Bull. Aust. Math. Soc. **96** (2017), 326–332
doi:10.1017/S0004972717000053

SELF-SHRINKERS WITH SECOND FUNDAMENTAL FORM OF CONSTANT LENGTH

QIANG GUANG

(Received 15 December 2016; accepted 18 December 2016; first published online 2 March 2017)

Abstract

We give a new and simple proof of a result of Ding and Xin, which states that any smooth complete self-shrinker in $\mathbb{R}^3$ with the second fundamental form of constant length must be a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{2-k}$ for some $k \leq 2$. Moreover, we prove a gap theorem for smooth self-shrinkers in all dimensions.

2010 *Mathematics subject classification*: primary 53C44.

Keywords and phrases: self-shrinker, second fundamental form, mean curvature flow.

1. Introduction

A one-parameter family of hypersurfaces $M_t \subset \mathbb{R}^{n+1}$ flows by mean curvature if

$$\partial_t x = -H\mathbf{n},$$

where H is the mean curvature, $\mathbf{n}$ is the outward pointing unit normal and x is the position vector.

We call a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ a *self-shrinker* if it satisfies

$$H = \frac{\langle x, \mathbf{n}\rangle}{2}.$$

Under the mean curvature flow (MCF), Σ is shrinking homothetically, that is, $\Sigma_t \equiv \sqrt{-t}\,\Sigma$ gives an MCF.

Self-shrinkers play a key role in the study of MCF. By Huisken's montonicity formula [12] and an argument of Ilmanen [14] and White [19], self-shrinkers provide all singularity models of the MCF. Although there are infinitely many of them, we only know a few embedded complete examples (see [2, 6, 15, 17, 18]). Moreover, numerical results show that it is impossible to give a complete classification of self-shrinkers in higher dimensions. Under certain conditions, there are many classification results for self-shrinkers.

In [12, 13], Huisken proved that the only smooth complete embedded self-shrinkers in $\mathbb{R}^{n+1}$ with polynomial volume growth, $H \geq 0$, and $|A|$ bounded are the generalised cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ (where $\mathbb{S}^k$ has radius $\sqrt{2k}$). Here, $|A|$ is the norm of the second fundamental form. Later, Colding and Minicozzi [8] showed that this result holds without the $|A|$ bound, which was crucial in their study of generic singularities. Moreover, after introducing the concept of entropy stability, they proved that the only stable self-shrinkers are $\mathbb{S}^k \times \mathbb{R}^{n-k}$, and all others can be perturbed away. Thus, the only generic singularities are the generalised cylinders. A natural question is to seek rigidity and gap results for the generalised cylinders. We are interested in conditions involving the norm of the second fundamental form, that is, $|A|$. First, we consider the following question.

Conjecture 1.1. *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a smooth complete embedded self-shrinker with polynomial volume growth. If the second fundamental form of Σ is of constant length, that is, $|A|^2 = constant$, then Σ is a generalised cylinder.*

The case $n = 1$ follows from a more general result by Abresch and Langer [1], which says that the only smooth complete and embedded self-shrinkers in $\mathbb{R}^2$ are the lines and a round circle. In $\mathbb{R}^3$, that is $n = 2$, the above conjecture was proved by Ding and Xin [11] using the identity

$$\tfrac{1}{2}\mathcal{L}|\nabla A|^2 = |\nabla^2 A|^2 + (1 - |A|^2)|\nabla A|^2 - 3\Xi - \tfrac{3}{2}|\nabla|A|^2|^2,$$

where $\mathcal{L}$ is the operator $\mathcal{L} = \Delta - \langle \frac{1}{2}x, \nabla\cdot\rangle$, h_{ij} is the second fundamental form and

$$\Xi = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml} - 2\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{im}h_{jl}.$$

In this note we give a new and simple proof of the above result without heavy computation. More precisely, we prove the following theorem.

Theorem 1.2. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth complete embedded self-shrinker with polynomial volume growth. If the second fundamental form of Σ^2 is of constant length, that is, $|A|^2 = constant$, then Σ^2 is a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{2-k}$ for $k \le 2$.*

The key idea in the proof of Theorem 1.2 is to analyse the point where $|x|$ achieves its minimum. Since Σ has polynomial volume growth and, thus, Σ is proper (see [5, 10]), such a point always exists. At this point, we have $\nabla H = 0$. Combining this with $|A| = constant$ implies that $|A|^2 \le 1/2$. Therefore, the conclusion follows directly from the fact that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \le 1/2$ must be a generalised cylinder. We remark that our method does not apply to higher dimensions to prove Conjecture 1.1.

For self-shrinkers, there are some gap phenomena for the norm of the second fundamental form. Cao and Li [4] proved that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \le \frac{1}{2}$ in arbitrary codimension is a generalised cylinder. Colding *et al.* [7] showed that generalised cylinders are rigid in the strong sense that any self-shrinker which is sufficiently close to one of the generalised cylinders on a large and compact set must itself be a generalised cylinder. Using this result, we prove that any self-shrinker with $|A|^2$ sufficiently close to $\frac{1}{2}$ must also be a generalised cylinder (cf. [11, Theorem 4.4]).

THEOREM 1.3. *Given n and λ_0, there exists $\delta = \delta(n, \lambda_0) > 0$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

(†) $\quad |A|^2 \leq \frac{1}{2} + \delta,$

then Σ^n is a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$.

REMARK 1.4. It is expected that one could remove the entropy bound in Theorem 1.3 when $n = 2$. In other words, the bound for $|A|^2$ may imply the entropy bound in $\mathbb{R}^3$. Let Σ^2 be a closed self-shrinker in $\mathbb{R}^3$ with $|A|^2 \leq C$, where C is a constant less than 1. By the *Gauss–Bonnet formula*, one can easily see that the genus of Σ^2 is 0 and, thus, obtain an entropy bound for Σ^2 (see [10]). Recently, Brendle [3] proved that the only compact embedded self-shrinker in $\mathbb{R}^3$ of genus 0 is the round sphere $\mathbb{S}^2(2)$.

2. Background

In this section we recall some background for self-shrinkers from [8]. Throughout this note we assume self-shrinkers to be smooth complete embedded, without boundary and with polynomial volume growth.

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a hypersurface, Δ its Laplacian operator, $\mathbf{n}$ its outward unit normal, $H=\mathrm{div}_\Sigma \mathbf{n}$ its mean curvature and A its second fundamental form. With this convention, the mean curvature H is n/r on the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius r.

First, recall the operators $\mathcal{L}$ and L defined by

$$\begin{aligned} \mathcal{L} &= \Delta - \tfrac{1}{2}\langle x, \nabla \cdot \rangle, \\ L &= \Delta - \tfrac{1}{2}\langle x, \nabla \cdot \rangle + |A|^2 + \tfrac{1}{2}. \end{aligned}$$

LEMMA 2.1 [8]. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth self-shrinker, then*

$$\begin{aligned} \mathcal{L}|x|^2 &= 2n - |x|^2, \\ \mathcal{L}H^2 &= 2(\tfrac{1}{2} - |A|^2)H^2 + 2|\nabla H|^2, \\ \mathcal{L}|A|^2 &= 2(\tfrac{1}{2} - |A|^2)|A|^2 + 2|\nabla A|^2. \end{aligned}$$

A direct consequence of Lemma 2.1 is the following corollary.

COROLLARY 2.2 [4, 16]. *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a smooth self-shrinker. If $|A|^2 \leq \frac{1}{2}$, then Σ is a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$. Moreover, if $|A|^2 < \frac{1}{2}$, then Σ is a hyperplane.*

Colding and Minicozzi [8] introduced the following notions of the F-functional and the entropy of a hypersurface.

DEFINITION 2.3. For $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$, the F-functional F_{x_0,t_0} of a hypersurface $M \subset \mathbb{R}^{n+1}$ is defined by

$$F_{x_0,t_0}(M) = (4\pi t_0)^{-n/2} \int_M \exp\Big(-\frac{|x - x_0|^2}{4t_0}\Big),$$

https://doi.org/10.1017/S0004972717000053 Published online by Cambridge University Press

and the entropy of M is given by

$$\lambda(M) = \sup_{x_0,t_0} F_{x_0,t_0}(M),$$

where the supremum is taken over all $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$.

3. Proof of Theorem 1.2

By Corollary 2.2, if $|A|^2 < \frac{1}{2}$, then Σ is a hyperplane in $\mathbb{R}^3$. Therefore, in the following we only consider the case where $|A|^2 \geq \frac{1}{2}$.

For any point $p \in \Sigma$, we can choose a local orthonormal frame $\{e_1, e_2\}$ such that the coefficients of the second fundamental form are $h_{ij} = \lambda_i\delta_{ij}$ for $i, j = 1, 2$. By definition,

$$|\nabla H|^2 = (h_{111} + h_{221})^2 + (h_{112} + h_{222})^2.$$

Since $|A| = \mathit{constant}$, Lemma 2.1 gives

$$h_{11}h_{111} + h_{22}h_{221} = h_{11}h_{112} + h_{22}h_{222} = 0 \tag{3.1}$$

and

$$|\nabla A|^2 = h_{111}^2 + h_{222}^2 + 3h_{112}^2 + 3h_{221}^2 = |A|^2(|A|^2 - \tfrac{1}{2}).$$

First, we prove that $|x| > 0$ on Σ. We argue by contradiction. Suppose that Σ goes through the origin. Then at the origin, we have $H = |\nabla H| = 0$. Therefore,

$$h_{11} + h_{22} = h_{111} + h_{221} = h_{112} + h_{222} = 0.$$

Combining this with (3.1) yields

$$h_{111} = h_{222} = h_{112} = h_{221} = 0.$$

This implies that

$$|\nabla A|^2 = |A|^2(|A|^2 - \tfrac{1}{2}) = 0, \quad \text{that is, } |A|^2 = \tfrac{1}{2}.$$

By Corollary 2.2, we conclude that Σ is $\mathbb{S}^2$ or $\mathbb{S}^1 \times \mathbb{R}$. However, this contradicts the assumption that Σ goes through the origin.

Note that Σ is proper since Σ has polynomial volume growth (see [10] and [5, Theorem 4.1]). By the maximum principle, Σ must intersect $\mathbb{S}^2(2)$. Hence, there exists a point $p \in \Sigma$ which minimises $|x|$.

Now, at the point p, we have $|x| > 0$ and $x^T = 0$, where x^T is the tangential projection of x. This implies that

$$4H^2(p) = |x|^2(p) \quad \text{and} \quad \nabla H(p) = 0.$$

Thus, we have

$$h_{111} + h_{221} = h_{112} + h_{222} = 0.$$

By (3.1),

$$h_{111}(h_{11} - h_{22}) = h_{222}(h_{11} - h_{22}) = 0.$$

https://doi.org/10.1017/S0004972717000053 Published online by Cambridge University Press

If $h_{111} = h_{222} = 0$, then we see that $|\nabla A|^2 = 0$ and $|A|^2 = \frac{1}{2}$. By Corollary 2.2, we conclude that Σ is a generalised cylinder.

If $h_{11} = h_{22}$, then

$$|A|^2 = 2h_{11}^2 = \frac{H^2(p)}{2} = \frac{|x|^2(p)}{8}. \tag{3.2}$$

Since every smooth complete self-shrinker must intersect the sphere $\mathbb{S}^2(2)$, we conclude that $|x|(p) \leq 2$. By (3.2), this gives

$$|A|^2 = \frac{|x|^2(p)}{8} \leq \frac{1}{2}.$$

The theorem follows immediately from Corollary 2.2.

4. Proof of Theorem 1.3

The proof of Theorem 1.3 relies on the following two ingredients from [7]. The first one is a rigidity theorem for the generalised cylinders, and the second one is a compactness theorem for self-shrinkers.

THEOREM 4.1 [7]. *Given n, λ_0 and C, there exists $R = R(n, \lambda_0, C)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

- *Σ is smooth in B_R with $H \geq 0$ and $|A| \leq C$ on $B_R \cap \Sigma$,*

then Σ is a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$.

LEMMA 4.2 [7]. *Let $\Sigma_i \subset \mathbb{R}^{n+1}$ be a sequence of F-stationary varifolds with $\lambda(\Sigma_i) \leq \lambda_0$ and such that*

$$B_{R_i} \cap \Sigma_i \text{ is smooth with } |A| \leq C,$$

where $R_i \to \infty$. Then there exists a subsequence Σ_i' that converges smoothly and with multiplicity one to a complete embedded self-shrinker Σ with $|A| \leq C$ and

$$\lim_{i\to\infty} \lambda(\Sigma_i') = \lambda(\Sigma).$$

REMARK 4.3. In the above lemma, the entropy bound is used to guarantee that the convergence is with finite multiplicity. Moreover, if the multiplicity is greater than one, then the limit is L-stable [9]. Using the fact that there are no complete L-stable self-shrinkers with polynomial volume growth (see [9, Theorem 0.5]) the multiplicity of the convergence must be one.

Now we are ready to give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. We will argue by contradiction, so suppose there is a sequence of smooth embedded self-shrinkers $\Sigma_i \neq \mathbb{S}^k \times \mathbb{R}^{n-k}$ $(k \leq n)$ with $\lambda(\Sigma_i) \leq \lambda_0$ and

$$|A|^2 \leq \frac{1}{2} + \frac{1}{i}. \tag{4.1}$$

https://doi.org/10.1017/S0004972717000053 Published online by Cambridge University Press

By Lemma 4.2, there exists a subsequence Σ_i (still denoted by Σ_i) that converges smoothly and with multiplicity one to a complete embedded self-shrinker Σ. By (4.1), we can conclude that Σ satisfies $|A|^2 \le \frac{1}{2}$, and thus, Σ is a generalised cylinder.

Now we choose the R as in Theorem 4.1. For N sufficiently large, Σ_m is very close to Σ on $B_{2R} \cap \Sigma_m$ for $m \ge N$, that is,

$$\Sigma_m \text{ satisfies } H \ge 0 \text{ and } |A| \le 1 \text{ on } B_R \cap \Sigma_m.$$

By the rigidity theorem for self-shrinkers, Theorem 4.1, Σ_m is a generalised cylinder. However, this contradicts our assumption that Σ_i is not a generalised cylinder, completing the proof. □

Theorem 4.1 and Lemma 4.2 also imply that any self-shrinker satisfying $|A|^2 \le 1/2$ on a large ball must be a generalised cylinder. This improves Corollary 2.2 and can be thought of as a quantitative version of Corollary 2.2. The result is as follows.

THEOREM 4.4. *Given n and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth embedded self-shrinker with entropy $\lambda(\Sigma) \le \lambda_0$ satisfying*

- $|A|^2 \le \frac{1}{2}$ *on $B_R \cap \Sigma$,*

then Σ is a generalised cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $k \le n$.

References

[1] U. Abresch and J. Langer, 'The normalized curve shortening flow and homothetic solutions', *J. Differential Geom.* **23**(2) (1986), 175–196.

[2] S. B. Angenent, 'Shrinking doughnuts', in: *Nonlinear Diffusion Equations and their Equilibrium States, 3 (Gregynog, 1989)*, Progress in Nonlinear Differential Equations and their Applications, 7 (Birkhäuser, Boston, 1992), 21–38.

[3] S. Brendle, 'Embedded self-similar shrinkers of genus 0', *Ann. of Math. (2)* **183**(2) (2016), 715–728.

[4] H. Cao and H. Li, 'A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension', *Calc. Var. Partial Differential Equations* **46**(3–4) (2013), 879–889.

[5] X. Cheng and D. Zhou, 'Volume estimate about shrinkers', *Proc. Amer. Math. Soc.* **141**(2) (2013), 687–696.

[6] D. Chopp, 'Computation of self-similar solutions for mean curvature flow', *Experiment. Math.* **3**(1) (1994), 1–15.

[7] T. H. Colding, T. Ilmanen and W. P. Minicozzi II, 'Rigidity of generic singularities of mean curvature flow', *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 363–382.

[8] T. H. Colding and W. P. Minicozzi II, 'Generic mean curvature flow I: Generic singularities', *Ann. of Math. (2)* **175**(2) (2012), 755–833.

[9] T. H. Colding and W. P. Minicozzi II, 'Smooth compactness of self-shrinkers', *Comment. Math. Helv.* **87**(2) (2012), 463–475.

[10] Q. Ding and Y. L. Xin, 'Volume growth, eigenvalue and compactness for self-shrinkers', *Asian J. Math.* **17**(3) (2013), 443–456.

[11] Q. Ding and Y. L. Xin, 'The rigidity theorems of self-shrinkers', *Trans. Amer. Math. Soc.* **366**(10) (2014), 5067–5085.

[12] G. Huisken, 'Asymptotic behavior for singularities of the mean curvature flow', *J. Differential Geom.* **31**(1) (1990), 285–299.

[13] G. Huisken, ‘Local and global behaviour of hypersurfaces moving by mean curvature’, in: *Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990)*, Proceedings of Symposia in Pure Mathematics, 54 (American Mathematical Society, Providence, RI, 1993), 175–191.

[14] T. Ilmanen, ‘Singularities of mean curvature flow of surfaces’, Preprint, 1997.

[15] S. Kleene and N. M. Møller, ‘Self-shrinkers with a rotational symmetry’, *Trans. Amer. Math. Soc.* **366**(8) (2014), 3943–3963.

[16] N. Q. Le and N. Sesum, ‘Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers’, *Comm. Anal. Geom.* **19**(4) (2011), 633–659.

[17] N. M. Møller, ‘Closed self-shrinking surfaces in $\mathbf{R}^3$ via the torus’, Preprint, 2011, arXiv:1111.7318.

[18] X. H. Nguyen, ‘Construction of complete embedded self-similar surfaces under mean curvature flow. I’, *Trans. Amer. Math. Soc.* **361**(4) (2009), 1683–1701.

[19] B. White, ‘Partial regularity of mean-convex hypersurfaces flowing by mean curvature’, *Int. Math. Res. Not. IMRN* **4** (1994), 185–192.

QIANG GUANG, Department of Mathematics,
University of California, Santa Barbara, CA 93106-3080, USA
e-mail: guang@math.ucsb.edu

https://doi.org/10.1017/S0004972717000053 Published online by Cambridge University Press