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# Notes on Diagonal Coinvariants of the Dihedral Group

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*Abstract.* The bigraded Hilbert function and the minimal free resolutions for the diagonal coinvariants of the dihedral groups are exhibited, as well as for all their bigraded invariant Gorenstein quotients.

## 1 Introduction

This paper was inspired by a number of lovely lectures given by François Bergeron during a series of encounters at Queen's University, Université du Quebec à Montréal, the University of Ottawa, and the Fields Institute. In these lectures Bergeron explained a part of the motivation behind the $n!$ theorem of Mark Haiman, which gives subtle and beautiful information about the diagonal representation of the symmetric group. Haiman's result gives the dimension of certain of these modules, and we hoped to extend his work to a discussion of the minimal free resolution of those same modules. This is rather a formidable problem, and so our first thought was to solve the analogous problems for some simpler Coxeter groups, *i.e.,* the dihedral groups. This paper is the result of those investigations.

We were kindly informed by Mark Haiman that E. Reiner and J. Alfano worked out the bigraded Hilbert function and character of the diagonal coinvariants for the dihedral groups in 1992–93 but never published their result. However, the results are referred to by Haiman in [Hai94].

Questions about minimal free resolutions for the diagonal representation of dihedral groups are discussed here for the first time.

## 2 Setup

We will assume that the field $k$ is algebraically closed of characteristic zero, and in fact we might as well assume that $k = \mathbf{C}$. We first recall some elementary (and well known) facts about the representation theory of the dihedral groups. The dihedral group $D_n$ is generated by two reflections $s_1$ and $s_2$ with the relation $(s_1s_2)^n = e$. We can also think of if as generated by one reflection $s = s_1$ and one rotation $\phi = s_1s_2$. For any integer $i$, we get a two-dimensional representation $E_i$ by

$$s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \phi \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix},$$

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where $\xi$ is a primitive $n$-th root of unity. Of course, $E_i = E_j$ if $i \equiv j \pmod{n}$. For $0 < i < n/2$, this representation is irreducible. For $i = n/2$ when $n$ is even, the representation $E_{n/2}$ splits into the sum of the two one-dimensional representations $E^{\pm}_{n/2}$

$$s \mapsto \pm 1 \qquad \phi \mapsto -1.$$

There is always the trivial one-dimensional representation (Triv). The alternating one-dimensional representation (Alt) is given by

$$s \mapsto -1 \qquad \phi \mapsto 1.$$

In fact $E_0 = E_n = \text{Triv} \oplus \text{Alt}$.

We let the dihedral group act on the polynomial ring $k[x_1, x_2]$ by $s.x_1 = x_2$, $s.x_2 = x_1$, $\phi.x_1 = \xi x_1$, and $\phi.x_2 = \xi^{-1} x_2$ For positive integers $i$, we can then realize the representation $E_i$ in $k[x_1, x_2]$ by the vector spaces spanned by $(x_1x_2)^d x_1^i$ and $(x_1x_2)^d x_2^i$ for any $d$. In fact this gives us the decomposition of $k[x_1, x_2]$ into irreducible representations using the splitting of $E_{n/2}$ and $E_n = E_0$.

### 2.1 The Invariants

Recall the following, which are special cases of results true for any reflection group.

***Theorem 2.1*** *The invariants under the action of the dihedral group on $k[x_1, x_2]$ are all polynomials in the basic invariants $x_1x_2$ and $x_1^n + x_2^n$.*

***Theorem 2.2*** *The ideal $I$ generated by the invariants in $S = k[x_1, x_2]$ is a complete intersection ideal, and the quotient $S/I$ is isomorphic to the regular representation.*

## 3 The Diagonal Action

We now move on to the diagonal action of the dihedral group on the polynomial ring $R = k[x_1, x_2, y_1, y_2]$ which is the tensor product $S \otimes_k S$.

***Proposition 3.1*** *The ideal generated by the invariants of $R$ under the diagonal action of the dihedral group is minimally generated by the quadratic forms*

$$q_{2,0} = x_1x_2, \quad q_{1,1} = x_1y_2 + x_2y_1, \quad q_{0,2} = y_1y_2,$$

*and the forms*

$$g_0 = x_1^n + x_2^n, \quad g_1 = x_1^{n-1}y_1 + x_2^{n-1}y_2,$$
$$\dots, \quad g_{n-1} = x_1y_1^{n-1} + x_2y_2^{n-1}, \quad g_n = y_1^n + y_2^n.$$

**Proof** The forms given in the proposition are clearly invariant under the diagonal action. Look at the ideal spanned by them and suppose that we have a form $f$ that is invariant but not in the ideal. Since a monomial is sent by $\phi$ to a multiple of itself, any such multiple has to be a unit multiple in order for $f$ to be invariant. Under the action of $s$, a monomial $x_1^a x_2^b y_1^c y_2^d$ is sent to $x_1^b x_2^a y_1^d y_2^c$. Thus $f$ can be written as a sum of terms of the form

$$x_1^a x_2^b y_1^c y_2^d + x_1^b x_2^a y_1^d y_2^c$$

where $a+c \equiv b+d \pmod{n}$. If $a$ and $b$ are both positive or $c$ and $d$ are both positive, such a term is in the ideal generated by the first two invariants. Thus we may assume that the terms are of the forms

$$x_1^a y_1^c + x_2^a y_2^c \quad \text{and} \quad x_1^a y_2^d + x_2^a y_1^d$$

Using the generator $x_1 y_2 + x_2 y_1$, we can reduce all the terms of the second kind to elements in the ideal. If $a \geq n$, we can use the generator $x_1^n + x_2^n$ to write

$$x_1^a y_1^c + x_2^a y_2^c = (x_1^n + x_2^n)(x_1^{a-n} y_1^c + x_2^{a-n} y_2^c) - x_1^n x_2^{a-n} y_2^c - x_2^a x_1 x^{a-n} y_1^c$$

which is clearly in the ideal.

We now need to prove that the set of generators is minimal. It is clear that we need the first three quadratic generators. If we add $x_1$ and $y_1$ to the ideal, we get the ideal

$$J = (x_1, y_1, x_2^n, x_2^{n-1} y_2, \dots, y_2^n)$$

which does need all these generators. Thus none of the generators of degree $n$ can be excluded from the generating set of the ideal $I$. ∎

The polynomial ring $R = k[x_1, x_2, y_1, y_2]$ is bigraded and the diagonal action preserves this bigrading. Thus the ideal $I$ generated by the invariants will be bihomogeneous and the quotient $A = R/I$ again bigraded. We will now determine the bigraded Hilbert function of $A$. We start by looking at the ideal $J$ generated by the quadratic forms in $I$.

***Lemma 3.2*** *The bigraded Hilbert function of*

$$A = R/(x_1 x_2, x_1 y_2 + x_2 y_1, y_1 y_2)$$

*is given by $H_A(i, j) = 2$, except for $H_A(0,0) = 1$ and $H_A(1,1) = 3$. The element $x_1 y_2 - x_2 y_1$ generates an alternating one-dimensional representation and is annihilated by the maximal ideal $(x_1, x_2, y_1, y_2)$.*

**Proof** Consider the ideal $J = (x_1 x_2, x_1 y_2 + x_2 y_1, y_1 y_2)$. Modulo this ideal, $x_1 y_2 - x_2 y_1$ is annihilated by $x_1$, since

$$x_1(x_1 y_2 - x_2 y_1) = x_1(x_1 y_2 + x_2 y_1) - 2x_1 x_2 y_1 \in J,$$

and, by symmetry, it is also killed by all the other variables. Thus it is in the socle of $R/J$, and in order to compute the bigraded Hilbert function in degrees different from $(1,1)$, we may as well add $(x_1 y_2 - x_2 y_1)$ to the ideal. We then get the ideal $J' = (x_1 x_2, x_1 y_2, x_2 y_1, y_1 y_2) = (x_1, y_1) \cap (x_2, y_2)$. The bigraded Hilbert function of this ideal is well known and equals two in all bidegrees different from $(0,0)$. In fact, the only monomials that are not in the ideal are $x_1^i y_1^j$ and $x_2^i y_2^j$ for all $(i, j)$. ∎

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**Proposition 3.3** *The bigraded Hilbert function of $A = R/I$ is given by*

$$H_A(i, j) = \begin{cases} 1 & \text{if } i = j = 0 \text{ or } i + j = n, \\ 2 & \text{if } 1 \le i + j \le n, \ (i, j) \neq (1, 1), \\ 3 & \text{if } (i, j) = (1, 1), \\ 0 & \text{otherwise,} \end{cases}$$

*or, more explicitly,*

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & \cdots & n-2 & n-1 & n \\ \hline 0 & 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & \cdots & 2 & 1 & \\ 2 & 2 & 2 & 2 & \cdots & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ n-2 & 2 & 2 & 1 & & & & \\ n-1 & 2 & 1 & & & & & \\ n & 1 & & & & & & \end{array}$$

**Proof** Using the lemma, we need only prove that the Hilbert function is one in bidegree $(i, n-i)$ and zero in higher bidegrees. In each bidegree $(i, n-i)$, $i = 0, 1, \ldots, n$, we add the generator $x_1^i y_1^{n-i} + x_2^i y_2^{n-i}$. Thus the Hilbert function drops from two to one in these degrees leaving the form $x_1^i y_1^{n-i} - x_2^i y_2^{n-i}$ outside the ideal. However, this element is killed by $x_1$ since

$$x_1(x_1^i y_1^{n-i} - x_2^i y_2^{n-i}) = x_1(x_1^i y_1^{n-i} + x_2^i y_2^{n-i}) - 2x_1 x_2^i y_2^{n-i} \in I,$$

and, by symmetry, it is annihilated by all the other variables. Thus these alternating forms are in the socle of $R/I$ and so $R/I$ must be zero in all higher bidegrees. ■

### 3.1 Decomposition into Irreducible Representations

We now go further and investigate the action of the dihedral group on the quotient $A$.

**Proposition 3.4** *The bigraded component $A_{i,j}$ is isomorphic to $E_{i+j}$ whenever $H_A(i, j) = 2$. For $i = 0, 1, \ldots, n$, the component $A_{i,n-i}$ is the alternating representation, $A_{0,0}$ is the trivial representation and $A_{1,1}$ is the sum of the alternating representation and $E_2$.*

*The representation $E_i$ occurs $n+2$ times for $0 < i < n/2$. For $n$ even, $E^{\pm}_{n/2}$ each occurs $(n+2)/2$ times. The trivial representation occurs once and the alternating representation occurs $n + 2$ times.*

**Proof** In degrees where the Hilbert function is 2, we have that the component $A_{i,j}$ is generated by $x_1^i y_1^j$ and $x_2^i y_2^j$, and since the group acts the same way on the two sets of variables, this is the same representation as $E_{i+j}$. In degree $(0, 0)$, the representation is trivial. In bidegrees $(i, n-i)$, the component $A_{i,j}$ is generated by the alternating element $x_1^i y_1^{n-i} - x_2^i y_2^{n-i}$ since $x_1^i y_1^{n-i} + x_2^i y_2^{n-i}$ is in the ideal. In bidegree $(1, 1)$, we

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have generators $x_1y_1$, $x_2y_2$ and $x_1y_2 - x_2y_1$. The first two generate $E_2$, and the third one is alternating.

For $0 < i < n/2$, we have that $E_i$ occurs in bidegrees $(j, i-j)$, for $j = 0, 1, \dots, i$. Since $E_i$ is isomorphic to $E_{n-i}$, it also occurs in bidegrees $(j, n-i-j)$, for $j = 0, 1, \dots, n-i$. Together, this makes $(i+1) + (n-i+1) = n+2$ times.

If $n$ is even, we have that $E_{n/2} = E^+_{n/2} \oplus E^-_{n/2}$ occurs in $A_{i,n/2-i}$, for $i = 0, 1, \dots, n/2$, *i.e.*, $(n+2)/2$ times. ∎

## 4 Resolution and Bigraded Betti Numbers

We have already found a minimal set of generators for the ideal $I$ generated by the diagonal invariants. We now find the syzygies and the bigraded Betti numbers of $I$.

We start by taking a look at the ideal $J$ generated by the quadratic invariants.

***Proposition 4.1*** *A minimal resolution of $R/J$, where $J = (x_1x_2, x_1y_2 + x_2y_1, y_1y_2)$, is given by*

$$0 \longrightarrow F_4 \xrightarrow{\Phi_4} F_3 \xrightarrow{\Phi_3} F_2 \xrightarrow{\Phi_2} F_1 \xrightarrow{\Phi_1} F_0 \longrightarrow R/J \longrightarrow 0,$$

*where the maps are given by the matrices*

$$\Phi_1 = \begin{pmatrix} x_1y_1 & x_1y_2 + x_2y_1 & y_1y_2 \end{pmatrix},$$

$$\Phi_2 = \begin{pmatrix} 0 & y_1y_2 & -x_1y_2 - x_2y_1 & y_1^2 & y_2^2 \\ -y_1y_2 & 0 & x_1x_2 & -x_1y_1 & -x_2y_2 \\ x_1y_2 + x_2y_1 & -x_1x_2 & 0 & x_1^2 & x_2^2 \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ y_1 & y_2 & x_2 & x_1 \\ 0 & 0 & y_2 & y_1 \\ -y_2 & 0 & 0 & x_2 \\ 0 & -y_1 & x_1 & 0 \end{pmatrix},$$

$$\Phi_4 = \begin{pmatrix} x_2 \\ -x_1 \\ -y_1 \\ y_2 \end{pmatrix},$$

*and the decomposition into irreducible representations is given by*

$$F_1 = R(-2,0) \oplus R(-1,-1) \oplus R(0,-2),$$

$$F_2 = R(-3,-1) \oplus (k \oplus E_2) \otimes R(-2,-2) \oplus R(-1,-3),$$

$$F_3 = E_1 \otimes R(-3,-2) \oplus E_1 \otimes R(-2,-3,)$$

$$F_4 = \mathrm{Alt} \otimes R(-3,-3.)$$

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**Proof** We can verify that this is actually a minimal free resolution using either of the computer algebra systems CoCoA [CoC] or Macaulay 2 [GS]. ■

The resolution of the ideal $J$ will be part of the resolution of the ideal $I$ as the new generators will come in higher degrees.

***Proposition 4.2*** *Apart from the syzygies of $J$, the first syzygies of $I$ are given by (for $i = 0, \dots, n-2$)*

$$y_1 g_i - x_1 g_{i+1} - x_2^{n-(i+1)} y_2^i q_{1,1} + 2x_2^{n-(i+2)} y_2^{i+1} q_{2,0} = 0,$$
$$y_2 g_i - x_2 g_{i+1} - x_1^{n-(i+1)} y_1^i q_{1,1} + 2x_1^{n-(i+2)} y_1^{i+1} q_{2,0} = 0,$$

*and*

$$y_1 g_{n-1} - x_1 g_n + y_2^{n-1} q_{1,1} - 2x_2 y_2^{n-2} q_{0,2} = 0,$$
$$y_2 g_{n-1} - x_2 g_n + y_1^{n-1} q_{1,1} - 2x_1 y_1^{n-2} q_{0,2} = 0,$$

*plus the Koszul relations*

$$(x_1^n + x_2^n) q_{2,0} - x_1 x_2 g_0 = 0, \quad (x_1^n + x_2^n) q_{1,1} - (x_1 y_2 + x_2 y_1) g_0 = 0,$$

*and, for $i = 0, \dots, n$, the Koszul relations*

$$(x_1^{n-i} y_1^i + x_2^{n-i} y_2^i) q_{0,2} - y_1 y_2 g_i = 0.$$

**Proof** The Koszul relations are obviously syzygies. Thus, for $i = 0, 1, \dots, n-2$, we have that

$$\begin{aligned} y_1 g_i - x_1 g_{i+1} &= y_1(x_1^{n-i} y_1^i + x_2^{n-i} y_2^i) - x_2(x_1^{n-i-1} y_1^{i+1} + x_2^{n-i-1} y_2^{i+1} \\ &= x_2^{n-(i+1)} y_2^i (x_1 y_2 + x_2 y_1) - 2x_2^{n-(i+2)} y_2^{i+1} x_1 x_2 \\ &= x_2^{n-(i+1)} y_2^i q_{1,1} - 2x_2^{n-(i+2)} y_2^{i+1} q_{2,0}, \end{aligned}$$

and also

$$\begin{aligned} y_1 g_{n-1} - x_1 g_n &= y_1(x_1 y_1^{n-1} + x_2 y_2^{n-1}) - x_2(y_1^n + y_2^n \\ &= x_2^{n-(i+1)} y_2^i (x_1 y_2 + x_2 y_1) - 2x_2^{n-(i+2)} y_2^{i+1} x_1 x_2 \\ &= x_2^{n-(i+1)} y_2^i q_{1,1} - 2x_2^{n-(i+2)} y_2^{i+1} q_{2,0}. \end{aligned}$$

By symmetry, the corresponding statement is true for $y_2 g_i - x_2 g_{i+1}$, $i = 0, 1, \dots, n$.

We now want to show that these relations are independent. The linear syzygies are independent modulo the syzygies coming from the quadratic generators. This can be seen by looking only at the coefficients of the $g_i$. The harder part is to verify that the Koszul syzygies given in the proposition are independent modulo the linear syzygies.

To do that we look at the matrix

$$
\begin{pmatrix}
y_1 & -x_1 & 0 & \cdots & 0 & 0 \\
y_2 & -x_2 & 0 & \cdots & 0 & 0 \\
0 & y_1 & -x_1 & \cdots & 0 & 0 \\
0 & y_2 & -x_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 & \vdots \\
0 & 0 & 0 & \cdots & y_1 & -x_1 \\
0 & 0 & 0 & \cdots & y_2 & -x_2 \\
\hline
x_1x_2 & 0 & 0 & \cdots & 0 & 0 \\
x_1y_2 + x_2y_1 & 0 & 0 & \cdots & 0 & 0 \\
y_1y_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & y_1y_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & y_1y_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & y_1y_2 & 0 \\
0 & 0 & 0 & \cdots & 0 & y_1y_2
\end{pmatrix}
$$

giving the coefficients of the degree $n$ generators in all of the $2n+n+3$ syzygies given above. In the last column, we have one element of degree $(0,2)$ and two elements of bidegree $(1,0)$. This means that the last row cannot be part of a syzygy. In the next to the last column, there is one element of bidegree $(0,2)$ in the lower part, and the only way to get such an element from the top part of the matrix is to use coefficients of bidegree $(0,1)$ on the last two rows. However, this always produces a non-zero element of bidegree $(1,1)$ in the last column. Thus we can conclude that we can take away the last row of the matrix and the last two rows of the first part of the matrix. This process repeats until we have only three rows of the lower part of the matrix and no rows in the upper. That finishes the proof. ■

In order to express the second syzygies of $I$, we use the following matrix form of the linear first syzygies given in the proposition above.

$$
\begin{pmatrix}
2x_2^{n-2}y_2 & 2x_1^{n-2}y_2 & 2x_2^{n-3}y_2^2 & 2x_1^{n-3}y_1^2 & \cdots & 0 & 0 \\
-x_2^{n-1} & -x_1^{n-1} & -x_2^{n-2}y_2 & -x_1^{n-2}y_1 & \cdots & y_2^{n-1} & y_1^{n-1} \\
0 & 0 & 0 & 0 & \cdots & -2x_2y_2^{n-2} & -2x_1y_1^{n-2} \\
y_1 & y_2 & 0 & 0 & \cdots & 0 & 0 \\
-x_1 & -x_2 & -y_1 & -y_2 & \cdots & 0 & 0 \\
0 & 0 & -x_1 & -x_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -x_1 & -x_2
\end{pmatrix}
$$

***Proposition 4.3*** *There are at most $n-1$ linear second syzygies involving the $2n$ linear first syzygies given in the previous proposition.*

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**Proof** It suffices to show that there are at most $n - 1$ linear syzygies among the columns of the following matrix.

$$\begin{pmatrix} y_1 & y_2 & 0 & 0 & \cdots & 0 & 0 \\ -x_1 & -x_2 & -y_1 & -y_2 & \cdots & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -x_1 & -x_2 \end{pmatrix}$$

The first row tells us that the first two coefficients of a syzygy must be $\lambda y_2$ and $-\lambda y_1$ for some element $\lambda$ of $k$. Subtracting this from the syzygy, we may assume that the first two coefficients are zero. By induction, we can assume that the smaller matrix, obtained by removing the first row and the first two columns, has at most $n-2$ syzygies. The base case for the induction is a $0 \times 0$-matrix which has no syzygies at all. In fact, the syzygies of the matrix (4) are given by the columns of the following matrix.

$$\begin{pmatrix} y_2 & 0 & 0 & \cdots & 0 \\ -y_1 & 0 & 0 & \cdots & 0 \\ -x_2 & y_2 & 0 & \cdots & 0 \\ x_1 & -y_1 & 0 & \cdots & 0 \\ 0 & -x_2 & y_2 & \cdots & 0 \\ 0 & x_1 & -y_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -x_2 \\ 0 & 0 & 0 & \cdots & -x_2 \end{pmatrix}$$

■

***Theorem 4.4*** *The graded Betti numbers of A are given by the following Betti diagram*

| | 1 | $n+4$ | $3n+8$ | $3n+8$ | $n+2$ |
|---|---|---|---|---|---|
| 0 | 1 | – | – | – | – |
| 1 | – | 3 | – | – | – |
| 2 | – | – | 5 | 4 | 1 |
| 3 | – | – | – | – | – |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1$ | – | $n+1$ | $2n$ | $n-1$ | – |
| $n$ | – | – | $n+3$ | $2n+4$ | $n+1$ |

**Proof** We already know the minimal set of generators from Proposition 3.1. This gives us the second column of the Betti diagram. We also know the last column, since it corresponds to the socle elements, which are exactly the alternating elements in $A$.

The Hilbert function of $A$ is given by

$$1, 4, 7, 8, 10, 12, \dots, 2n, m+1, 0, 0, \dots,$$

and its fourth difference is

$$1, 0, -3, 0, 5, -4, 1, 0, \dots, 0, -(n+1), 2n, 4, -(2n+4), n+1, 0, \dots.$$

Thus we can see that the only entries of the Betti diagram that we do not yet know for sure are the syzygies in degree $n+2$ where there is an overlap between the two numbers giving the difference 4. But, in Proposition 4.2, we have found $n+3$ independent first syzygies in degree $n+2$ and, in Proposition 4.3, we have proved that there are no more than $n-1$ linearly independent second syzygies of degree $n+2$. Since the fourth difference of the Hilbert function gives us that the difference $\beta_{2,n+2} - \beta_{3,n+2} = 4$, and in fact, $(n+3) - (n-1) = 4$, we have proved that the inequalities are equalities in both cases. ∎

## 5 Gorenstein Quotients Given by Socle Elements of the Ring of Covariants

As we have seen, there are $n+2$ elements of the socle of $A$ given by the alternating elements of $A$ *i.e.*, $h = x_1y_2 - x_2y_1$ and $h_m = x_1^m y_2^{n-m} - x_2^m y_1^{n-m}$, for $m = 0, 1, \dots, n$. For each such $m$ we can find a unique Gorenstein quotient, $A_m = A/\bar{I}_m = R/I_m$, of $A$, where $h_m$ generates the socle.

***Proposition 5.1*** *The bigraded Hilbert function of the quotient $A_m$ is given by*

$$H(i,j) = \begin{cases} 1 & \text{if } (i,j) = (0,0) \text{ or } (i,j) = (m, n-m), \\ 2 & \text{if } 0 \le i \le m,\ 0 \le j \le n-m \text{ and } 0 < i+j < n, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** It is clear from the bigrading and Hilbert function of $A$ that everything in $A$ of bidegree $(i,j)$ where $i > m$ or $j > n-m$ is zero. From the structure of $A$ that we noted earlier, the only other element that annihilates $h_m$ is the socle element, $h$ above, of $A$ in degree $(1,1)$. That finishes the proof. ∎

Notice that in the case of the symmetric groups, $S_n$, the Gorenstein quotients all have multiplicity $n!$, which is the order of $S_n$ (Haiman's remarkable $n!$-theorem, [Hai01]). In the case of the dihedral groups $D_n$, we see that the multiplicities of the Gorenstein quotients are always greater than or equal to the order of $D_n$, which is $2n$.

***Proposition 5.2*** *If we set $m' = n - m$, then*

(i) $A_m = R/I_m$, *where*

$$I_m = \langle x_1x_2, x_1y_2, x_2y_1, y_1y_2, x_1^{m+1}, x_2^{m+1}, y_1^{m'+1}, y_2^{m'+1}, x_1^m y_1^{m'} + x_2^m y_2^{m'} \rangle;$$

(ii) *the resolution of $A_m$, as an $R$-module, is*

$$0 \longrightarrow F_4 \xrightarrow{\Phi_4} F_3 \xrightarrow{\Phi_3} F_2 \xrightarrow{\Phi_2} F_1 \xrightarrow{\Phi_1} F_0 \longrightarrow R/I_m \longrightarrow 0.$$

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*where the maps are given by the following matrices.*

$$\Phi_1 = \begin{pmatrix} x_1x_2 & x_1y_2 & x_2y_1 & y_1y_2 & x_1^{m+1} & x_2^{m+1} & y_1^{m'+1} & y_2^{m'+1} & x_1^m y_1^{m'} + x_2^m y_2^{m'} \end{pmatrix}$$

$$\Phi_2^t = \left(\begin{array}{ccccccccc}
-y_2 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_1 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -y_1 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -y_2 & x_2 & 0 & 0 & 0 & 0 & 0 \\
\hline
-x_1^m & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 \\
0 & -x_1^m & 0 & 0 & y_2 & 0 & 0 & 0 & 0 \\
-x_2^m & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & -x_2^m & 0 & 0 & y_1 & 0 & 0 & 0 \\
0 & 0 & -y_1^{m'} & 0 & 0 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & -y_1^{m'} & 0 & 0 & y_2 & 0 & 0 \\
0 & -y_2^{m'} & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 0 & 0 & -y_2^{m'} & 0 & 0 & 0 & y_1 & 0 \\
\hline
-x_2^{m-1}y_2^{m'} & 0 & 0 & 0 & -y_1^{m'} & 0 & 0 & 0 & x_1 \\
-x_1^{m-1}y_1^{m'} & 0 & 0 & 0 & 0 & -y_2^{m'} & 0 & 0 & x_2 \\
0 & 0 & 0 & -x_2^m y_2^{m'-1} & 0 & 0 & -x_1^m & 0 & y_1 \\
0 & 0 & 0 & -x_1^m y_1^{m'-1} & 0 & 0 & 0 & -x_2^m & y_2
\end{array}\right)$$

$$\Phi_3 = \left(\begin{array}{ccccccccc}
y_1 & x_1^m & 0 & 0 & 0 & 0 & 0 & x_2^{m-1}y_2^{m'} & 0 \\
-y_2 & 0 & x_2^m & 0 & 0 & 0 & x_1^{m-1}y_1^{m'} & 0 & 0 \\
x_2 & 0 & 0 & 0 & y_2^{m'} & 0 & 0 & x_1^m y_1^{m'-1} & 0 \\
-x_1 & 0 & 0 & y_1^{m'} & 0 & 0 & x_2^m y_2^{m'-1} & 0 & 0 \\
\hline
0 & -y_2 & 0 & 0 & 0 & -y_1^{m'} & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 & 0 & -y_1^{m'} & 0 \\
0 & 0 & -y_1 & 0 & 0 & y_2^{m'} & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & 0 & -y_2^{m'} & 0 & 0 \\
0 & 0 & 0 & -y_2 & 0 & 0 & x_1^m & 0 & 0 \\
0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & -x_1^m \\
0 & 0 & 0 & 0 & -y_1 & 0 & 0 & x_2^m & 0 \\
0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_2^m \\
\hline
0 & 0 & 0 & 0 & 0 & -x_2 & 0 & -y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & x_1 & -y_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & -y_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1
\end{array}\right)$$

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$$\Phi_4 = \begin{pmatrix} x_1^m y_1^{m'} + x_2^m y_2^{m'} \\ -y_1^{m'+1} \\ y_2^{m'+1} \\ x_1^{m+1} \\ -x_2^{m+1} \\ y_1 y_2 \\ x_1 y_2 \\ -x_2 y_1 \\ x_1 x_2 \end{pmatrix}$$

**Proof** (i) From Proposition 4.1, we know that the three quadratic generators $x_1x_2, x_1y_2 + x_2y_1, y_1y_2$ are in the ideal $I_m$. The socle element of $A$ in degree two, $h = x_1y_2 - x_2y_1$, must be in the ideal $I_m$ since $A_m$ is Gorenstein with socle in degree $n > 2$. This now gives the generators of total degree two, guaranteed by Proposition 5.1. The bigraded Hilbert function of the quotient of $R$ by the ideal generated by these four bigraded forms is given by 1 in bidegree $(0, 0)$ and 2 elsewhere. Thus, to obtain the Hilbert function of Proposition 5.1 requires two generators of bidegree $(m + 1, 0)$, two generators in bidegree $(0, m' + 1)$, and one generator in bidegree $(m, m')$. A simple calculation of monomials shows that the generators of bidegrees $(m+1, 0)$ and $(0, m' + 1)$ are the ones given above. Finally, one notices that the form $x_1^m y_1^{m'} + x_2^m y_1^{m'}$ is already in the ideal $I$, but not in the ideal generated by the previous generators.

(ii) Since $A_m$ is Gorenstein with known Hilbert function, given by Proposition 5.1, and since we now know the degrees of the minimal generators of the ideal $I_m$, there is only one possible Betti diagram for $A_m$. That diagram is

| | 1 | 9 | 16 | 9 | 1 |
|---|---|---|---|---|---|
| 0 | 1 | – | – | – | – |
| 1 | – | 4 | 4 | 1 | – |
| 2 | – | – | – | – | – |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $m$ | – | 2 | 4 | 2 | – |
| $m+1$ | – | – | – | – | – |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $m'-1$ | – | – | – | – | – |
| $m'$ | – | 2 | 4 | 2 | – |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-2$ | – | – | – | – | – |
| $n-1$ | – | 1 | 4 | 4 | – |
| $n$ | – | – | – | – | 1 |

Since the matrices given above contain no non-zero constants and since they have size and degrees of entries consistent with this Betti diagram, it is enough to show that they form a complex. We leave this tedious but straightforward calculation to the reader. ■

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## References

[CoC] CoCoATeam, CoCoA*: a system for doing Computations in Commutative Algebra.* Available at http://cocoa.dima.unige.it.

[GS] D R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry.* Available at http://www.math.uiuc.edu/Macaulay2/.

[Hai94] M. D. Haiman, *Conjectures on the quotient ring by diagonal invariants.* J. Algebraic Combin. **3**(1994), no. 1, 17–76. doi:10.1023/A:1022450120589

[Hai01] ———, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture.* J. Amer. Math. Soc. **14**(2001), no. 4, 941–1006. doi:10.1090/S0894-0347-01-00373-3

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