)(h_1)}{x(h_1)}vy_1.$$

Since v is an isomorphism $(vb)(h_1) = 0$ so $vb = 0$ and $b = 0$.

For some proofs of the next assertions see the remarks following the proof of Theorem 6 below. If H is infinite, compact, and extremally disconnected, and if $h_1, \ldots, h_k$ are distinct points of H one can choose open and closed neighbourhoods V_i of h_i such that $V_i \cap V_j = \emptyset$ if $i \neq j$ and such that $H - (\bigcup_{i \leqslant k} V_i)$ is infinite. If h_{k+1} is not in $\bigcup_{i \leqslant k} V_i$ then an open and closed neighbourhood V_{k+1} of h can be chosen so that $V_{k+1} \cap (\bigcup_{i \leqslant k} V_i) = \emptyset$ and $H - (\bigcup_{i \leqslant k+1} V_i)$ is infinite. Thus one can choose a sequence V_i of open, closed, and mutually disjoint sets so that $H - (\bigcup_{i \leqslant k} V_i)$ is infinite for each k. James **(8)** shows that m can be embedded as a subspace m_1 of $C(H)$ (and so (B,P_1) implies B is finite dimensional or not separable) of functions constant on each V_i and vanishing off $\overline{\bigcup_i V_i}$. If $f \in m_1$ corresponds to $x \in m$, then $f(h) = x_i$ if $h \in V_i$.

THEOREM. 4 *If H is compact and infinite and if H contains a convergent sequence of distinct elements or if H is extremally disconnected then a complement of a finite dimensional subspace in $C(H)$ is isomorphic to $C(H)$.*

Proof. Let $h_j' \to h_0$, where h_j' is a convergent sequence of distinct points. If $C(H) = X \oplus B$ where X is finite dimensional and B is closed, then by Theorem 3, B is isomorphic to

$$C_{\{h_1, \ldots, h_k\}}(H),$$

for some $h_1, \ldots, h_k$ and, clearly, these h_j may be chosen so that $h_0 \neq h_j$, $j = 1, \ldots, k$. By dropping to a subsequence if necessary we can further assume that $h_0, \ldots, h_k \notin \{h_j'\}$. Let h'' be the sequence

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$$h_j'' = \begin{cases} h_j & \text{if } j \leqslant k. \\ h_{j-k}' & \text{if } j > k. \end{cases}$$

The subspace of c_0 of those x such that $x_j = 0$ if $j \leqslant k$ is isomorphic to c_0 and so by Corollary 2.1

$$C_{\{h_1, \ldots, h_k\}}(H)$$

is isomorphic to $C(H)$.

If H is compact and extremally disconnected construct the sets V_i such that $h_i \in V_i$, $i \leqslant k$. Since those elements of m vanishing on the first k co-ordinates form an isomorphic subspace of m, again

$$C_{\{h_1, \ldots, h_k\}}(H)$$

is isomorphic to $C(H)$, by Theorem 2.

Remarks. The following properties of a compact, extremally disconnected H are needed below.

If U is open, then $\bar{U}$ is open. Equivalently: if U and V are disjoint open sets, then $\bar{U} \cap \bar{V} = \phi$. This property defines an extremally disconnected space.

If U is an infinite open and closed set in H and if $h \in U$, then $U - \{h\}$ contains an infinite open and closed set.

Proof. If, for each neighbourhood V of h such that $V \subset U$, $U - V$ is finite, then each sequence $\{h_n\} \subset U - \{h\}$ of distinct elements is open and $\overline{\{h_n\}} = \{h_n\} \cup \{h\}$. Thus two such sequences which are disjoint are open and do not have disjoint closures. Hence there is a neighbourhood V of h such that $U - V$ is infinite. If $f \in C(H)$ takes the value 1 on $U - V$, 0 at h, and 0 off U, then

$$\overline{\{h' \mid f(h') > \tfrac{1}{2}\}}$$

is infinite, open, and closed.

THEOREM 5. *If H is compact and extremally disconnected and if m is embedded in $C(H)$ as a space of functions $\bar{m}$ constant on each V_i where $\{V_i\}$ is a sequence of mutually disjoint open and closed sets, let $h_i \in V_i$. Suppose $f(h_i) = x_i$ if f corresponds to x in the embedding. Then a subspace B of $C(H)$ complementary to $\bar{m}$ is isomorphic to $C(H)$ or is finite dimensional.*

To prove this theorem we use the following:

LEMMA. *If $X = X_1 \oplus X_2 \oplus X_3$ where X is a Banach space and X_1, X_2, X_3 are closed subspaces, and if $X_2 \oplus X_3$ is isomorphic to X_2, then $X_1 \oplus X_2$ is isomorphic to X.*

Proof. Let u be an isomorphism of X_2 onto $X_2 \oplus X_3$. Identifying an element x_j of X_j with the element $x_2 \oplus 0$ or $0 \oplus x_3$ in $X_2 \oplus X_3$ one has u^{-1} defined on X_j to X_2. Let p_i be the projection of X to X_i given by the decomposition $X = X_1 \oplus X_2 \oplus X_3$ and let w be defined on $X_1 \oplus X_2$ by $w = p_1 + up_2$. Then w is linear and continuous.

Suppose $wf = 0 = p_1f + up_2f$. Since p_1f and up_2f are in complementary subspaces of X, $p_1f = 0 = up_2f$ and so $p_2f = 0$ since u is an isomorphism. Since $f \in X_1 \oplus X_2$, $p_1f + p_2f = f = 0$ so that w is $1 - 1$.

It remains to show w is onto. Let $x = \bar{x} + x_3$ where $\bar{x} \in X_1 \oplus X_2$ and $x_3 \in X_3$. For some $x_2 \in X_2$, $ux_2 = x_3$ and let $f = p_1\bar{x} + u^{-1}p_2\bar{x} + x_2$. Then $wf = = p_1{}^2\bar{x} + p_1u^{-1}p_2\bar{x} + p_1x_2 + up_2p_1\bar{x} + up_2u^{-1}p_2\bar{x} + up_2x_2$. Since $p_1{}^2\bar{x} = p_1\bar{x}$, $p_1u^{-1}p_2 = 0$, $p_1x_2 = 0$, $p_2p_1 = 0$, then $up_2p_1\bar{x} = 0$, $p_2u^{-1} = u^{-1}$, $up_2u^{-1}p_2\bar{x} = p_2\bar{x}$. Finally $p_2x_2 = x_2$ so that $up_2x_2 = ux_2 = x_3$. Thus the equation reduces to $wf = p_1\bar{x} + p_2\bar{x} + x_3 = \bar{x} + x_3 = x$. Hence w is onto.

Proof of Theorem 5. Let w be the embedding of m to $\bar{m}$. If p is defined by $pf = wx$, where $x_j = f(h_j)$, then p is a projection of $C(H)$ onto $\bar{m}$. Clearly $pf = 0$ if and only if $f \in C_{\{h_i\}}(H)$ so that $C_{\{h_i\}}(H)$ is complementary to m. Let u be defined on B by $ub = f$, where $b = f + x$ and $f \in C_{\{h_i\}}(H)$, $x \in \bar{m}$. Then u is linear, continuous, and $1 - 1$ (if $b = x$, then $b = 0 = x$ since B and $\bar{m}$ are complementary). If $f \in C_{\{h_i\}}(H)$ and if $f = b + x$, where $b \in B$ and $x \in \bar{m}$, then $b = f + (-x)$ so that u is onto. Hence B is isomorphic to $C_{\}h_i\}}(H)$ and it is enough to show $C_{\{h_i\}}(H)$ is isomorphic to $C(H)$.

If we can write $C(H) = A \oplus m_1 \oplus \bar{m}$ where A, m_1 are closed subspaces of

$$C_{\{h_i\}}(H),\ C_{\{h_i\}}(H) = A \oplus m_1$$

and m_1 is isomorphic to m, then it is easily seen that $m_1 \oplus \bar{m}$ is isomorphic to m, so the lemma will conclude the proof. Since $f \in C_{\{h_i\}}(H)$ if and only if

$$f \in C_{\overline{\{h_i\}}}(H)$$

and since $C_{\{h_i\}}(H)$ is infinite dimensional, it follows that $H - \overline{\{h_i\}}$ is infinite.

Suppose now that if V is an open set such that $\overline{\{h_i\}} \subset V$, then $H - V$ is finite. Then $H - \overline{\{h_i\}}$ is a discrete set and let $\{h_n'\}$ be a sequence of distinct points in $H - V$. Embed c_0 in $C(H)$ by letting $ux(h) = 0$ if $h \notin \{h_n'\}$ and x_n if $h = h_n'$.

To show $ux \in C(H)$, clearly ux is continuous at h if $h \in H - \overline{\{h_i\}}$. If $h \in \overline{\{h_i\}}$ choose k such that $n > k$ implies $x_n < \epsilon$. Then $H - \{h_1', \ldots, h_n'\}$ is a neighbourhood of any point in $\overline{\{h_j\}}$ and $ux(h) < \epsilon$ if $h \in H - \{h_1', \ldots, h_n'\}$. Thus ux is continuous at every point so that $ux \in C(H)$.

Clearly u is an isomorphism of c_0 into $C(H)$ and we can project from $C\{_{h_i}\}(H)$ onto uc_0, say by q. Then $q(1 - p)$ is a projection of $C(H)$ onto uc_0, contradicting Grothendieck's Theorem (see the Introduction).

Hence there is an open set V containing $\overline{\{h_i\}}$ such that $H - V$ is infinite. If $f \in C(H)$ is such that $f(h) = 0$ if $h \in \overline{\{h_i\}}$ and $f(h) = 1$ if $h \in H - V$, then $W = \{h|f(h) > \frac{1}{2}\}$ is an infinite open and closed set in $H - \overline{\{h_i\}}$.

Now W in the relative topology is compact and extremally disconnected so that we can embed m in $C(W)$. $C(W)$ can then be embedded in

$$C_{\overline{\{h_i\}}}(H) \subset C(H)$$

by letting

$$uf(h) = \begin{cases} f(h) & \text{if } h \in W \\ 0 & \text{if } h \notin W. \end{cases}$$

Thus m can be embedded isomorphically in

$$C_{\overline{\{h_i\}}}(H).$$

Remarks. If H is compact and extremally disconnected, then $C_{\{h_i, \ldots, h_n\}}$ and $C_{\{h_i\}}(H)$ are complete lattices. They do not have units however, and hence they are not in the class considered in Theorem 1, unless $\{h_1, \ldots, h_n\}$, $\overline{\{h_i\}}$ are open and closed.

We conclude this section with a sufficient condition that a subspace of m be isomorphic to m.

THEOREM 6. *Let $m = A \oplus B$ where A and B are closed subspaces of m. Then there are subspaces $\bar{m}$ and A_1 of m isomorphic to m and A respectively and such that $m = A_1 \oplus \bar{m}$.*

COROLLARY. If (X,P_s) and if $\dim_1(X) = \dim_1(m)$, then X and m are isomorphic.

Proof. Since $\dim_1(X) = \dim_1(m)$, X is isomorphic to a subspace A of m and m is isomorphic to a subspace m_1 of X. Both A and m_1 are P_t spaces for some t so we can write $m = A \oplus B$ and $X = m_1 \oplus Y$ for closed subspaces B of m and Y of X.

Theorem 6 promises that $m = A_1 \oplus \bar{m}$ where A_1 and $\bar{m}$ are isomorphic to A and m respectively. Then X and A_1 are isomorphic, say under u, $uX = A_1$. Then $A_1 = um_1 \oplus uY = m_2 \oplus B_1$ where m_2 is isomorphic to m.

Thus we can write $m = B_1 \oplus m_2 \oplus \bar{m}$. Since m_2 is isomorphic to $m_2 \oplus \bar{m}$, Theorem 4 asserts that $B_1 \oplus m_2$ is isomorphic to m.

Proof of Theorem 6. Loosely, the proof proceeds thus.
$m = A \oplus B = m_1 \oplus m_2 \oplus \ldots = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) \oplus \ldots =$
$= A_1 \oplus (B_1 \oplus A_2) \oplus (B_2 \oplus A_3) \oplus \ldots = A_1 \oplus \bar{m}_1 \oplus \bar{m}_2 \oplus \ldots =$
$= A_1 \oplus \bar{m}$. The A_j, B_j, m_j, $\bar{m}_j$, $\bar{m}$ are isomorphic to A, B, m, m, m respectively.

Choose subsequences $a_i = \{n_1^i, n_2^i, \ldots\}$ of N (the positive integers) such that $a_i \cap a_j = \phi$ if $i \neq j$, $i = 1,2, \ldots$; and such that $\bigcup_i a_i = N$. Define q_i, s_i on m to m by $q_i f(j) = 0$ if $j \notin a_i$ and $f(j)$ if $j \in a_i$; $s_i f(n_j^k) = 0$ if $i \neq k$ and $f(j)$ if $i = k$. Then $q_i s_i = s_i$ and s_i is an isomorphism of $m_i = q_i(m)$ with m.

Let $A_i = s_i(A)$ and $B_i = s_i(B)(m = A \oplus B)$. Then $m_i = A_i \oplus B_i$. If p is the projection $pm = A$, $(1 - p)m = B$, then $r_i = s_i p s_i^{-1} q_i$ and $v_i = s_i(1 - p)s_i^{-1} q_i$ are projections of m onto A_i and B_i respectively.

Let $\bar{m} = \{f | r_1(f) = 0\}$.

Since $m = A_1 \oplus \bar{m}$ and since A_1 is isomorphic to A the proof is finished if $\bar{m}$ is shown to be isomorphic to m.

Define v, u on m to m by $vf(j) = v_i f(j)$ if $j \in a_i$ and $uf(n_j{}^i) = f(n_j{}^{i+1})$. Then $w = (1 - v)u + v$ is the isomorphism desired, as follows. One easily shows that $q_i v = v_i$ and then that $v^2 = v$ so that v and $1 - v$ are projections. The following identities are also easily proved. $r_i f(j) = (1 - v)f(j)$ if $j \in a_i$; $us_{i+1} = s_i$; $q_i u(1 - v) = ur_{i+1}$; $q_i uv = uv_{i+1}$. Hence $(1 - v)u(1 - v) = u(1 - v)$; $uvu = uv$. For example,

$$vuvf(n_j^i) = v_i uvf(n_j^i) = s_i(1 - p)s_i^{-1}q_i uvf(n_j^i) = s_i(1 - p)s_i^{-1}uv_{i+1}f(n_j^i)$$

(since $q_i uv = uv_{i+1}$)

$$= s_i(1 - p)s_i^{-1}us_{i+1}(1 - p)s_{i+1}^{-1}q_{i+1}f(n_j^i)$$

$$= s_i(1 - p)s_i^{-1}s_i(1 - p)s_{i+1}^{-1}q_{i+1}f(n_j^i)$$

(since $us_{i+1} = s_i$)

$$= s_i(1 - p)s_{i+1}^{-1}q_{i+1}f(n_j^i) = us_{i+1}(1 - p)s_{i+1}^{-1}q_{i+1}f(n_j^i)$$

$$= uv_{i+1}f(n_j^i) = v_{i+1}f(n_j^{i+1}) = vf(n_j^{i+1}) = uvf(n_j^i).$$

Clearly w is linear and continuous.

w is $1 - 1$: Let $wf = 0 = vf + (1 - v)uf$. Then vf and $(1 - v)uf$ are in complementary subspaces of m so $vf = 0 = (1 - v)uf$. Then $(1 - v)f = f$ and so $(1 - v)uf = (1 - v)u(1 - v)f = u(1 - v)f = uf$. Thus $uf = 0$. Since $uf(n_j{}^i) = 0 = f(n_j{}^{i+1})$ it follows that $f \in m_1$. ($uf = 0$ if and only if $f \in m_1$). Since $(1 - v)f = f$, then

$$(1 - v)f(n_j^i) = (1 - v_i)f(n_j^i) = f(n_j^i)$$

$$= \begin{cases} 0 & \text{if } i > 1 \\ (1 - v_1)f(n_j^i) & \text{if } i = 1 \end{cases} = \begin{cases} (1 - v_1)f(n_j^i) & \text{if } i > 1 \\ (1 - v_1)f(n_j^i) & \text{if } i = 1 \end{cases} = (1 - v_1)f(n_j^i),$$

and so $(1 - v)f = (1 - v_1)f = f$ or $v_1 f = 0$. Then

$$(r_1 + v_1)f = s_1 p s_1^{-1}q_1 f + s_1(1 - p)s_1^{-1}q_1 f = s_1((p + 1 - p)s_1^{-1}q_1 f) = q_1 f = f$$

since $f \in m_1$. Thus $r_1 f = f$ and since $f \in \bar{m}$, $r_1 f = 0$. Thus w is $1 - 1$.

w is onto: Let $f \in m$ and define h by $h(n_j{}^i) = (1 - v)f(n_j{}^{i-1})$ if $i > 1$ and 0 if $i = 1$. Then $uh = (1 - v)f$ and let $g = h - vh + vf$. Then $r_1 g = r_1 h - r_1 vh + r_1 vf = 0$ as follows: $r_1 h = s_1 p s_1{}^{-1} q_1 h = 0$ since $q_1 h = 0$ (h vanishes on a_1). $r_1 vf_1(j) = (1 - v)vf_1(j) = 0$ for every f_1 and j, so $r_1 vh = r_1 vf = 0$. Finally

$$wg = wh - wvh + wvf = vh - vh + vf + (1 - v)uh - (1 - v)uvh + (1 - v)uvh = vf + (1 - v)f = f$$

once it is known that $(1 - v)uv = 0$ which was shown above. Q.E.D.

3. Separable Banach spaces and property S.

THEOREM 7. *The following are equivalent if B is separable.*

(a) (B,S)

(b) *If $X \supset B$ and if X is separable, then there is a continuous projection from X onto B.*

(c) (B,S_t) *for some t.*

(d) *For every embedding $u(B)$ of B into $C([0,1])$ there is a continuous projection from $C([0,1])$ onto $u(B)$.*

Proof. If (a), $X \supset B$, and X is separable, then the identity map I from B to B has a continuous extension u from x to B which is then a continuous projection of X onto B. If (b) and if u is an isometry from B onto B_1, let X_1 be separable and $X_1 \supset B_1$. Then there is an $X \supset B$ and an isometry u_1 of X with X_1 which agrees with u on B **(6**, pp. 90, 91**)**. If p is a projection of X onto B, then $u_1 p u_1^{-1}$ is a projection of X_1 onto B_1. Hence (b) is preserved up to isometry. Since B is separable, it can be embedded isometrically in m **(2**, p. 187**)**, say under u. Suppose there is no t for which (B,S_t). Then, for every positive integer n, there is a space X_n, a separable space $Y_n \supset X_n$, and a map u_n from X_n to B, such that $|u_n| = 1$ and if w_n is a map from Y_n to B, which extends u_n, then $|w_n| > n$. The maps uu_n from X_n to uB are also maps from X_n to m and hence have extensions w_n from Y_n to m with $|w_n| = |uu_n| = 1$. Each $w_n Y_n$ is separable. Hence the sets uB and $\bigcup_n w_n Y_n$ generate a separable subspace Y of m and, from the above calculation, there is a projection p of Y onto uB. The map $u^{-1}pw_n$ is an extension to Y_n of u_n and $|u^{-1}pw_n| \leqslant |p|$. This contradicts the assumption that $|u^{-1}pw_n|$ must be greater than n, for every n. Hence (b) implies (c). Clearly c implies a.

If uB is an embedding of B into $C([0,1])$, then u^{-1} has an extension w. Then uw is a continuous projection of $C([0,1])$ on uB. Now assume (d). If $Y \supset B$ and Y is separable we can embed Y in $C([0,1])$ **(2**, p. 185**)**, and let u be such an embedding. By (d) there is a continuous projection p from $C([0,1])$ onto uB, $u^{-1}pu$ is a continuous projection of Y onto B. Q.E.D.

The next theorem shows that no infinite dimensional separable Banach space has property S_1.

THEOREM 8. *Let B have the following property. If $Y \supset X$ and if Y/X is one dimensional, then a continuous linear map u from X to B has an extension u_1 such that $|u_1| = |u|$. Then (B,P_1).*

Proof. Suppose $A \supset X$ and $u:X \to B$ is continuous and linear. Let F denote the set of pairs (Y,w) such that $Y \supset X$ and w is an extension of u, $w:Y \to X$, such that $|w| = |u|$. Order F by saying $(Y,w) \geqslant (Y_1,w_1)$ if $Y \supset Y_1$ and $w_1 = w$ on Y_1. One easily shows a simply ordered subset of F has an upper bound; so by Zorn's Lemma choose a maximal element (Y,w). If $Y \neq A$ and if $a \in A - Y$, then there is an extension of w, w_1, from Y_1 to B, with $|w_1| = |w|$ where Y_1 is the subspace of A generated by Y and a (Y_1/Y is one dimensional). This contradicts maximality of (Y,w) so $Y = A$. Since $|w| = |u|$ and since A, X, and u are arbitrary, we have (B,P_1).

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COROLLARY 8.1. *If B is separable and (B,S_1), then B is finite dimensional.*

Proof. Let u be an isometric embedding of B in m, and suppose that Y/X is one dimensional and $v:X \to B$. We can write $Y = (y) \oplus X$ for some $y \in Y$, where (y) is the subspace of Y generated by y. Since (m,P_1) uv has an extension $v_1:Y \to m$ such that $|v_1| = |vu|$. v_1Y is contained in the subspace Z of m generated by uB and v_1y. This subspace is separable. If (B,S_1), then (uB,S_1) and there is a projection p from Z to uB such that $|p| = 1$. Then $u^{-1}pv_1$ is an extension of v such that $|u^{-1}pv_1| = |v|$. Thus B has the property of Theorem 8. The only separable such B are finite dimensional. Q.E.D.

The space c of convergent sequences has a variant of property S_1; if $c \subset X$ and if X is separable, then there is a subspace c_1 of c, isometric to c, and a projection p of X onto c_1 with $|p| = 1$.

Sobczyk **(15)** proved that if $c_0 \subset X \subset m$ where X is separable then there is a projection p from X onto c_0 such that $|p| \leqslant 2$. McWilliams **(11)** proved an analogous result for c, the space of convergent sequences with supremum norm, with $|p| \leqslant 3$. In both cases the authors showed $t = 2$ and $t = 3$ were the best possible t. From Theorem 7 it follows easily that (c_0,S_2) and (c,S_3).

These results are proved below, with the help of Theorem 7, as corollaries to:

THEOREM 9. *Let $H = [0,1]$ and let K be a closed subset of H. Then there are projections p and r of $C(H)$ onto $C_K(H)$ and X respectively, where X is the subspace of $C(H)$ of functions constant on K. Moreover p and r can be chosen so that $|p| \leqslant 2$, $|r| \leqslant 3$.*

Proof. $H - K$ is open and so is a countable union of sets (h_ik_i) where h_i and k_i are in K; and h is in $H - K$ if $h_i < h < k_i$ for some i. Let

$$(qf)h = \begin{cases} f(h) & \text{if } h \in K \\ \dfrac{f(k_i) - f(h_i)}{k_i - h_i}(h - h_i) + f(h_i) & \text{if } h \in (h_ik_i). \end{cases}$$

Then $|qf| \leqslant \sup \{|f(h)| \mid h \in K\} \leqslant |f|$ and $q^2f = qf$. Hence q is a projection of norm 1. If $qf = 0$, then $f(h) = 0$ if h is in K and if $f = 0$ on K, then $qf = 0$. Hence $I - q = p$ is a projection of $C(H)$ onto $C_K(H)$ of norm at most two.

Let e be the identically one function on H. Then $qe = e$ so $pe = 0$. Define a projection p_1 of $C(H)$ onto (e) by $p_1f(h) = f(k)e$, where k is fixed in K. Then $|p_1| = 1$ and $p_1f = 0$ for every $f \in C_K(H)$. Since $pf = 0$ for every $f \in (e)$ we have that $pp_1 = p_1p = 0$ and so $p + p_1 = r$ is a projection with $|r| \leqslant |p| + |p_1| \leqslant 3$, of $C(H)$ onto X.

COROLLARY 9.1. (c_0,S_2), (c,S_3).

Proof. Let c_1 be either c_0 or c and let w embed c_1 isometrically into $C(H)$. Then w' is an isomorphism of $(wc_1)'$ with c_1' and $|w'x'| = |x'|$ for every $x' \in (wc_1)'$. If $d_i \in c_1'$ is defined by $d_i(x) = x_i$ for every $x \in c_1$, let $e_i \in (wc_1)'$ such that $w'e_i = d_i$. Then $|e_i| = |d_i| = 1$ and each e_i is an extreme point of the

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unit ball of $(wc_1)'$. Hence (**14**, p. 104) e_i can be extended to an extreme point f_i' of the unit ball of $(C(H))'$. Then f_i' is of the form $\pm e_{h_i}$ for some h_i, where $e_{h_i}(f) = f(h_i)$ for every $f \in C(H)$ (**4**, p. 85). For $x \in c_1$,

$$wx(h)_i = e_{h_i}(wx) = \pm f_i'(wx) = \pm e_i(wx) = \pm d_i(x) = \pm x_i.$$

Let $K = \overline{\{h_i\}} - \{h_i\}$. Then $K \neq \emptyset$ since a convergent subsequence of h_i converges to a point in $H - \{h_i\}$ (if $x_i = 1/i$, then $wx(h_i) = \pm 1/i$ while $wx(h) = 0$ if $h \in K$).

If $c_1 = c_0$ let p be a projection of $C(H)$ onto $C_K(H)$ such that $|p| \leqslant 2$. If $f \in C_K(H)$ one easily shows that $f_i'(f) \to 0$ as $i \to \infty$ and we define $v{:}C_K(H) \to c_0$ by $(vf)_i = f_i'(f)$. Then wvp is the desired projection of $C(H)$ onto wc_0 and $|wvp| \leqslant |w|\ |v|\ |p| = |p| \leqslant 2$.

If $c_1 = c$, let r be a projection of $C(H)$ onto X, the subspace of $C(H)$ of functions constant on K, such that $|r| \leqslant 3$. Again one shows $f_i'(f)$ converges $(i \to \infty)$ and that if v is defined by $(vf)_i = f_i'(f)$, then wvr is a projection with norm at most three from $C(H)$ onto wc.

From Theorem 7 (d) the corollary follows.

COROLLARY 9.2. *Let Y be separable and let $X \subset Y$. If $\{x_n'\} \subset X'$ is such that $x_i'(x) \to x'(x)$ for every $x \in X$, then the sequence $\{x_i'\}$ can be extended to a sequence $\{y_i'\}$ such that $y_i'(y) \to y'(y)$ for every $y \in Y$ (and so y' is an extension of x'). Moreover the extensions y_i' can be chosen so that $|y_i'| \leqslant 3\ |x_i'|$.*

Proof. The mapping u from X to c defined by $(ux)_i = x_i'(x)$ for every $x \in X$ has an extension u_1 to Y such that $|u_1| \leqslant 3\ |u|$. Let $y_i' = u_1'd_1$. One easily shows the y_i' have the desired properties and converge pointwise on Y (weak-star) to a $y' \in Y'$ which extends x'.

Remarks. One can reverse the steps of Corollary 9.2 to show (c,S_3). McWilliams' result that 3 is the best t possible so that (c,S_t) then shows that the 3 in the corollary is the best possible. Since c is P_t for no t one cannot in general extend sequences of pointwise convergent linear functionals so that the extensions are pointwise convergent.

If Y is separable, $X \subset Y$, and $x_n' \in X'$ is a pointwise convergent sequence; choose extensions y_n' and a subsequence $f_i' = y_{n_i}'$ such that $n_i\uparrow$, f_i' is a pointwise convergent sequence and $|y_i'| = |x_i'|$ for every i. Using such sequences we can prove

THEOREM 10. *If $Y \supset c$ and if Y is separable, then there is a subspace c_1 of c such that c_1 is isometric to c and a projection p of Y onto c_1, such that $|p| = 1$.*

Proof. Each d_i (as in the proof of the above corollary) can be extended to a linear functional y_i' in Y' such that $|y_i'| = |d_i| = 1$. Since Y is separable choose a subsequence $\{y_{n_i}'\}$ of $\{y_i'\}$ which is pointwise convergent and so that $n_{i+1} > n_i$ for each i. Define u: $Y \to c$ by $(uy)_n = y_{n_i}'(y)$ if $n_i \leqslant n < n_{i+1}$. Let c_1 be the subspace of c of sequences f for which $f_{n_i} = f_{n_i+1} = \ldots = f_{n_{i+1}-1}$

for every i. Clearly $uY \subset c_1$. If $f \in c_1$, then $(uf)_n = y_{n_i}'(f)$, $n_i \leqslant n < n_{i+1}$, $= d_{n_i}(f) = f_{n_i} = f_n$ so that $uf = f$ and u is a projection of Y onto c_1 with $|u| = 1$.

It remains to show c_1 is isometric to c. Define v from c to c_1 by $(vf)_n = f_i$ if $n_i \leqslant n < n_{i+1}$. Then $vf \in c_1$ and $|vf| = |f|$. If $f \in c_1$ let g be that element of c defined by $g(i) = f(n_i)$. Then $(vg)_n = g_i = f_{n_i} = f_n$ if $n_i \leqslant n \leqslant n_{i+1}$. Thus $vg = f$ and v is onto.

4. Involutions of norm one in $C(K)$ where K is compact and extremally disconnected. Kelley constructs a compact, extremally disconnected H from the extreme points of the unit ball of B' if (B,P_1) and shows that B is isometric to $C(H)$. In this section it is shown that if B is "conveniently situated in a $C(K)$ space, with K compact and extremally disconnected, then the representation space H can be taken to be an open and closed subset of K.

The following theorem is due to Stone (**4**, p. 86). Eilenberg (**5**) established the theorem for arbitrary topological H.

THEOREM (Stone). *If u is an isometry from $C(L)$ onto $C(K)$, where L and K are compact, then there is a homeomorphism π of K with L, and an element a of $C(K)$ such that $(uf)(k) = a(k)f(\pi k)$ and a takes only the values ± 1.*

If $K = L$, then π is a homeomorphism of K with K. This is the case considered below.

If $\pi^2 = 1$ (the identity mapping), then π induces a linear mapping u of $C(K)$ onto $C(K)$ such that $|u| = 1$ and $u^2 = 1$ (such a u is called an involution). The map $p = (1 - u)/2$ is a projection and $|p| = |1 - p| = 1$. Moreover $p(C(K)) = B$, where B is the subspace of $C(K)$ for which $b \in B$ if and only if $ub = b$, $(1 - p)C(H) = X$ is the subspace $x \in X$ if and only if $ux = -x$.

LEMMA. *With the notation above there are disjoint subsets H and W of K such that $H \cup W = K$ and B,X are isometric to $C_W(K)$ and $C_H(K)$ respectively.*

Before proceeding with the proof an example will show why K is chosen to be extremally disconnected. Let K be the set of rationals of the form $1/n$, n a positive integer, and 0 using the relative topology of the reals. Let π be defined by

$$\pi(0) = 0,\ \pi\left(\frac{1}{2n}\right) = \frac{1}{2n-1},\ \pi\left(\frac{1}{2n-1}\right) = \left(\frac{1}{2n}\right)$$

for $n \geqslant 1$. Then π is a homeomorphism of K and $\pi^2 = I$. Let u be the induced involution. Then $(uf)h = f(\pi h)$ and $|u| = 1$. Both B and X are infinite dimensional, and so W and H must both be infinite. The space K does not permit such a decomposition though it is a totally disconnected space.

Proof of the Lemma. Let $\mathfrak{F}$ be the set of $U \subset K$ such that U is open and there is an $x \in X$ such that $x(k) > 0$ if $k \in U$. Order $\mathfrak{F}$ by inclusion. If F is a simply ordered subset of $\mathfrak{F}$ let $V = \bigcup_{U \in F} U$.

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For each $U \in F$ choose x_U such that $x_U(k) > 0$ if $k \in U$, $|x_U| \leqslant 1$, and $x_U \in X$. The collection x_U, $U \in F$ is bounded above in $C(K)$, and since $C(K)$ is a complete lattice, let y be the least upper bound of this collection. Clearly $y(k) > 0$ if $k \in V$ so that $y(k) \geqslant 0$ if $k \in \bar{V}$ which is an open set.

Now $\pi V \cap V = \emptyset$ as follows: If $k \in \pi V \cap V$, then $\pi k \in V \cap \pi V$. Let $k \in U_1$, $U_1 \in F$. Then $\pi k \in U_2$ where $U_2 \in F$ for some U_2. So either $U_2 \supset U_1$ or $U_1 \supset U_2$ since F is simply ordered. Suppose $U_2 \supset U_1$. Then πk, $k \in U_2$ and $x_{U_2}(k) = -x_{U_2}(\pi k)$ (since $ux_{U_2} = -x_{U_2}$) which is a contradiction to $x_{U_2}(k') > 0$ if $k' \in U_2$.

Since πV and V are open and disjoint, $\overline{\pi V} \cap \bar{V} = \emptyset$. One easily checks that $\overline{\pi V} = \pi \bar{V}$. Define f by

$$f(k) = \begin{cases} y(k) \text{ if } k \in \bar{V} \\ -y(\pi k) \text{ if } k \in \pi \bar{V} \\ 0 \text{ otherwise,} \end{cases}$$

Then it is easily seen that $f \in X$ and $f(k) > 0$ if $k \in V$. Thus F has an upper bound and by Zorn's lemma let W be a maximal element of $\mathscr{F}$.

As above $W \cap \pi W = \emptyset$ and W and πW are open. Hence $\bar{W} \cap \overline{\pi W} = \emptyset$ and $\overline{\pi W} = \pi \bar{W}$. Define f by

$$f(k) = \begin{cases} 1 \text{ if } k \in \bar{W} \\ -1 \text{ if } k \in \pi \bar{W} \\ 0 \text{ otherwise,} \end{cases} \qquad \text{Then } f \in C(K)$$

and $f \in X$. Moreover $f(b) > 0$ if $k \in \bar{W}$ and since $\bar{W}$ is open and W is maximal, $W = \bar{W}$.

The next step is to show $x(k) = 0$ if $k \notin W \cup \pi W$ ($K - (W \cup \pi W)$ is the set of fixed points of π). Assume, by way of contradiction, that $k \in W \cup \pi W$ exists such that $x(k) > 0$. Now $K - (W \cup \pi W)$ is open and closed and we choose an open and closed subset L of $K - (W \cup \pi W)$ such that $x(k) > 0$ on L. Letting

$$x_1(k) = \begin{cases} x(k) \text{ if } k \in L \cup \pi L \\ 0 \text{ otherwise,} \end{cases}$$

one checks that $x_1 \in X$ and $(x_1 + f)(k) > 0$ if $k \in \bar{W} \cup L$, where f is the function

$$f(k) = \begin{cases} 1 \text{ if } k \in \bar{W} \\ -1 \text{ if } k \in \pi \bar{W} \\ 0 \text{ otherwise.} \end{cases}$$

Since $x_1 + f$ is in X, this contradicts the choice of W as maximal. If $x(k) < 0$ repeat the above using $-x$. Thus $x(k) = 0$ if $k \notin W \cup \pi W$.

Let $H = K - W$ so that H is open and closed. Define v on X by

$$(vx)\ (k) = \begin{cases} x(k) \text{ if } k \in W \\ 0 \text{ if } k \notin W \end{cases}$$

and let v on B be defined by

$$vb(k) = \begin{cases} b(k) \text{ if } k \in H \\ 0 \text{ if } k \notin H. \end{cases}$$

If $f \in C_H(K)$, then let

$$x(k) = \begin{cases} f(k) \text{ if } k \in W \\ -\mathrm{f}(\pi k) \text{ if } k \in \pi W \\ 0 \text{ otherwise.} \end{cases}$$

Then $x \in X$ and $vx = f$ so v is onto. If $f \in C_W(K)$ let

$$b(k) = \begin{cases} f(k) \text{ if } k \in H \\ f(\pi k) \text{ if } k \in W. \end{cases}$$

Then $b \in B$ and $vb = f$. One easily checks that $|vx| = |x|$, $|vb| = |b|$ if $x \in X$, $b \in B$.

Eilenberg (**5**, p. 577) showed that for any topological H if $C(H) = B \oplus X$ and if $|f|$ is the maximum of $|b|$ and $|x|$, where b is in B, x is in X, and $f = b + x$, then there are sets K and M such that $K \cap M$ is empty, $K \cup M = H$, and $b \in C_K(H)$, $x \in C_M(X)$. In this case the map u defined by $u(b + x) = b - x$, for x in X and b in B, is an involution and $|u| = 1$. Not every u with $|u| = 1$ yields a decomposition of this type. As an example let H be the set of integers. Define u on $m(H)$ by $(uf)h = f(-h)$. There is a $C(K)$ isometric to $m(H)$ with K compact and extremally disconnected. The decomposition of $C(K)$ induced by the involution of $C(K)$ which corresponds to u is not the above type. From the lemma we can prove the following:

THEOREM 11. *Let K be compact and extremally disconnected and u an involution of $C(K)$ with $|u| = 1$. Let p be the projection $(I + u)/2$, $pC(K) = B$ and $(I - p)\ C(K) = X$. Then there is an H and V with $H \cap V$ empty, $H \cup V = K$, and B is isometric to $C_V(K)$ while X is isometric to $C_H(K)$.*

Proof. Let $uf(k) = a(k)f(\pi k)$ where $a(k) = \pm 1$ for every k (see Stone's theorem above). Let $U = \{k|a(k) = 1\}$, $W = \{k|a(k) = -1\}$. Then U, W are open and closed, disjoint, and $U \cup W = K$. Also if $k \in U$ and $\pi k \in W$, then $u^2 f(k) = a(k)uf(\pi k) = a(k)a(\pi k)f(k)$ (since $\pi^2 k = k$) $= -f(k)$ which is a contradiction to $u^2 = I$ if we choose f such that $f(k) \neq 0$. Hence $\pi U = U$, $\pi W = W$. Define w_1 on B by

$$w_1 b(k) = \begin{cases} b(k) \text{ if } k \in U \\ 0 \text{ if } k \in W. \end{cases}$$

Then $w_1 b \in B$ (using that $\pi U = U$, $\pi W = W$) and we denote $w_1 B$ by B_1. Similarly, define w_1 on X and denote the image $w_1 X$ by X_1. Every $f \in C_W(K)$ is clearly of the form $b_1 + x_1$ for some $b_1 \in B_1$ and $x_1 \in X_1$ and u restricted to $C_W(K)$ is such that $u^2 = I$ and $|u| = 1$. Identify $C_W(K)$ with $C(U)$ by letting $\bar{f}(h) = f(h)$ if $h \in U$, $\bar{f}(h) = 0$ if $h \notin U$, where $f \in C(U)$; there are subsets V_1, H_1, of U which are open, closed, disjoint and $V_1 \cup H_1 = U$, and such that B_1 and X_1 are isometric to $C_{V_1 \cup W}(K)$ and $C_{H_1 \cup W}(K)$ respectively.

Defining w_2 on B by

$$w_2 b(k) = \begin{cases} b(k) \text{ if } k \in W \\ 0 \text{ if } k \in U, \end{cases}$$

and similarly on X, and denoting $w_2 B$ by B_2 and $w_2 X$ by X_2, then u restricted to $C_U(K)$ is such that $u^2 = I$. Here the set of fixed points of u is X_2 and $B_2 = \{f \in C_U(K) | uf = -f\}$. Reversing the roles of B and X in the lemma there are subsets V_2 and H_2 of W which are open, closed, disjoint, and $V_2 \cup H_2 = W$ and such that B_2 is isometric to $C_{V_2 \cup U}(K)$ and X_2 is isometric to $C_{H_2 \cup U}(K)$.

Let v_1 and v_2 be isometric mappings of B_1 and B_2 onto $C_{V_1 \cup W}(K)$ and $C_{V_2 \cup U}(K)$ respectively. Now $B = B_1 \oplus B_2$ and if $b = b_1 + b_2$ with $b_1 \in B_1$ and $b_2 \in B_2$, then $|b| = \max(|b_1|, |b_2|)$. Define v on B by $vb = v_1 b_1 + v_2 b_2$. Then v is onto $C_{V_1 \cup W}(K) \oplus C_{V_2 \cup U}(K) = C_{V_1 \cup V_2}(K)$ and $|v_1 b_1 + v_2 b_2| = \max(|vb_1|, |vb_2|) = |b|$. Similarly, X is isometric to $C_{H_1 \cup H_2}(K)$. Put $V = V_1 \cup V_2$, H $= H_1 \cup H_2$.

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