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# SOME PROPERTIES OF ISOLATING BLOCKS FOR PLANAR SYSTEMS

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**Abstract.** In this paper, some qualitative properties of trajectories inside an isolating block for planar differential equations are obtained.

## §1. Introduction

Consider the differential system defined in the plane

$$\frac{dx}{dt} = X(x,y), \qquad \frac{dy}{dt} = Y(x,y). \tag{1.1}$$

Suppose $X, Y \in C^1$. Let the vector field $V \equiv (X, Y)$ define a flow $f(p,t)$. Let $B \subset R^2$ be the closure of a bounded and connected open set with the boundary $\partial B$. In general, $B$ is assumed to be multiply connected. Let $L_1, \cdots, L_n$ denote its boundary components, where $L_i \cap L_j = \phi$ for $i \neq j$, and $L_1$ the external boundary. Each of them is a smooth simple closed curve. Let int$B$ denote the interior of $B$. We define three subsets $b^+, b^-$ and $\tau$ as follows:

$$b^+ = \{p \in \partial B | \exists \varepsilon > 0 \quad \text{with} \quad f(p, (-\varepsilon, 0)) \cap B = \phi\},$$

$$b^- = \{p \in \partial B | \exists \varepsilon > 0 \quad \text{with} \quad f(p, (0, \varepsilon)) \cap B = \phi\},$$

$$\tau = \{p \in \partial B | V \quad \text{is tangent to} \quad B \text{ at } p\}.$$

Definition 1.1. ([1]) If $b^+ \cap b^- = \tau$ and $b^+ \cup b^- = \partial B$, then $B$ is called an isolating block for the flow defined by (1.1).

It follows from the above definition that if $B$ is an isolating block, then all the tangencies to $B$ must be external.

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DEFINITION 1.2. Suppose $B \subset R^2$ is an isolating block for the flow defined by (1.1). If a trajectory $\Gamma_1$ of (1.1) enters $B$ at $M_1$ (a strict entrance point) of the external boundary $L_1$ of $B$ and then leaves $B$ at $M_2$ (a strict exit point) of $L_1$ so that int$B$ is divided into two disconnected regions $B_1$ and $B_2$, then $\Gamma_1$ is called a cut trajectory of $B$.

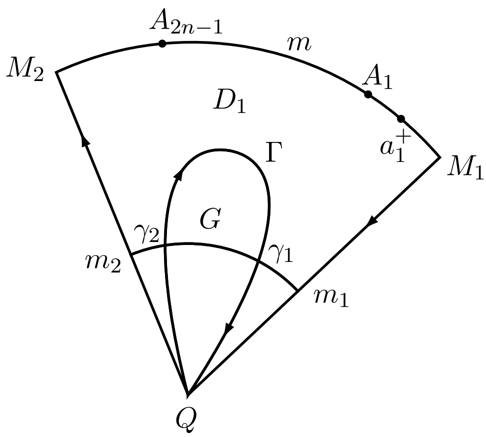
Conley and Easton in [2] have studied generally properties of the isolating block. A special property of planar isolating blocks and an application to the existence of connecting orbits have discussed in [3] (See Lemma 1 of [3]). In the present paper, we shall discuss qualitative properties of planar flows inside the isolating block and give some new results.

## §2. The main results

Suppose $B$ is an isolating block for flow defined by (1.1) and a critical point $Q \in \text{int}B$. Our main aim is to study the existence of elliptic regions of $Q$ (See [4, p.295] for the definition). Therefore, we consider a bounded sectorial region $D$ contained in $B$ with boundary consisting of the critical point $Q$, two semi-trajectory arcs $f(M_1, R^+), f(M_2, R^-)$ and the closed subarc $M_1mM_2$ of $\partial B$ from $M_1$ to $M_2$, and such that when $t \to +\infty$ (or $-\infty$), $f(M_1, t)$ (or $f(M_2, t)$) tends to $Q$, and $M_1 \in b^+, M_2 \in b^-$. Such a sectorial region is said to be adjacent to $\partial B$.

DEFINITION 2.1. Suppose $D$ is a bounded sectorial region adjacent to $\partial B$, as stated above. $D$ is said to be inadmissible if there are a trajectory $\Gamma \subset D$ which tends to $Q$ as $t \to \pm\infty$ and a circle $\rho$ of radius $r$ small enough with the centre $Q$ such that the interior of each of the curvilinear triangles $Qm_1\gamma_1$ and $Qm_2\gamma_2$ is a parabolic sector of $Q$ ([5,p.164]) in $\rho$, where it is assumed that $\rho$ intersects $QM_1, \Gamma$ and $QM_2$ at $m_1, \gamma_1, \gamma_2$ and $m_2$, respectively (Fig.1).

THEOREM 2.1. *Let $B$ be an isolating block for flow defined by* (1.1) *and a critical point $Q \in intB$. Let $D$ be an inadmissible sectorial region adjacent to $\partial B$ and let $D$ do not contain any internal boundary components of $\partial B$. Let $D_1 = D \backslash \overline{G}$, where $G$ is the region enclosed by $\Gamma$ and $Q$. Then there must be at least one critical point of* (1.1) *in $D_1$* (Fig.1).




(Figure 1)

*Proof.* Since $M_1 \in b^+$ and $M_2 \in b^-$, there must be at least one tangency to $B$ on the segmental arc $M_1mM_2$. Let $A_1, A_2, \cdots, A_{2n-1}$ be the tangencies to be arranged in numerical order on the arc $M_1mM_2$. Then the points on the segmental arcs $M_1A_1, A_2A_3, \cdots, A_{2n-2}A_{2n-1}$ (In what follows, we shall denote these arcs by $1^+, 2^+, \cdots, n^+$ respectively) of $M_1mM_2$ are the strict entrance points of $B$, while the points on the segmental arcs $A_1A_2, A_3A_4, \cdots, A_{2n-1}M_2$ (We shall denote them by $1^-, 2^-, \cdots, n^-$ respectively) of $M_1mM_2$ are the strict exit points of $B$ ([5, p.37]). Take arbitrarily an integer $k$ such that $1 < k < n$. Consider three consecutive tangencies $A_{2k}, A_{2k-1}, A_{2k-2}$, and four relevant segmental arcs $(k+1)^+, k^-, k^+, (k-1)^-$. It is easy to see that the positive semi-trajectory originating from every point in a small neighbourhood of $A_{2k-1}$ on $k^+$ must leave $D_1$ from some point on the segmental arc $k^-$, while the positive semi-trajectory originating from every point in a small neighbourhood of $A_{2k-2}$ on $k^+$ must leave $D_1$ from some point on the segmental arc $(k-1)^-$. Therefore, there must be a nonempty set $\beta_k^+ \subset k^+$ such that for each point $x \in \beta_k^+$, the positive semi-trajectory $f(x, R^+)$ leaves $D_1$ neither from a point on $k^-$ nor from a point on $(k-1)^-$ for increasing time. Similarly, there must be a nonempty set $\beta_k^- \subset k^-$ such that for each point $x \in \beta_k^-$, the negative semi-trajectory $f(x, R^-)$ leaves $D_1$ neither from a point on $k^+$ nor from a point on $(k+1)^+$ for all $t < 0$. In other words, each of these semi-trajectories can not leave $D_1$ from a point on the adjacent segmental arcs. Consider case $k = 1$. It is easy to see that if the positive semi-trajectory $f(x, R^+)$ originating from a

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point $x \in 1^+$ tends to $Q$ in $D$ as $t \to +\infty$, then it must enter the parabolic sector $Qm_1\gamma_1$(see Fig.1). So, from the continuity (the solutions depend continuously on initial conditions) it follows that the positive semi-trajectory originating from every point in a small neighbourhood of $x$ on $1^+$ also tends to $Q$ in $D$ as $t \to +\infty$. The same argument implies that the positive semi-trajectory through a point $x$ of $M_1A_1$ sufficiently close to $M_1$ must tend to $Q$ in $D$ as $t \to +\infty$. Thus there is a maximal open segmental arc $M_1h$ of $M_1A_1$ such that for every point $x \in M_1h$, the positive semi-trajectory $f(x, R^+)$ tends to $Q$ in $D$ as $t \to +\infty$, while the positive semi-trajectory $f(h, R^+)$ does not tend to $Q$ in $D$ as $t \to +\infty$. On the other hand, the same argument used in case $1 < k < n$ implies that the positive semi-trajectory originating from every point in a small neighbourhood of $A_1$ on $1^+$ must leave $D_1$ from some point on the segmental arc $1^-$. Moreover, the set of such points is an open set on the segmental arc $M_1A_1$. This implies that $h \neq A_1$, and the positive semi-trajectory $f(h, R^+)$ can neither tend to $Q$ in $D$ nor leave $D_1$ from a point on $1^-$ for increasing time. Hence, for $k = 1$, we have proved that there is a nonempty set $\beta_1^+ \subset 1^+$ such that for each point $x \in \beta_1^+$, the positive semi-trajectory $f(x, R^+)$ can neither tend to $Q$ in $D$ nor leave $D_1$ from a point on $1^-$ for increasing time. For $k = n$, a similar conclusion holds.

Choose arbitrarily $n$ points $a_i^+ \in \beta_i^+ (i = 1, 2, \cdots, n)$. We now can prove that there is at least one among the positive semi-trajectories $\{f(a_i^+, R^+)|i = 1, 2, \cdots, n\}$ such that it stays in $D_1$ for all $t > 0$ and does not tend to $Q$ as $t \to +\infty$. The following proof proceeds by reduction to absurdity. Suppose that each of the semi-trajectories $\{f(a_i^+, R^+)|i = 1, 2, \cdots, n\}$ either leaves $D_1$ from some point on the segmental arc $M_1mM_2$ for increasing time or tends to $Q$ as $t \to +\infty$. Thus, since the positive semi-trajectory $f(a_1^+, R^+)$ can not tend to $Q$(note $a_1^+ \in \beta_1^+$), it must leave $D_1$ from a point on $M_1mM_2$ for increasing time. Let it leave $D_1$ from some point on $k^-$, where $1 < k \leq n$. But this means that each of $\{f(a_i^+, R^+)|i = 2, \cdots, k\}$ can not tend to $Q$ as $t \to +\infty$ for, otherwise it must meet $f(a_1^+, R^+)$ at a point for increasing time and which contradicts uniqueness of solutions. Therefore, each of $\{f(a_i^+, R^+)|i = 2, \cdots, k\}$ must leave $D_1$ from a point on $M_1mM_2$ for increasing time. However, we note that the semi-trajectory $f(a_2^+, R^+)$ can not leave $D_1$ from a point on $1^-$ or $2^-$ because $a_2^+ \in \beta_2^+$, hence, it can only leave $D_1$ from a point on $i^-(i \geq 3)$. This implies that the positive semi-trajectory $f(a_3^+, R^+)$ can not leave $D_1$ from a point on $1^-$ for increasing time for, otherwise it must meet $f(a_2^+, R^+)$ and which contradicts uniqueness of solutions, hence, it can only leave $D_1$ from a point on $i^-(i \geq 4)$. Further, this also implies that the semi-trajectory $f(a_4^+, R^+)$ can not leave $D_1$ from a point on $2^-$, hence it can only leave $D_1$ from a point on $i^-(i \geq 5)$ for increasing time. Repeating an argument used above, one implies that the semi-trajectory $f(a_i^+, R^+)$ can only leave $D_1$ from a point on $j^-(k \geq j \geq i+1)$ (i.e., on the segmental arc with greater subscript). From this, it follows that the semi-trajectory $f(a_k^+, R^+)$ can not leave $D_1$ from a point on $M_1mM_2$ for increasing time. Moreover, as stated above, it can not also tend to $Q$ as $t \to +\infty$, therefore, this contradicts the preceding hypothesis. Thus, there must be a positive semi-trajectory $\gamma^+$ such that it can neither tend to $Q$ nor leave $D_1$ from a point on $M_1mM_2$ for increasing time. By the Poincaré-Bendixson theory of planar systems, the $\omega$-limit set of $\gamma^+$ must contain critical points or closed orbits. Further, since a closed orbit contains at least one critical point of (1.1) in its interior, this implies that there must be at least one critical point of (1.1) in $D_1$. Hence Theorem 2.1 is proved.

COROLLARY 1. *If the sectorial region $D$ in Theorem* 2.1 *contains the internal boundary components $L_{i_1}, \cdots, L_{i_k}$ of $\partial B$, then the conclusion of Theorem* 2.1 *still holds provided we set $D_1 = D \backslash (\overline{G} \cup \overline{G}_{i_1} \cup \cdots \cup \overline{G}_{i_k})$, where $G_{i_1}, \cdots, G_{i_k}$ are the regions enclosed by $L_{i_1}, \cdots, L_{i_k}$ respectively.*

*Proof of Corollary* 1. We know from the proof of Theorem 2.1 that there is at least one among the positive semi-trajectories $\{f(a_i^+, R^+) | i = 1, 2, \cdots, n\}$, say $f(a_j^+, R^+)$, such that it stays in $D \backslash \overline{G}$ for all $t > 0$ and does not tend to $Q$ as $t \to +\infty$, where $a_j^+ \in \beta_j^+ \subset j^+, A_{2j-1}$ and $A_{2j-2}$ are two tangencies close to $a_j^+$.

Suppose $f(a_j^+, R^+)$ intersects $L_{i_0}$ at $b_j$, where $L_{i_0}$ is one of the internal boundary components $\{L_{i_1}, \cdots, L_{i_k}\}$ and $b_j$ is a strict exit point of $B$. Consider the segmental arc $A_{2j-2}a_j^+A_{2j-1}$ and its segmental subarc $a_j^+A_{2j-1}$. Let $\tilde{a} = \{x \in A_{2j-2}a_j^+A_{2j-1} |$ the point where $f(x, R^+)$ intersects $L_{i_0}$ is a strict exit point of $B\}$. From the theorem of continuity (the solutions depend continuously on initial conditions) it follows that $\tilde{a}$ is an open set on the segmental arc $A_{2j-2}a_j^+A_{2j-1}$. Since $A_{2j-1}$ is a tangency to $B$, the positive semi-trajectory originating from every point in a small neighbourhood of $A_{2j-1}$ on $a_j^+A_{2j-1}$ must leave $B$ from some point on the segmental arc $M_1mM_2$. Hence there must be at least one boundary point of the set $\tilde{a}$ on $a_j^+A_{2j-1}$. Let $a_0$ be a boundary point close to $a_j^+$. Then either there is a

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point $p \in f(a_0, R^+)$ such that $p$ is a tangency to $L_{i_0}$ or $f(a_0, R^+)$ tends to a critical point of (1.1) in $D_1$ as $t \to +\infty$. In the former case, it follows that there is an internal tangency to $B$. But this is impossible because $B$ is an isolating block. In the latter case, it follows that Corollary 1 holds.

If $f(a_j^+, R^+)$ does not meet any one of the internal boundary components $\{L_{i_1}, \cdots, L_{i_k}\}$, then by the Poincaré-Bendixson theory of planar systems it follows that there must be at least one critical point of (1.1) in $D_1$, where $D_1 = D \backslash (\overline{G} \cup \overline{G}_{i_1} \cup \cdots \cup \overline{G}_{i_k})$. Corollary 1 is proved.

*Remark* 1. Suppose $Q$ is a unique critical point of (1.1) in $B$. Then, Theorem 2.1 means that the fact that there are no internal tangencies to $B$ can imply that there are no certain type of elliptic regions of the critical point $Q$.

Using exactly the same argument used in the proof of Theorem 2.1, we can prove the following theorem. We suppose that the symbols $a_i^{\pm}, i^{\pm}, \beta_i^{\pm}$ have the same meanings as in Theorem 2.1.

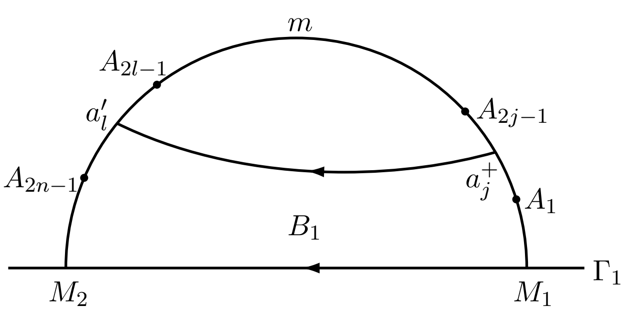
THEOREM 2.2. *Let $B$ be an isolating block for flow defined by* (1.1). *Let $B_1$ be the region enclosed by the trajectory arc $M_1M_2$ of the cut trajectory $\Gamma_1$ of $B$ and the segmental arc $M_1mM_2$ of the external boundary $L_1$ of $B$* (Fig.2). *Let $A_1, A_2, \cdots, A_{2n-1}$ be the tangencies to be arranged in numerical order on the arc $M_1mM_2$. If $n \geq 2$, then there must be a point $a_i^+ \in i^+$ and a point $a_i^- \in i^-$ such that the semi-trajectories $f(a_i^+, R^+)$ and $f(a_i^-, R^-)$ stay in $B_1$ for all $t > 0$ and $t < 0$ respectively* $(i = 1, 2, \cdots, n)$.

*Proof.* First we note, by Definition 1.2, it follows that the positive semi-trajectory originating from every point in a small neighbourhood of $M_1$ on $1^+$ must leave $B_1$ from a point on $n^-$ for increasing time. For $k = n$, a similar conclusion holds. Thus, one can consider $1^+$ and $n^-$ as two adjacent segmental arcs.

We proceed by induction. First suppose $n = 2$. That is, there are three tangencies $A_1, A_2, A_3$ on the arc $M_1mM_2$. It is easy to see that the positive (or negative) semi-trajectory originating from any point on $\beta_i^+$ (or $\beta_i^-$) $(i = 1, 2)$ stays in $B_1$ for all $t > 0$ (or $t < 0$). So, when $n = 2$, Theorem 2.2 holds.

Let $k > 2$ be an arbitrary positive integer. Let us now make the inductional hypothesis that Theorem 2.2 is true for $2 \leq n \leq k - 1$ (i.e., for all those odd numbers which are not greater than $2k - 3$). We need to show that it is also true for $n = k$ (i.e., for the odd number $2k - 1$).

In fact, from $n = k-1$ to $n = k$, two tangencies $A_{2k-2}$ and $A_{2k-1}$ are added to the arc $M_1mM_2$. The following proof proceeds by reduction to absurdity. Suppose that there is some segmental arc $j^+$ such that for every point $x \in j^+$, the positive semi-trajectory $f(x, R^+)$ leaves $B_1$ from a point on $M_1mM_2$ for increasing time. Take arbitrarily a point $a_j^+ \in \beta_j^+$, then, the positive semi-trajectory $f(a_j^+, R^+)$ must leave $B_1$ from the point $a_l'$ on $l^-$ for increasing time. The trajectory arc $a_j^+a_l'$ divides the segmental arc $M_1mM_2$ into three segmental arcs: The segmental arcs $a_j^+a_l'$, $M_1a_j^+$ and $M_2a_l'$ (Fig.2). From the fact that the semi-trajectory $f(a_j^+, R^+)$ leaves $B_1$ neither from a point on the adjacent segmental arcs nor from a point on any entrance segmental arc $i^+$ for increasing time, it follows that there are at least three tangencies on the segmental arc $a_j^+a_l'$ of $M_1mM_2$, while the amount of tangencies on the arcs $M_1a_j^+$ and $M_2a_l'$ of $M_1mM_2$ is not less than 2. Thus the number of tangencies on the arc $a_j^+a_l'$ of $M_1mM_2$ is not greater than $2k-1-2 = 2k-3$. Furthermore, since $a_j^+$ is a strict entrance point of $B_1$ and $a_l'$ is a strict exit point of $B_1$, the trajectory arc $a_j^+a_l'$ possesses the same property as the arc $M_1M_2$ of $\Gamma_1$. By the inductional hypothesis it follows that there must be a point $q$ on the arc $a_j^+A_{2j-1}$ of $M_1mM_2$ such that the positive semi-trajectory $f(q, R^+)$ stays for all $t > 0$ in the region enclosed by the segmental arc $a_j^+a_l'$ of $M_1mM_2$ and the trajectory arc $a_j^+a_l'$, hence in $B_1$. But since $a_j^+A_{2j-1} \subset j^+$, this contradicts the preceding hypothesis. Hence we have proved that for each $i \in \{1, 2, \cdots, n\}$, there must be a point $a_i^+ \in i^+$ such that $f(a_i^+, R^+)$ stays in $B_1$ for all $t > 0$. Similarly, we can prove that for each $i \in \{1, 2, \cdots, n\}$, there must be a point $a_i^- \in i^-$ such that $f(a_i^-, R^-)$ stays in $B_1$ for all $t < 0$. Thus Theorem 2.2 is proved.




(Figure 2)

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