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ON COMMUTATIVITY AND STRONG COMMUTATIVITY-PRESERVING MAPS

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ABSTRACT. If R is a ring and $S \subseteq R$, a mapping $f\colon R \longrightarrow R$ is called *strong commutativity-preserving* (scp) on S if $[x,y] = [f(x),f(y)]$ for all $x, y \in S$. We investigate commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is scp on a nonzero right ideal.

The recent literature includes several papers on commutativity in prime or semiprime rings with commutator constraints involving elements of the ring and images of elements under suitable maps (See [1] for a partial list of references). There is also a growing literature on commutativity-preserving maps f, defined by the property that whenever x and y are commuting elements of the ring, so are $f(x)$ and $f(y)$. (See [3] for references.)

In this paper, which can be thought of as belonging to both of these categories, we study commutativity in rings admitting a special kind of commutativity-preserving map. Specifically, if R is a ring and $S \subseteq R$, a map $f\colon R \longrightarrow R$ is called *strong commutativity-preserving* (scp) *on* S if $[x,y] = [f(x),f(y)]$ for all $x, y \in S$.

Our first theorem, which extends Theorem 5 of [2], establishes a near-commutativity property for semiprime rings admitting a derivation which is scp on a nonzero right ideal. The remaining theorems deal with prime or semiprime rings admitting endomorphisms which are scp on certain subsets. The final section gives examples of scp maps.

1. **Notation and preliminaries.** For the ring R, Z or $Z(R)$ will denote the center of R; and for $S \subseteq R$, $A_r(S)$ will denote the right annihilator of S. As usual, for elements $x, y \in R$, the commutator $xy - yx$ will be written as $[x,y]$; and for $S \subseteq R$, the set of all commutators of elements of S will be written as $[S,S]$. A right ideal K of R will be called *essential* if $K \cap I \neq \{0\}$ for all nonzero right ideals I.

We recall some known results on prime and semiprime rings.

LEMMA 1. *(a) If R is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R.*

(b) If R is a prime ring, the centralizer of any nonzero one-sided ideal is equal to the center of R. Thus, if R has a nonzero central right ideal, R must be commutative.

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LEMMA 2 [4, LEMMA 1]. *Let R be a semiprime ring and I a nonzero ideal of R. If $z \in R$ and z centralizes $[I, I]$, then z centralizes I.*

LEMMA 3 [5]. *Let R be a prime ring. If a, b are elements of R such that $axb = bxa$ for all $x \in R$, and if $a \neq 0$, then $b = \lambda a$ for some λ in the extended centroid of R.*

LEMMA 4 [1]. *Let R be a prime ring, U a nonzero right ideal of R, and T an endomorphism of R. If $T(u) = u$ for all $u \in U$, then T is the identity map on R.*

2. A theorem on scp derivations.

THEOREM 1. *Let R be a semiprime ring and U a nonzero right ideal. If R admits a derivation D which is* scp *on U, then $U \subseteq Z$.*

PROOF. For all $x, y \in U$, we have $[x, xy] = [D(x), D(xy)]$, from which it follows easily that

$$[D(x), x]D(y) + D(x)[D(x), y] = 0 \quad \text{for all } x, y \in U. \tag{1}$$

Replacing y by yr gives

$$[D(x), x]\big(yD(r) + D(y)r\big) + D(x)\big(y[D(x), r] + [D(x), y]r\big) = 0,$$

which on comparison with (1) yields

$$[D(x), x]yD(r) + D(x)y[D(x), r] = 0 \quad \text{for all } x, y \in U \text{ and } r \in R. \tag{2}$$

Letting $r = D(x)$, we see that

$$[D(x), x]UD^2(x) = \{0\} = [D(x), x]URD^2(x) \quad \text{for each } x \in U. \tag{3}$$

Since R is semiprime, it must contain a family $\mathcal{P} = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap P_\alpha = \{0\}$. If P is a typical member of $\mathcal{P}$ and $x \in U$, (3) shows that

$$D^2(x) \in P \text{ or } [D(x), x]U \subseteq P. \tag{4}$$

Suppose that $D^2(x) \in P$. Then for each $y \in U$, $[x, yD(x)] = \big[D(x), D\big(yD(x)\big)\big]$; hence $[x, y]D(x) + y[x, D(x)] = y[D(x), D^2(x)] + [D(x), y]D^2(x) + [D(x), D(y)]D(x)$ and therefore $y[x, D(x)] = y[D(x), D^2(x)] + [D(x), y]D^2(x)$. Thus $U[x, D(x)] \subseteq P$ and $UR[x, D(x)] \subseteq P$, so that either $U \subseteq P$ or $[x, D(x)] \in P$. Either of these conditions implies $[x, D(x)]U \subseteq P$; and recalling (4) gives $[x, D(x)]U \subseteq P$ for all $x \in U$ and all $P \in \mathcal{P}$. Since $\bigcap P_\alpha = \{0\}$, we have

$$[x, D(x)]U = \{0\} \quad \text{for all } x \in U.$$

It now follows from (2) that $D(x)UR[D(x), r] = \{0\}$ for each $x \in U$ and $r \in R$; hence for each $P \in \mathcal{P}$ and each $x \in U$,

$$D(x)U \subseteq P \text{ or } [D(x), R] \subseteq P.$$

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For fixed P, the sets of $x \in U$ for which these two conditions hold are additive subgroups of U whose union is U; therefore

(5) $$D(U)U \subseteq P \text{ or } [D(U), R] \subseteq P.$$

Suppose that $D(U)U \subseteq P$. For arbitrary $x, y, z \in U$, the condition $[x, yz] = [D(x), D(yz)]$ reduces to $[D(x), y]D(z) = -D(y)[D(x), z]$; and since the right side of the latter equation is in P, we have $yD(x)D(z) \in P$. Thus $U[D(x), D(z)] = U[x, z] \subseteq P$ for all $x, z \in U$; and primeness of P implies that either $U \subseteq P$ or $[x, z] \in P$ for all $x, z \in U$. In either event $[U, U] \subseteq P$. Returning to (5), we note that the second alternative gives $[D(U), D(U)] \subseteq P$ and hence $[U, U] \subseteq P$. Now using the fact that $\bigcap P_\alpha = \{0\}$, we conclude that U is a commutative right ideal; and since R is semiprime, Lemma 1(a) shows that $U \subseteq Z$.

COROLLARY 1. *If R is a semiprime ring admitting a derivation D which is* scp *on R, then R is commutative.*

3. **Two theorems on** scp **endomorphisms.**

THEOREM 2. *Let R be a prime ring and U an essential right ideal of R. If R admits a non-identity endomorphism T which is* scp *on U, then R is commutative.*

PROOF. For all $x, y \in U$ we have $[x, xy] = [T(x), T(xy)]$, from which it follows that $\big(T(x) - x\big)[x, y] = 0$. Replacing y by yr, $r \in R$, we get

$$\big(T(x) - x\big)U[x, r] = \{0\} = \big(T(x) - x\big)UR[x, r] \quad \text{for all } x \in U,\ r \in R;$$

thus, for each $x \in U$, either $x \in Z$ or $\big(T(x) - x\big)U = \{0\}$. The sets of $x \in U$ for which these alternatives hold are additive subgroups of U, hence either $U \subseteq Z$ or $\big(T(x) - x\big)U = \{0\}$ for all $x \in U$. If $U \subseteq Z$, R is commutative by Lemma 1(b); thus, we can assume that

(6) $$\big(T(x) - x\big)U = \{0\} \quad \text{for all } x \in U.$$

Now use the fact that $[x, yx] = [T(x), T(yx)]$—that is, $[x, y]\big(T(x) - x\big) = 0$—for all $x, y \in U$; and replace y by yw, $w \in U$, thereby obtaining

(7) $$[x, y]U\big(T(x) - x\big) = \{0\} = [x, y]UR\big(T(x) - x\big) \quad \text{for all } x, y \in U.$$

By Lemma 4, T cannot be the identity on U; and it follows easily from (7) that

(8) $$[x, y]U = \{0\} \quad \text{for all } x, y \in U.$$

Let $V = U \cap T^{-1}(U)$, and note that V contains all commutators $[x, y]$ for $x, y \in U$. If U is commutative, R is commutative by Lemma 1(b); hence we may assume that U is not commutative and $V \neq \{0\}$.

Consider any $b \in V \setminus \{0\}$. By (8) we have $[bx, by]b = 0$ for all $x, y \in R$—*i.e.* $bxbyb = bybxb$ for all $x, y \in R$. Thus for fixed $x \in R$, Lemma 3 gives us an element $\lambda = \lambda(x)$ in the

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extended centroid of R such that $bxb = \lambda b$; and it follows that $[bxb, b] = 0 = b[xb, b]$ for all $x \in R$. Now if b is not a left zero divisor, then b centralizes the nonzero left ideal Rb; hence by Lemma 1(b), b is central and therefore regular. But by (8), b is a right zero divisor; consequently b must be a left zero divisor and $A_r(b) \neq \{0\}$. Since U is an essential right ideal, there exists $a \in U \setminus \{0\}$ for which $ba = 0$. The fact that T is scp on U now gives $ab = T(a)T(b)$, and by (6) we get $ab = aT(b)$ or $a\big(b - T(b)\big) = 0$. Since a may be replaced by ar for any $r \in R$, we conclude that $b - T(b) = 0$. Thus, T is the identity on V, contradicting Lemma 4; and we have eliminated the possibility that U, and hence R, is not commutative.

Note that in a prime ring, any nonzero two-sided ideal is an essential right ideal. Thus we have

COROLLARY 2. *Let R be a prime ring and U a nonzero two-sided ideal. If R admits a non-identity endomorphism which is* scp *on U, then R is commutative.*

Our final theorem, for semiprime rings, may be regarded as an extension of Corollary 2.

THEOREM 3. *Let R be a semiprime ring and* $U \neq \{0\}$ *a two-sided ideal. If R admits an endomorphism T which is* scp *on U and not the identity on the ideal* $U \cap T^{-1}(U)$*, then R contains a nonzero central ideal.*

PROOF. As in the proof of Theorem 1, let $\mathcal{P} = \{P_\alpha \mid \alpha \in \Lambda\}$ be a family of prime ideals such that

$$\bigcap P_\alpha = \{0\}. \tag{9}$$

Note that (7) holds under the present hypotheses; hence for each $P_\alpha \in \mathcal{P}$, we either have $T(x) - x \in P_\alpha$ for all $x \in U$ or $[x, y]U \subseteq P_\alpha$ for all $x, y \in U$. In the latter case, using the fact that U is a left ideal, we get $[x, y]RU \subseteq P_\alpha$ and hence $[x, y] \in P_\alpha$ for all $x, y \in U$. Now invoking (9), together with the fact that the P_α are ideals, we get

$$\big(T(x) - x\big)[y, u] = 0 = [y, u]\big(T(x) - x\big) \quad \text{for all } x, y, u \in U. \tag{10}$$

Let $W = U \cap T^{-1}(U)$. If $x \in W$, we see from (10) and Lemma 2 that $T(x) - x \in Z(U)$; and it follows by Lemma 1(a) that

$$T(x) - x \in Z(R) \quad \text{for all } x \in W. \tag{11}$$

Choose $x_0 \in W$ such that $T(x_0) - x_0 \neq 0$, and let $K = U\big(T(x_0) - x_0\big)$. In view of (11), K is a two-sided ideal; moreover, $K \neq \{0\}$, since otherwise $U \cap A_r(U)$ would be a nonzero nilpotent ideal. Since $T(x_0) - x_0 \in Z(R)$, it is immediate from (10) that $[y, u]K = K[y, u] = \{0\}$ for all $y, u \in U$; and application of Lemmas 1(a) and 2 gives $K \subseteq Z(R)$.

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4. Some examples.

EXAMPLE 1. Let R be a 3-dimensional algebra over a field of characteristic 2, with basis $\{u_0, u_1, u_2\}$ and multiplication defined by

$$u_i u_j = \begin{cases} u_0 & \text{if } (i,j) = (1,2) \\ 0 & \text{otherwise} \end{cases}.$$

Let d be the linear transformation on R defined by $d(u_0) = 0$, $d(u_1) = u_1$, $d(u_2) = u_2$. It is easily verified that d is a derivation which is scp on R. This example shows that in Theorem 1, the hypothesis of semiprimeness cannot be omitted.

The derivation d is not an inner derivation. Indeed, it is easy to show that any ring R admitting an inner derivation which is scp on R must be commutative.

EXAMPLE 2. Let $R = R_1 \oplus R_2$, where R_1 is a non-commutative prime ring with derivation d_1 and R_2 is a commutative domain. Define $d: R \longrightarrow R$ by $d\big((r_1, r_2)\big) = \big(d_1(r_1), 0\big)$. Then R is a semiprime ring, and d is a derivation which is scp on the ideal U consisting of elements of form $(0, r_2)$. Thus, under the hypotheses of Theorem 1 we cannot prove that R must be commutative.

EXAMPLE 3. Let S be any ring, let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \,\middle|\, a, b \in S \right\}$, and let $U = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \,\middle|\, b \in S \right\}$. Define $T: R \longrightarrow R$ by $T\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. Then R is a ring under the usual operations, U is an ideal, and T is an endomorphism which is scp on U and not the identity on $U = U \cap T^{-1}(U)$. But for appropriate choices of S (*e.g.* S a non-commutative division ring), R has no nonzero central ideals. Thus, in Theorem 3 the hypothesis of semiprimeness is crucial.

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