EXTENSIONS OF PSEUDOMETRICS

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1. Introduction. If γ is an infinite cardinal number, a subset S of a topological space X is said to be *P^{γ}-embedded* in X if every γ-separable continuous pseudometric on S can be extended to a γ-separable continuous pseudometric on X. (A pseudometric d on X is *γ-separable* if there exists a subset G of X such that $|G| \leqslant \gamma$ and such that G is dense in X relative to the pseudometric topology $\mathfrak{T}_d$. A pseudometric d is *continuous* if d is continuous relative to the product topology on $X \times X$.) We say that S is *P-embedded* in X if every continuous pseudometric on S can be extended to a continuous pseudometric on X.

The problem of extending a metric from a subspace was first studied by F. Hausdorff in 1930 (**13**). He showed that a continuous metric on a closed subset of a metric space can be extended to a continuous metric on the whole space. Bing (**3**) and Arens (**1**) rediscovered this result independently. The concept "P^{γ}-embedded" was introduced in 1953 by Arens (who called it "γ-normally embedded") in connection with a generalization of the Tietze extension theorem. In this paper we generalize some of Arens' results and continue a study of P^{γ}-embedding and P-embedding.

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In §2 we prove several necessary and sufficient conditions for a subset to be P^{γ}-embedded. We then discuss the concept of P-embedding with respect to other concepts, including measurable cardinal numbers, C-embedding, C^*-embedding, pseudocompactness, normality, and collectionwise normality. We show that if S is P-embedded in X, then S is C-embedded in X, but that the converse does not hold. However, if S is C-embedded in X, then S is $P^{\aleph_0}$-embedded in X. This generalizes a result of Arens (**2**, Theorem 3.1). However, our proof is entirely different from Arens' in that ours uses a new characterization of $P^{\aleph_0}$-embedding.

In §5 we show that a topological space X is collectionwise normal if and only if every closed subset is P-embedded in X. This is analogous to the well-known result that a topological space X is normal if and only if every closed subset is C-embedded in X.

The notation and terminology will follow that of (**11**). For the definition of a normal cover, consult (**25**, p. 46). For the definition of collectionwise normal, the reader is referred to (**4**, p. 176). Other terms used in this paper are defined below.

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If $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of a space X, then by $\mathfrak{U}|S$ we mean the family $(U_\alpha \cap S)_{\alpha \in I}$. We say that $\mathfrak{U}$ has *power at most* γ (γ an infinite cardinal number) if $|I| \leqslant \gamma$. We say that the family $(U_\alpha)_{\alpha \in I}$ is *locally finite* if for each $x \in X$ there exist a neighbourhood G of X and a finite subset J of I such that $G \cap U_\alpha = \emptyset$ for every $\alpha \notin J$. When a family of subsets of a subspace S of X is said to be *open*, *locally finite*, etc., this refers to the topology of S.

Next let X be a topological space, let $S \subset X$, let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ be a cover of X, and let $\mathfrak{V} = (V_\beta)_{\beta \in J}$ be a cover of S. Then $\mathfrak{U}$ is an *extension* of $\mathfrak{V}$ if $I = J$ and $U_\alpha \cap S = V_\alpha$ for all $\alpha \in I$.

Following Kelley **(16)**, we say that X is *paracompact* if X is regular and if every open cover of X has a locally finite open refinement. Thus a pseudometric space is paracompact.

If f is a real-valued continuous function on X, set

$$Z(f) = Z_X(f) = \{x \in X : f(x) = 0\}.$$

Call $Z(f)$ the *zero-set of* f. The complement of $Z(f)$ is called the *cozero-set of* f. If $S \subset X$, then S is a *zero-set* if $S = Z(f)$, and S is a *cozero-set* if $S = X - Z(f)$. We denote the collection of all zero-sets in X by $\mathbf{Z}(X)$. We say that $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is a *cozero-set cover* of X if $\mathfrak{U}$ is a cover of X and if U_α is a cozero-set for each $\alpha \in I$. We define a *zero-set cover* of X in an analogous manner.

Suppose that $(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is a finite sequence of covers of a set X, and that $\mathfrak{A}_i = (A_i(\alpha))_{\alpha \in J_i}$ for each $i = 1, \ldots, n$. Then by $\wedge_{i=1}^n \mathfrak{A}_i$ we mean the family

$$(A_1(\alpha_1) \cap \ldots \cap A_n(\alpha_n))_{(\alpha_i, \ldots, \alpha_n) \in J_1 \times \ldots \times J_n}.$$

2. Equivalent formulations. In this section we characterize P^γ-embedding in terms of locally finite cozero-set covers, normal cozero-set covers, normal open covers, locally finite normal open covers, and uniformly locally finite open covers.

2.1. THEOREM. *Suppose that X is a topological space, that $S \subset X$, and that γ is an infinite cardinal number. Then the following statements are equivalent:*

(1) *S is P^γ-embedded in X.*

(2) *Every γ-separable bounded continuous pseudometric on S can be extended to a γ-separable bounded continuous pseudometric on X.*

(3) *Every γ-separable bounded continuous pseudometric on S can be extended to a continuous pseudometric on X.*

(4) *Every normal locally finite cozero-set cover of S of power at most γ has a refinement that can be extended to a normal open cover of X.*

(5) *Every normal open cover of S of power at most γ has a refinement that can be extended to a normal locally finite cozero-set cover of X of power at most γ.*

Before proving Theorem 2.1 we state some preliminary results that are interesting in themselves. Propositions 2.2 and 2.3 are clear but are worthy of note since they state the relationship between the topological structure induced

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by a pseudometric and the given topology on the space. Theorem 2.4 states that every normal sequence of open covers determines a continuous pseudometric. This was first shown by Tukey (**25**, Theorem 7.1) using results due to A. H. Frink (**10**). It can also be proved using (**5**, Chap. IX, §1, 4, Proposition 2). Proposition 2.5 states a useful basic fact about open covers.

2.2. PROPOSITION. *Suppose that $(X, \mathfrak{T})$ is a topological space and that d is a pseudometric on X. Then d is continuous if and only if $\mathfrak{T}_d \subset \mathfrak{T}$.*

2.3. PROPOSITION. *Suppose that $(X, \mathfrak{T})$ is a topological space and that d is a continuous pseudometric on X. If G is an open subset of X relative to $\mathfrak{T}_d$, then G is a cozero-set relative to $\mathfrak{T}$.*

If $(\mathfrak{U}_n)_{n \in \mathbf{N}}$ is a normal sequence of open covers of a space X and if d is a pseudometric on X, then we say that d is *associated with* $(\mathfrak{U}_n)_{n \in \mathbf{N}}$ if the following three conditions are satisfied:

(1) d is bounded by **1**.

(2) If $k \in \mathbf{N}$ and if $d(x, y) < 2^{-(k+1)}$, then $x \in st(y, \mathfrak{U}_k)$.

(3) If $k \in \mathbf{N}$ and if $x \in st(y, \mathfrak{U}_k)$, then $d(x, y) < 2^{-(k-3)}$.

2.4. THEOREM. *If $(\mathfrak{U}_n)_{n \in \mathbf{N}}$ is a normal sequence of open covers of a topological space X, then there exists a continuous pseudometric on X that is associated with $(\mathfrak{U}_n)_{n \in \mathbf{N}}$.*

2.5. PROPOSITION. *Suppose that X is a topological space, that $S \subset X$, that $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is an open cover of S, and that $\mathfrak{V}$ is a normal open cover of X such that $\mathfrak{V}|S$ refines $\mathfrak{U}$. Then there exists a normal locally finite cozero-set cover $\mathfrak{W} = (W_\alpha)_{\alpha \in I}$ of X such that $W_\alpha \cap S \subset U_\alpha$ for each $\alpha \in I$.*

Proof. Suppose that $\mathfrak{V}$ is a normal open cover of X such that $\mathfrak{V}|S$ refines $\mathfrak{U}$. By (**23**, Theorem 1.2), there exists a locally finite cozero-set cover $\mathfrak{A} = (A_\beta)_{\beta \in J}$ of X such that $\mathfrak{A}$ refines $\mathfrak{V}$. Since $\mathfrak{A}|S$ refines $\mathfrak{U}$, there exists a function $\pi: J \to I$ such that $A_\beta \cap S \subset U_{\pi(\beta)}$ for each $\beta \in J$. For each $\alpha \in I$, let

$$W_\alpha = \bigcup_{\beta \in \pi^{-1}(\alpha)} A_\beta.$$

Then one easily verifies that $(W_\alpha)_{\alpha \in I}$ is a locally finite cozero-set cover of X such that $W_\alpha \cap S \subset U_\alpha$ for each $\alpha \in I$. Since every locally finite cozero-set cover of X is normal (**23**, Theorem 1.2), the proof is now complete.

2.6. LEMMA. *Suppose that $(X, \mathfrak{T})$ is a topological space and that $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is a normal open cover of X of power at most γ (γ an infinite cardinal number). Then there exists a normal sequence $(\mathfrak{V}_i)_{i \in \mathbf{N}}$ of open covers of X such that $\mathfrak{V}_1$ refines $\mathfrak{U}$ and such that, for each $i \in \mathbf{N}$, $\mathfrak{V}_i$ has power at most γ.*

Proof. By hypothesis, there exists a normal sequence $(\mathfrak{U}_i)_{i \in \mathbf{N}}$ of open covers of X such that $\mathfrak{U}_1$ refines $\mathfrak{U}$. Let d be a continuous pseudometric associated with $(\mathfrak{U}_i)_{i \in \mathbf{N}}$ (2.4). For each $\alpha \in I$, let

$$A_\alpha = \bigcup \{S_d(x, 2^{-3}) : S_d(x, 2^{-3}) \subset U_\alpha\}.$$

Then $\mathfrak{A} = (A_\alpha)_{\alpha \in I}$ is an open cover of X relative to $\mathfrak{T}_d$ (the topology on X determined by d) such that $A_\alpha \subset U_\alpha$ for each $\alpha \in I$. By 2.5, since $(S_d(x, 2^{-3}))_{x \in X}$ is a normal open cover of $(X, \mathfrak{T}_d)$ that refines $\mathfrak{A}$, there is a normal locally finite cozero-set cover $\mathfrak{W} = (W_\alpha)_{\alpha \in I}$ of $(X, \mathfrak{T}_d)$ such that $W_\alpha \subset A_\alpha$ for each $\alpha \in I$. We therefore have a locally finite open cover of the normal space $(X, \mathfrak{T}_d)$. A repeated application of (**22**, Theorem 1.2), and the observation that the covers constructed therein are of power at most γ, give us a normal sequence $(\mathfrak{V}_i)_{i \in \mathbf{N}}$ of open covers of X, relative to $\mathfrak{T}_d$, such that $\mathfrak{V}_1$ refines $\mathfrak{W}$ and such that, for each $i \in \mathbf{N}$, $\mathfrak{V}_i$ has power at most γ. Since $\mathfrak{T}_d \subset \mathfrak{T}$ and since $\mathfrak{W}$ refines $\mathfrak{U}$, the result now follows.

Proof of 2.1. (1) *implies* (2). Let d be a γ-separable bounded continuous pseudometric on S. By (1), there exists a γ-separable continuous pseudometric e on X such that $e|S \times S = d$. Now $d \leqslant \boldsymbol{\alpha}$ for some $\alpha \in \mathbf{R}^+$. Then one easily verifies that $\bar{d} = e \wedge \boldsymbol{\alpha}$ is a γ-separable bounded continuous pseudometric on X such that $\bar{d}|S \times S = d$. Thus (2) holds.

(2) *implies* (3). This implication is immediate.

(3) *implies* (4). Assume (3) and suppose that $\mathfrak{U}$ is a normal locally finite cozero-set cover of S of power at most γ. By 2.6, there exists a normal sequence $(\mathfrak{V}_i)_{i \in \mathbf{N}}$ of open covers of S such that $\mathfrak{V}_1$ refines $\mathfrak{U}$ and such that, for each $i \in \mathbf{N}$, $\mathfrak{V}_i$ has power at most γ. Then, by 2.4, there exists a continuous pseudometric d on S that is associated with $(\mathfrak{V}_i)_{i \in \mathbf{N}}$ and d is γ-separable. Therefore, by (3), there is a continuous pseudometric $\bar{d}$ on X such that $\bar{d}|S \times S = d$. Let $\mathfrak{W}' = (S_{\bar{d}}(x, 2^{-4}))_{x \in X}$. Since $(X, \bar{d})$ is a pseudometric space, it is paracompact, so there is a locally finite open cover $\mathfrak{W}$ of X such that $\mathfrak{W}$ refines $\mathfrak{W}'$. By 2.2, 2.3, and the fact that a locally finite cozero-set cover is normal, it follows that $\mathfrak{W}$ is a normal open cover of X relative to the given topology on X and one easily verifies that $\mathfrak{W}|S$ refines $\mathfrak{U}$.

(4) *implies* (5). This follows from (**23**, Theorem 1.2) and 2.5.

(5) *implies* (1). Assume (5). Let $\mathfrak{T}$ be the given topology on X and suppose that d is a γ-separable continuous pseudometric on S. For each $m \in \mathbf{N}$, let

$$\mathfrak{G}_m = (S_d(x, 2^{-(m+3)}))_{x \in S}.$$

Now consider any $m \in \mathbf{N}$. Since (S, d) is a γ-separable pseudometric space, there exists a locally finite open cover $\mathfrak{W}^m$ of S of power at most γ such that $\mathfrak{W}^m$ refines $\mathfrak{G}_m$. Moreover, $\mathfrak{W}^m$ is normal relative to $\mathfrak{T}_d$ and therefore, by (5) and 2.5, there exists a refinement of $\mathfrak{W}^m$ that extends to a locally finite normal cozero-set cover $\mathfrak{V}^m$ of X of power at most γ. Then there exists a normal sequence $(\mathfrak{V}_i^m)_{i \in \mathbf{N}}$ of open covers of X such that $\mathfrak{V}_1^m$ refines $\mathfrak{V}^m$ and such that, for each $i \in \mathbf{N}$, $\mathfrak{V}_i^m$ has power at most γ.

Now for all $i, m \in \mathbf{N}$, let

$$\mathfrak{U}^m = \wedge_{j=1}^m \mathfrak{V}^j$$

and

$$\mathfrak{U}_i^m = \wedge_{j=1}^m \mathfrak{V}_i^j.$$

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Then for all $i, m \in \mathbf{N}$, one easily verifies that

(i) $\mathfrak{U}^m$ and $\mathfrak{U}_i^m$ are open covers of X of power at most γ,

(ii) $\mathfrak{U}_{i+1}^m <^* \mathfrak{U}_i^m$ and $\mathfrak{U}_1^m$ refines $\mathfrak{U}^m$,

(iii) $\mathfrak{U}_i^{m+1}$ refines $\mathfrak{U}_i^m$ and $\mathfrak{U}^{m+1}$ refines $\mathfrak{U}^m$, and

(iv) $\mathfrak{U}^m|S$ refines $\mathfrak{G}_m$.

Now again consider any $m \in \mathbf{N}$. It follows from (i) and (ii) that $(\mathfrak{U}_i^m)_{i\in\mathbf{N}}$ is a normal sequence of open covers of X. Then, by 2.4, there exists a continuous pseudometric r_m on X that is associated with $(\mathfrak{U}_i^m)_{i\in\mathbf{N}}$ and r_m is γ-separable. By (ii) and (iv), we also have

(*) *If $x, y \in S$ and if $r_m(x, y) < 2^{-3}$, then $d(x, y) < 2^{-(m+2)}$.*

Define $r: X \times X \to \mathbf{R}^+$ by $r(x, y) = \sum_{m\in\mathbf{N}} 2^{-m} r_m(x, y)$. One easily verifies that r is a continuous γ-separable pseudometric on X.

From (*) it follows that

(**) *If $x, y \in S$, if $i > 3$, and if $r(x, y) < 2^{-i}$, then $d(x, y) < 2^{-(i-1)}$.*

Define a relation R on X as follows:

$$x\,R\,y \quad \text{if } r(x, y) = 0 \qquad (x, y \in X).$$

Observe that R is an equivalence relation on X. Let $X^* = X/R$ be the quotient space of X modulo R and let $\tau: X \to X^*$ be the canonical map. Then the formula

$$r^*(\tau(x), \tau(y)) = r(x, y) \qquad (x, y \in X)$$

determines a well-defined map $r^*: X^* \times X^* \to \mathbf{R}^+$ and one easily verifies that (X^*, r^*) is a metric space, that $\mathfrak{T}_{r^*}$ is the quotient topology on X^*, and that the canonical map $\tau: X \to X^*$ is an isometry. Since X^* is a continuous image of X (i.e. with respect to $\mathfrak{T}_r$ and $\mathfrak{T}_{r^*}$) and since X has a dense subset A with $|A| \leqslant \gamma$, then X^* also has a dense subset (namely $\tau(A)$) with $|\tau(A)| \leqslant \gamma$.

Let $S^* = \tau(S)$. By (**) it follows that we can define a map $d^*: S^* \times S^* \to \mathbf{R}^+$ as follows:

$$d^*(\tau(a), \tau(b)) = d(a, b) \qquad (a, b \in S).$$

Then one easily verifies that d^* is a pseudometric on S^*.

Let $\mathfrak{D}^*$ be the metric uniform structure on S^* whose base consists of the single metric $r^*|S^* \times S^*$ (**11**, 15.3). By (**) it follows that $d^* \in \mathfrak{D}^*$ and therefore d^* is a uniformly continuous function from $S^* \times S^*$ into $\mathbf{R}^+$ (**11**, 15N.1). Moreover, by (**11**, 15.11), d^* can be extended to a uniformly continuous function from $\operatorname{cl} S^* \times \operatorname{cl} S^*$ into $\mathbf{R}^+$. One easily verifies that this extension (which we shall again denote by d^*) is a pseudometric on $\operatorname{cl} S^*$.

It now follows by (**1**, Theorem 3.4) that d^* can be extended to a continuous pseudometric e on X^*. Define $\bar{d}$ on $X \times X$ by $\bar{d} = e \circ (\tau \times \tau)$. Since τ is

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continuous relative to $\mathfrak{T}$, $\bar{d}$ is a continuous pseudometric on X (2.2). Moreover, if $x, y \in S$, then

$$\bar{d}(x, y) = e(\tau(x), \tau(y)) = d^*(\tau(x), \tau(y)) = d(x, y).$$

Therefore $\bar{d}|S \times S = d$. Since $\bar{d}$ is continuous relative to $\mathfrak{T}_r$ and since r is γ-separable, it follows that $\bar{d}$ is γ-separable. Therefore (1) holds.

The proof is now complete.

Remark 1. The only other mention of P^γ-embedding that we are aware of is in Arens **(1, 2)**. Arens called the concept "γ-normally embedding" and proved the equivalence of (1) and (5) of Theorem 2.1 for the case in which X is a normal topological space and S is a closed subset of X **(2**, Theorem 2.4**)**.

Remark 2. The following example shows that in the proof that (5) implies (1) care must be taken in choosing the normal sequence that refines $\mathfrak{G}_m$.

Let X be an uncountable discrete space, let S be a countable subset of X, and let d be the discrete metric on S. Then d is $\aleph_0$-separable on S. For each $m \in \mathbf{N}$, let $\mathfrak{G}_m = (\{x\})_{x \in S}$, let $\mathfrak{B}^m = \mathfrak{G}_m \cup (X - S)$, and let $\mathfrak{B}_i{}^m = (\{x\})_{x \in X}$ for each $i \in \mathbf{N}$. If r_m is a pseudometric on X that is associated with $(\mathfrak{B}_i{}^m)_{i \in \mathbf{N}}$, then r_m is discrete for each $m \in \mathbf{N}$, and therefore

$$r = \sum_{m \in \mathbf{N}} 2^{-m} r_m$$

is discrete. It follows that, in this case, $\bar{d}$ can be chosen to be discrete and therefore not $\aleph_0$-separable.

Thus the proof of **(2**, Theorem 2.4**)** must be modified to obtain the required γ-separable continuous pseudometric on X that extends a given γ-separable continuous pseudometric on S.

Remark 3. If X is a topological space, if $S \subset X$, and if γ is an infinite cardinal number, then we can define S to be *$P^{\gamma*}$-embedded* in X if every γ-separable bounded continuous pseudometric on S can be extended to a γ-separable bounded continuous pseudometric on X. Similarly, we can define S to be *P^*-embedded* in X if every bounded continuous pseudometric on S can be extended to a bounded continuous pseudometric on X. However, in Theorem 2.1 we saw that S is P^γ-embedded in X if and only if S is $P^{\gamma*}$-embedded in X. Moreover, in Theorem 2.8 we shall show that S is P-embedded in X if and only if S is P^*-embedded in X. Thus it is unnecessary to define the concepts of $P^{\gamma*}$-embedding and P^*-embedding since they do not really differ from P^γ-embedding and P-embedding respectively.

We now introduce some terminology. Suppose that X is a topological space and that $(F_\alpha)_{\alpha \in I}$ and $(G_\beta)_{\beta \in J}$ are two families of subsets of X. We say that $(F_\alpha)_{\alpha \in I}$ is *finite with respect* to $(G_\beta)_{\beta \in J}$ if for each $\beta \in J$ there exists a finite subset K_β of I such that $F_\alpha \cap G_\beta = \emptyset$ if $\alpha \notin K_\beta$. We say that $(F_\alpha)_{\alpha \in I}$ is *uniformly locally finite* in X if there exists a locally finite open cover $(U_\lambda)_{\lambda \in K}$ of X such that $(F_\alpha)_{\alpha \in I}$ is finite with respect to $(U_\lambda)_{\lambda \in K}$. We say that $(F_\alpha)_{\alpha \in I}$ is *γ-uniformly* *locally finite* in X (γ an infinite cardinal number) if there exists a locally finite open cover $(U_\lambda)_{\lambda \in K}$ of X of power at most γ such that $(F_\alpha)_{\alpha \in I}$ is finite with respect to $(U_\lambda)_{\lambda \in K}$.

M. Katětov originally defined "uniformly locally finite" in **(15)**. It is evident that a uniformly locally finite family is locally finite. However, it follows from **(15**, Theorem 5.4**)** that the converse does not hold.

2.7. THEOREM. *Suppose that X is a normal space, that S is a closed subset of X, and that γ is an infinite cardinal number. Then the following statements are equivalent:*

(1) *S is P^γ-embedded in X.*

(2) *Every locally finite open cover of S of power at most γ has a refinement that can be extended to a locally finite open cover of X of power at most γ.*

(3) *If $(F_\alpha)_{\alpha \in I}$ is a γ-uniformly locally finite family of subsets of S, then $(F_\alpha)_{\alpha \in I}$ is γ-uniformly locally finite in X.*

(4) *If $(H_\alpha)_{\alpha \in I}$ is a locally finite family of open subsets of S of power at most γ and if $(F_\alpha)_{\alpha \in I}$ is a family of closed subsets of S such that $F_\alpha \subset H_\alpha$ for each $\alpha \in I$, then there exists a locally finite family $(G_\alpha)_{\alpha \in I}$ of open subsets of X such that $F_\alpha \subset G_\alpha \cap S \subset H_\alpha$ for each $\alpha \in I$.*

(5) *If $(H_\alpha)_{\alpha \in I}$ is a uniformly locally finite open cover of S of power at most γ, then there exists a uniformly locally finite open cover $(G_\alpha)_{\alpha \in I}$ of X such that $G_\alpha \cap S = H_\alpha$ for each $\alpha \in I$.*

Proof. (1) *implies* (2). Since a locally finite open cover of a normal space is normal, this implication follows from 2.1.

(2) *implies* (3). By hypothesis $\mathfrak{F} = (F_\alpha)_{\alpha \in I}$ is a γ-uniformly locally finite family of subsets of S, so there exists a locally finite open cover $\mathfrak{U} = (U_\beta)_{\beta \in J}$ of S of power at most γ such that $\mathfrak{F}$ is finite with respect to $\mathfrak{U}$. By (2) and 2.5, there exists a locally finite open cover $\mathfrak{V} = (V_\beta)_{\beta \in J}$ of X such that $V_\beta \cap S \subset U_\beta$ for each $\beta \in J$. Clearly $\mathfrak{V}$ has power at most γ and one easily verifies that $\mathfrak{F}$ is finite with respect to $\mathfrak{V}$.

(3) *implies* (4). Assume (3) and suppose that $(H_\alpha)_{\alpha \in I}$ is a locally finite family of open subsets of S of power at most γ and that $(F_\alpha)_{\alpha \in I}$ is a family of closed subsets of S such that $F_\alpha \subset H_\alpha$ for each $\alpha \in I$. Then, by **(15**, Theorem 5.1**)** and the fact that the cover constructed in the proof therein is of power at most γ, $(F_\alpha)_{\alpha \in I}$ is γ-uniformly locally finite in S. Therefore, by (3), $(F_\alpha)_{\alpha \in I}$ is γ-uniformly locally finite in X. Hence, by **(15**, Theorem 5.1**)**, there exists a locally finite family $(G'_\alpha)_{\alpha \in I}$ of open subsets of X such that $\mathrm{cl}_X F_\alpha \subset G'_\alpha$ for each $\alpha \in I$. Let $(H'_\alpha)_{\alpha \in I}$ be a family of open subsets of X such that $H'_\alpha \cap S = H_\alpha$ for each $\alpha \in I$. Set $G_\alpha = G'_\alpha \cap H'_\alpha$ ($\alpha \in I$) and observe that $(G_\alpha)_{\alpha \in I}$ is a locally finite family of open subsets of X such that $F_\alpha \subset G_\alpha \cap S \subset H_\alpha$ for each $\alpha \in I$. Therefore (4) holds.

(4) *implies* (5). We may assume that S is not empty. From **(15**, Theorem 5.1**)** and the fact that S is normal, it follows that there exists a uniformly locally finite open cover $(A_\alpha)_{\alpha \in I}$ of S such that $\mathrm{cl}_S H_\alpha \subset A_\alpha$ for each $\alpha \in I$. By

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(4), there exists a locally finite family $(U_\alpha)_{\alpha \in I}$ of open subsets of X such that $\mathrm{cl}_S H_\alpha \subset U_\alpha \cap S \subset A_\alpha$ for each $\alpha \in I$. Since $\mathrm{cl}_S H_\alpha = \mathrm{cl}_X H_\alpha$ and since X is normal, there exists, for each $\alpha \in I$, an open set V_α of X such that

$$\mathrm{cl}_X H_\alpha \subset V_\alpha \subset \mathrm{cl}_X V_\alpha \subset U_\alpha.$$

For each $\alpha \in I$, let W_α be an open subset of X such that $W_\alpha \cap S = H_\alpha$. Now choose an arbitrary $\beta \in I$ and define $(G_\alpha)_{\alpha \in I}$ as follows: set

$$G_\beta = (V_\beta \cap W_\beta) \cup (X - S);$$

and if $\alpha \neq \beta$, let $G_\alpha = V_\alpha \cap W_\alpha$. Then one easily verifies that $(G_\alpha)_{\alpha \in I}$ is a uniformly locally finite open cover of X such that $G_\alpha \cap S = H_\alpha$ for each $\alpha \in I$.

(5) *implies* (1). Assume (5) and suppose that $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ is a normal locally finite cozero-set cover of S of power at most γ. Since S is normal, there exists an open cover $\mathfrak{H} = (H_\alpha)_{\alpha \in I}$ of S such that $\mathrm{cl}_S H_\alpha \subset U_\alpha$ for each $\alpha \in I$. Moreover, $\mathfrak{H}$ is uniformly locally finite, so, by (5), there exists a uniformly locally finite open cover $\mathfrak{G} = (G_\alpha)_{\alpha \in I}$ of X such that $G_\alpha \cap S = H_\alpha$ for each $\alpha \in I$. Thus $\mathfrak{G}$ is a locally finite open cover of X of power at most γ such that $\mathfrak{G}|S$ refines $\mathfrak{U}$. Since $\mathfrak{G}$ is normal, it follows that 2.1(4) holds, whence S is P^γ-embedded in X. The proof is now complete.

Remark. The equivalence of (1) and (2) of Theorem 2.7 was originally proved by Arens **(2)**.

2.8. THEOREM. *If X is a topological space and if $S \subset X$, then the following statements are equivalent:*

(1) *S is P-embedded in X.*

(2) *S is P^γ-embedded in X for all infinite cardinal numbers γ.*

This result is an immediate consequence of the equivalence of (1) and (3) of Theorem 2.1.

Remark. From 2.8 it follows that the equivalences of Theorems 2.1 and 2.7 remain valid if all mention of the infinite cardinal number γ is deleted from them.

3. Relation of P-embedding to some other topological properties.

In Theorem 3.2 we shall show that a P-embedded subset is necessarily C-embedded, and in Theorem 3.3 we shall show that the converse holds if S is dense in X and $|S|$ is non-measurable. In Theorem 3.5 we shall give an equivalent formulation of Ulam's problem, and in Theorems 3.6–3.8 we shall investigate some of the relationships between P-embedding and pseudocompact spaces. In 3.9 and 3.10, absolutely P-embedded completely regular spaces will be defined and characterized.

3.1. LEMMA. *Suppose that S is P-embedded in X. If $f \in C(S)$, if $Z_S(f) \neq \emptyset$, and if $f \geqslant \mathbf{0}$, then there exists $g \in C(X)$ such that $g|S = f$.*

Proof. Let $f \in C(S)$ and suppose that $f \geqslant \mathbf{0}$ and that $Z = Z_S(f) \neq \emptyset$. Define ψ_f: $S \times S \to \mathbf{R}$ by

$$\psi_f(x, y) = |f(x) - f(y)| \qquad (x, y \in S).$$

Then ψ_f is a continuous pseudometric on S. Since S is P-embedded in X, there exists a continuous pseudometric d on X such that $d|S \times S = \psi_f$. Let g: $X \to \mathbf{R}$ be defined by $g(x) = \inf_{y \in Z} d(x, y)$ $(x \in X)$. Then $g \in C(X)$ and $g|S = f$.

3.2. THEOREM. *Suppose that X is a topological space. If S is P-embedded in X, then S is C-embedded in X.*

Proof. We may assume that $S \neq \emptyset$. Let $f \in C(S)$. Fix an arbitrary $a \in S$; let $f(a) = \alpha$, and let $g = (f \vee \boldsymbol{\alpha}) - \boldsymbol{\alpha}$ and $h = -((f \wedge \boldsymbol{\alpha}) - \boldsymbol{\alpha})$. By 3.1, there exist $\bar{g}, \bar{h} \in C(X)$ such that $\bar{g}|S = g$ and $\bar{h}|S = h$. Let $k = (\bar{g} - \bar{h}) + \boldsymbol{\alpha}$. Then one easily verifies that $k \in C(X)$ and that $k|S = f$.

Remark. In §5 we shall show that the converse of Theorem 3.2 does not hold. In Theorems 4.9 and 5.4 we shall show that the converse holds if S is separable, or if $|X|$ is non-measurable and there exists a collectionwise normal space Y such that $X \subset Y \subset vX$. We now prove another partial converse.

3.3. THEOREM. *Suppose that X is completely regular, that S is dense in X, and that $|S|$ is non-measurable. Then the following statements are equivalent:*

(1) *S is C-embedded in X.*

(2) *$\mathfrak{C}(S)$ is the relative uniform structure on S obtained from $\mathfrak{C}(X)$.*

(3) *S is P-embedded in X.*

Proof. (1) *implies* (2). If $f \in C(Y)$, let $\psi_f: Y \times Y \to \mathbf{R}$ be defined as follows:

$$\psi_f(x, y) = |f(x) - f(y)| \qquad (x, y \in Y).$$

Let $\mathfrak{S}(S) = \{\psi_f : f \in C(S)\}$ and let $\mathfrak{S}(X)|S = \{\psi_f|S \times S : f \in C(X)\}$. Then $\mathfrak{S}(S)$ is a subbase for $\mathfrak{C}(S)$ and $\mathfrak{S}(X)|S$ is a subbase for the relative uniform structure on S obtained from $\mathfrak{C}(X)$. One easily verifies that $\mathfrak{S}(X)|S = \mathfrak{S}(S)$ and it follows that (2) holds.

(2) *implies* (3). Equip S and X with the uniform structures $\mathfrak{C}(S)$ and $\mathfrak{C}(X)$ respectively **(11**, 15.5**)**, let $\mathfrak{P}(S)$ denote the universal uniform structure on S **(11**, 15G.4**)**, and let $\gamma(S, \mathfrak{P}(S))$ denote the completion of $(S, \mathfrak{P}(S))$ **(11**, 15.8**)**. Since $|S|$ is non-measurable, we have, by **(11**, 15.21**)**, that $\gamma(S, \mathfrak{P}(S))$ is realcompact and hence $\gamma(S, \mathfrak{P}(S)) = vS$. Since S is C-embedded (and dense) in X **(11**, 15P.4**)**, it follows that $vS = vX$ **(11**, 8.6**)**.

Suppose that d is a continuous pseudometric on S. By **(11**, 15N.4**)**, d can be extended to a continuous pseudometric $\bar{d}$ on $\gamma(S, \mathfrak{P}(S)) = vX$. Let $e = \bar{d}|X \times X$. Then e is a continuous pseudometric on X and $e|S \times S = d$. Therefore S is P-embedded in X.

(3) *implies* (1). This is 3.2.

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Remark. In Theorem 3.3 the hypothesis that $|S|$ is non-measurable was only necessary for applying **(11**, 15.21**)**. For this reason we can replace this hypothesis by the apparently weaker hypothesis "for each continuous pseudometric d on S, the cardinal of every d-discrete set in S is non-measurable." For the definition of d-discrete, see **(11**, 15.15**)**.

3.4. THEOREM. *If X is a discrete space, then the following statements are equivalent:*

(1) *$|X|$ is non-measurable.*

(2) *X is P-embedded in υX.*

Proof. (1) *implies* (2). This is immediate by 3.3.

(2) *implies* (1). Assume (2) and suppose that $|X|$ is measurable. Since X is a discrete topological space, $X \neq \upsilon X$ **(11**, Theorem 12.2**)**. Let $p \in \upsilon X - X$ and let $\mathfrak{U} = (\{x\})_{x \in X}$. Then $\mathfrak{U}$ is a locally finite cozero-set cover of X, and therefore, by 2.1 and 2.8, $\mathfrak{U}$ has a refinement, which must be $\mathfrak{U}$ itself, that can be extended to a locally finite cozero-set cover of υX. Therefore $(\{x\})_{x \in X}$ is locally finite in υX, so there exists a neighbourhood U of p in υX such that $U \cap X$ is finite. Thus $U \cap X$ is closed in υX and it follows that $(\upsilon X - (U \cap X)) \cap U$ is a neighbourhood of p in υX. Since X is dense in υX, there exists an $x \in X$ such that $x \in (\upsilon X - (U \cap X)) \cap U$, a contradiction.

As an immediate consequence of 3.3 and 3.4 we have:

3.5. THEOREM. *The following statements are equivalent:*

(1) *Every cardinal is non-measurable.*

(2) *If X is completely regular and if S is a dense C-embedded subset of X, then S is P-embedded in X.*

The next three results investigate some of the relationships between P-embedding and pseudocompact spaces.

3.6. THEOREM. *Suppose that X is completely regular and that $|X|$ is non-measurable. Then X is P-embedded in βX if and only if X is pseudocompact.*

Proof. This result follows from 3.3 and **(11**, 8A.4**)**.

3.7. THEOREM. *Suppose that X is completely regular and that S is a pseudocompact subset of X. If S is C^*-embedded in X, then S is P-embedded in X.*

Proof. We may assume that $S \neq \emptyset$. Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ be a locally finite cozero-set cover of S. Since S is pseudocompact, there exists a non-empty finite subset K of I such that $(U_\alpha)_{\alpha \in K}$ is a cover of S **(14**, Theorem 1**)**. (Actually, assuming that no U_α is empty, I must itself be finite.) By hypothesis, S is C^*-embedded in X, so for each $\alpha \in K$ there exists a cozero-set V_α in X such that $V_\alpha \cap S = U_\alpha$. Since $X - (\bigcup_{\alpha \in K} V_\alpha)$ is a zero-set in X and since S is a pseudocompact C^*-embedded subset of X, there exists a cozero-set G of X such that $S \cap G = \emptyset$ and $X - (\bigcup_{\alpha \in K} V_\alpha) \subset G$. Fix an arbitrary $\beta \in K$ and define

$\mathfrak{W} = (W_\alpha)_{\alpha \in K}$ as follows: set $W_\beta = G \cup V_\beta$; and if $\alpha \neq \beta$, let $W_\alpha = V_\alpha$. Then $\mathfrak{W}$ is a (finite) cozero-set cover of X such that $\mathfrak{W}|S$ refines $\mathfrak{U}$. Therefore S is P-embedded in X.

As an immediate consequence of 3.2, 3.7, and the fact that a C-embedded subset of a pseudocompact space is pseudocompact, we have:

3.8. THEOREM. *If X is a completely regular pseudocompact space and if $S \subset X$, then the following statements are equivalent:*
(1) *S is P-embedded in X.*
(2) *S is C-embedded in X.*
(3) *S is C^*-embedded in X and S is pseudocompact.*

3.9. DEFINITION. Suppose that X is completely regular. We say that X is *absolutely P-embedded* if every embedding of X in a completely regular space is a P-embedding. We say that X is *almost compact* if $|\beta X - X| \leqslant 1$. For a discussion of almost compact spaces, see (**11**, 6J and 15R).

3.10. THEOREM. *If X is completely regular, then the following statements are equivalent:*
(1) *X is absolutely P-embedded.*
(2) *X is almost compact.*

Proof. (1) *implies* (2). Suppose that $X \subset Y$ and that Y is completely regular. Since X is P-embedded in Y, it is C-embedded in Y (3.2). Therefore (**11**, 6J.5) holds, and it follows that X is almost compact.

(2) *implies* (1). Suppose that $X \subset Y$ and that Y is completely regular. Since X is almost compact, it follows, by (**11**, 6J), that X is C-embedded in βX, and hence X is pseudocompact. Again, by (**11**, 6J), X is C-embedded in Y. Therefore, by 3.7, X is P-embedded in Y.

4. $P^{\aleph_0}$-embedding and normal spaces. This section deals with normal spaces and the particular case of P^γ-embedding in which $\gamma = \aleph_0$. First we state four elementary results concerning cozero-set covers. In Theorem 4.5 we shall give several necessary and sufficient conditions for a topological space to be normal. (Note that a normal space need not be T_1.) These criteria for normality begin with Tukey (**25**), and although some of these conditions have appeared in one place or another in the literature, we are aware of no other attempt to organize them all into one theorem. Then, in Theorem 4.6, using earlier results of this section, we are able to sharpen many of the equivalences of Theorem 2.1. In 4.7 we shall prove that if S is C-embedded in X, then S is $P^{\aleph_0}$-embedded in X.

4.1. THEOREM. *Every countable cozero-set cover of a topological space X has a countable locally finite cozero-set refinement.*

The idea for this theorem was first contained in the proof of (**8**, Theorem 2) and later in the proof of (**18**, Theorem 1). The reader is referred to these papers for the proof of Theorem 4.1.

4.2. COROLLARY. *Every countable cozero-set cover of a topological space is normal.*

Proof. This follows from 4.1 and (**23**, Theorem 1.2).

Using 4.1, 4.2, and modifying the proof of (**21**, Theorem 3) we have:

4.3. THEOREM. *Every countable cozero-set cover of a topological space X has a countable star-finite normal cozero-set refinement.*

4.4. THEOREM. *Every countable cozero-set cover of a topological space X has a countable star-finite normal cozero-set star-refinement.*

Proof. Suppose that $\mathfrak{U}$ is a countable cozero-set cover of X. By 4.3, there exists a countable star-finite normal cozero-set cover $\mathfrak{V}$ such that $\mathfrak{V}$ refines $\mathfrak{U}$. Hence, there is a sequence $(\mathfrak{V}_i)_{i \in \mathbf{N}}$ of open covers of X such that $\mathfrak{V}_1$ refines $\mathfrak{U}$ and $\mathfrak{V}_3 <^* \mathfrak{V}_2 <^* \mathfrak{V}_1$ (i.e. $\mathfrak{V}_3 <^{**} \mathfrak{U}$). By (**23**, Theorem 1.2), there exists a locally finite cozero-set cover $\mathfrak{W}$ such that $\mathfrak{W}$ refines $\mathfrak{V}_3$. Thus $\mathfrak{W} <^{**} \mathfrak{U}$ and $\mathfrak{U}$ is countable, so, by (**12**, Theorem 1.2), there exists a countable open cover $\mathfrak{A}$ such that $\mathfrak{A} <^* \mathfrak{U}$. The proof of (**12**, Theorem 1.2) shows that $\mathfrak{A}$ is a cozero-set cover if $\mathfrak{W}$ is a locally finite cozero-set cover. Therefore, by 4.3, there exists a countable star-finite normal cozero-set cover $\mathfrak{B}$ of X such that $\mathfrak{B}$ refines $\mathfrak{A}$. Clearly $\mathfrak{B}$ is the desired countable star-finite normal cozero-set star-refinement of $\mathfrak{U}$.

Remark. The author is indebted to the referee for pointing out this result as well as its use in Theorem 4.5.

4.5. THEOREM. *If X is a topological space, then the following statements are equivalent:*

(1) *X is normal.*

(2) *If $(U_\alpha)_{\alpha \in I}$ is a point-finite open cover of X, then there exists a cozero-set cover $(V_\alpha)_{\alpha \in I}$ of X such that $V_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

(3) *If $(U_\alpha)_{\alpha \in I}$ is a star-finite open cover of X, then there exists a cozero-set cover $(V_\alpha)_{\alpha \in I}$ of X such that $V_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

(4) *Every countable point-finite open cover of X has a countable cozero-set refinement.*

(5) *Every binary open cover of X is normal.*

(6) *Every countable point-finite normal open cover of X has a countable star-finite normal cozero-set star-refinement.*

Proof. (1) *implies* (2). Assume (1) and let $(U_\alpha)_{\alpha \in I}$ be a point-finite open cover of X. By (**17**, Theorem 33.4), there exists an open cover $(A_\alpha)_{\alpha \in I}$ of X such that $\operatorname{cl} A_\alpha \subset U_\alpha$ for each $\alpha \in I$. Since X is normal, for each $\alpha \in I$ there exists $f_\alpha \in C(X)$ such that $f_\alpha(\operatorname{cl} A_\alpha) \subset \{0\}$, $f_\alpha(X - U_\alpha) \subset \{1\}$, and $\mathbf{0} \leqslant f_\alpha \leqslant 1$.

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Let $V_\alpha = f_\alpha^{-1}$ ([0, 1/2)). Then $(V_\alpha)_{\alpha \in I}$ is a cozero-set cover of X such that $V_\alpha \subset U_\alpha$ for each $\alpha \in I$.

(2) *implies* (4). This implication is immediate.

(4) *implies* (5). This implication follows from 4.2.

(5) *implies* (1). This follows from **(25**, p. 47**)**.

(2) *implies* (3). This implication is immediate.

(3) *implies* (1). Assume (3). Let F_1 and F_2 be disjoint closed subsets of X and let $U_i = X - F_i$ $(i = 1, 2)$. Then $\mathfrak{U} = (U_1, U_2)$ is a star-finite open cover of X. Hence there exists a cozero-set cover $\mathfrak{V} = (V_1, V_2)$ of X such that $V_i \subset U_i$ $(i = 1, 2)$. Now $X - V_1$ and $X - V_2$ are disjoint zero-sets and are therefore completely separated. Hence there exists $f \in C(X)$ such that $f(X - V_1) \subset \{0\}$, $f(X - V_2) \subset \{1\}$, and $\mathbf{0} \leqslant f \leqslant \mathbf{1}$. Let

$$W_1 = \{x \in X : f(x) < 1/3\} \quad \text{and} \quad W_2 = \{x \in X : f(x) > 2/3\}.$$

Then W_1 and W_2 are disjoint open sets such that $F_i \subset W_i$ $(i = 1, 2)$. It follows that X is normal.

(4) *implies* (6). This follows from 4.4.

(6) *implies* (4). This implication is immediate.

The proof is now complete.

Remark 1. Let X be a topological space and let us consider the statement:

(*) *Every point-finite open cover of X is normal.*

Clearly, if (*) holds, then X is normal; but the converse is not valid. To see this latter assertion, let us consider the normal, non-collectionwise normal space G of **(20**, Example 2**)**. This example is due to Bing **(4)**. However, Michael shows that every open cover of G has a point-finite open refinement. If G normal implies that (*) holds, then G is paracompact and therefore collectionwise normal, a contradiction.

Thus a topological space X is normal if and only if any one of the following statements holds:

(1) *Every countable star-finite open cover of X is normal.*

(2) *Every star-finite open cover of X is normal.*

(3) *Every countable locally finite open cover of X is normal.*

(4) *Every locally finite open cover of X is normal.*

(5) *Every countable point-finite open cover of X is normal.*

However we have shown that the word "countable" cannot be omitted in part (5).

Remark 2. If the word "point-finite" is deleted from condition (4) of Theorem 4.5 and from condition (5) of Remark 1 above, then each becomes a characterization of a normal and countably paracompact space **(23**, Theorems 1.1 and 1.2**)**.

4.6. THEOREM. *If X is a topological space and if $S \subset X$, then the following statements are equivalent:*

(1) *S is $P^{\aleph_0}$-embedded in X.*

(2) *Every countable cozero-set cover of S has a refinement that can be extended to a countable star-finite cozero-set cover of X.*

(3) *Every countable star-finite cozero-set cover of S has a refinement that can be extended to a star-finite cozero-set cover of X.*

(4) *Every countable star-finite cozero-set cover of S has a refinement that can be extended to a countable cozero-set cover of X.*

Proof. That (1) *implies* (2) follows from 2.1 and 4.3; obviously (2) *implies* (3); and that (3) *implies* (4) follows from 2.5.

(4) *implies* (1). Assume (4) and suppose that $\mathfrak{U}$ is a countable normal locally finite cozero-set cover of S. Then, by 4.3, there exists a countable star-finite cozero-set cover $\mathfrak{V}$ of S such that $\mathfrak{V}$ refines $\mathfrak{U}$. By (4), $\mathfrak{V}$ has a refinement that can be extended to a countable cozero-set cover $\mathfrak{W}$ of X, and, by 4.1, there exists a countable locally finite cozero-set cover $\mathfrak{A}$ of X such that $\mathfrak{A}$ refines $\mathfrak{W}$. Since $\mathfrak{A}|S$ refines $\mathfrak{U}$ and since a locally finite cozero-set cover is normal, 2.1(4) holds and it follows that S is P-embedded in X.

Remark 1. From Theorem 4.6 it follows that S is $P^{\aleph_0}$-embedded in X if and only if every countable cozero-set cover of S has a refinement that can be extended to a countable cozero-set cover of X. The omission of the word "countable" from this condition renders it invalid for the characterization of P-embedding. If it were valid, the C-embedded subset constructed in 5.3 below would have to be P-embedded, which we shall show to be false.

Remark 2. Although we strongly suspect that conditions similar to (2) through (4) of 4.6 are not equivalent to P^{γ}-embedding for arbitrary cardinal numbers $\gamma > \aleph_0$, we have been unable to prove or find a counterexample to this assertion.

In §5 we shall show that if S is C-embedded in X, then S need not be P-embedded in X. However, we shall now show that if S is C-embedded in X, then S is $P^{\aleph_0}$-embedded in X. This theorem generalizes a result of R. Arens, which we state as a corollary in 4.8.

4.7. THEOREM. *Suppose that X is a topological space and that $S \subset X$. If S is C-embedded in X, then S is $P^{\aleph_0}$-embedded in X.*

Proof. Let $\mathfrak{U} = (U_i)_{i \in \mathbf{N}}$ be a countable star-finite cozero-set cover of S. Thus, for each $i \in \mathbf{N}$, there exists $f_i \in C(S)$ such that $f_i(x) \neq 0$ if and only if $x \in U_i$. By hypothesis S is C-embedded in X, so for each $i \in \mathbf{N}$, there exists $\bar{f}_i \in C(X)$ such that $\bar{f}_i|S = f_i$. Let $V_i = \{x \in X : \bar{f}_i(x) \neq 0\}$. Since S is C-embedded in X and since $X - (\bigcup_{i \in \mathbf{N}} V_i)$ is a zero-set in X that is disjoint from S, there exists a cozero-set V such that $X - (\bigcup_{i \in \mathbf{N}} V_i) \subset V$ and $V \cap S = \emptyset$. Let $W_1 = V \cup V_1$ and $W_i = V_i$ for $i = 2, 3, \ldots$. Since $(W_i)_{i \in \mathbf{N}}$

is a countable cozero-set cover of X that extends $\mathfrak{U}$, 4.6(4) holds, so S is $P^{\aleph_0}$-embedded in X.

4.8. COROLLARY (Arens, 2). *If X is normal and if S is a closed subset of X, then S is $P^{\aleph_0}$-embedded in X.*

Although S being C-embedded in X is not sufficient for S to be P-embedded in X, the next result shows that a special case of this situation does hold.

4.9. THEOREM. *If X is a topological space and if S is a separable subset of X, then the following statements are equivalent:*

(1) *S is P-embedded in X.*

(2) *S is C-embedded in X.*

(3) *S is $P^{\aleph_0}$-embedded in X.*

Proof. That (1) implies (2) is just 3.2; 4.7 states that (2) implies (3); and that (3) implies (1) follows from the fact that a continuous pseudometric on a separable space is $\aleph_0$-separable.

4.10. *Example.* We show that if S is C^*-embedded in X, then S need not be $P^{\aleph_0}$-embedded in X.

Let $X = \beta\mathbf{R} - (\beta\mathbf{N} - \mathbf{N})$ (**11**, 6P) and let $S = \mathbf{N}$. Then S is C^*-embedded in X but S is not C-embedded in X (**11**, 6P.4). Since S is not C-embedded in X and since S is separable, it follows, by 4.9, that S is not $P^{\aleph_0}$-embedded in X.

5. P-embedding and collectionwise normal spaces. In this section we state two theorems that lead us to the speculation that, roughly speaking, P-embedding relates to a collectionwise normal space as C-embedding does to a normal space. Specifically, Theorems 5.2 and 5.4 remain true if we replace "collectionwise normal" by "normal" and "P-embedded" by "C-embedded." In order to prove these theorems we shall first state (Theorem 5.1) several conditions that are necessary and sufficient for a normal space to be collectionwise normal. These characterizations are due to Katětov (**15**).

5.1. THEOREM (Katětov, **15**). *If X is normal, then the following statements are equivalent:*

(1) *X is collectionwise normal.*

(2) *If $(F_\alpha)_{\alpha\in I}$ is a locally finite family of closed subsets of X with finite order, then there exists a locally finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in I$.*

(3) *For every closed $S \subset X$, if $(H_\alpha)_{\alpha\in I}$ is a locally finite family of open subsets of S and if $(F_\alpha)_{\alpha\in I}$ is a family of closed subsets of S such that $F_\alpha \subset H_\alpha$ for each $\alpha \in I$, then there exists a locally finite family $(G_\alpha)_{\alpha\in I}$ of open subsets of X such that $F_\alpha \subset G_\alpha \cap S \subset H_\alpha$ for each $\alpha \in I$.*

(4) *For every closed $S \subset X$, if $(H_\alpha)_{\alpha\in I}$ is a uniformly locally finite open cover of S, then there exists a uniformly locally finite open cover $(G_\alpha)_{\alpha\in I}$ of X such that $G_\alpha \cap S = F_\alpha$ for each $\alpha \in I$.*

(5) *For every closed $S \subset X$, if $(F_\alpha)_{\alpha \in I}$ is a uniformly locally finite family of subsets of S, then $(F_\alpha)_{\alpha \in I}$ is uniformly locally finite in X.*

For the proof of Theorem 5.1, the reader is referred to **(15)**.

5.2. THEOREM. *If X is a topological space, then the following statements are equivalent:*

(1) *X is collectionwise normal.*

(2) *Every closed subset of X is P-embedded in X.*

Proof. (1) *implies* (2). Let S be a closed subset of X. It suffices to show that 2.7(4) holds for all infinite cardinal numbers γ. But this is immediate by 5.1(3).

(2) *implies* (1). Note first that since all closed subsets are P-embedded, they are C-embedded (3.2). Therefore X is normal. To show that X is collectionwise normal, we need only show that 5.1(5) holds. But this is immediate from 2.7(3) and 2.8.

Theorem 5.2 is implicitly proved by C. H. Dowker in **(9)** as is stated in Mathematical Reviews by E. Michael **(19)** and by A. H. Stone **(24)**. However, the above is an explicit proof.

5.3. *Remark.* If X is a topological space and if S is a closed subset of X, then S being C-embedded in X does not imply that S is P-embedded in X. For, let X be a normal topological space that is not collectionwise normal; cf. **(4**, Example G**)**. Then, by 5.2, there exists a closed subset S of X that is not P-embedded in X; and, since S is a closed subset of a normal space, S is C-embedded in X.

5.4. THEOREM. *Suppose that X is a completely regular space, that $S \subset X$, and that $|S|$ is non-measurable. If there exists a collectionwise normal space Y such that $X \subset Y \subset \upsilon X$, then the following statements are equivalent:*

(1) *S is C-embedded in X.*

(2) *$\mathfrak{C}(S)$ is the relative uniform structure on S obtained from $\mathfrak{C}(X)$.*

(3) *S is P-embedded in X.*

Proof. (1) *implies* (2). The proof proceeds exactly as in the proof of the implication "(1) *implies* (2)" of 3.3.

(2) *implies* (3). Let d be a continuous pseudometric on S. By 3.3, there exists a continuous pseudometric $\bar{d}$ on υS such that $\bar{d}|S \times S = d$. Moreover, by **(11**, 15P.4**)**, S is C-embedded in X, and hence, by **(11**, 8.10(a)**)**, $\mathrm{cl}_{\upsilon X} S = \upsilon S$. Since υS is closed in υX, $\upsilon S \cap Y$ is closed in Y. Let $e = \bar{d}|(\upsilon S \cap Y) \times (\upsilon S \cap Y)$. Then e is a continuous pseudometric on $\upsilon S \cap Y$ and therefore, by 5.2, there exists a continuous pseudometric $\bar{e}$ on Y such that $\bar{e}|(\upsilon S \cap Y) \times (\upsilon S \cap Y) = e$. Let $p = \bar{e}|X \times X$. Then p is a continuous pseudometric on X and $p|S \times S = d$. Thus S is P-embedded in X.

(3) *implies* (1). This is just 3.2.

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We shall now show that Theorem 5.4 remains true if we replace "collectionwise normal" by "normal" and "P-embedded" by "C-embedded."

5.5. THEOREM. *Suppose that X is a completely regular space and that $S \subset X$. If there exists a normal space Y such that $X \subset Y \subset \upsilon X$, then the following statements are equivalent:*
(1) *S is C-embedded in X.*
(2) *$\mathfrak{C}(S)$ is the relative uniform structure on S obtained from $\mathfrak{C}(X)$.*

Proof. (1) *implies* (2). The proof, like that of "(1) *implies* (2)" of 5.4, proceeds exactly as in the implication "(1) *implies* (2)" of 3.3.

(2) *implies* (1). Equip S and X with the uniform structures $\mathfrak{C}(S)$ and $\mathfrak{C}(X)$ respectively and let γS and γX denote the completions of $(S, \mathfrak{C}(S))$ and $(X, \mathfrak{C}(X))$ respectively. By (**11**, 15C.3), $\gamma S = \text{cl}_{\gamma X} S$ and, by (**11**, 15.13a), $\gamma S = \upsilon S$ and $\gamma X = \upsilon X$.

Suppose that f is a continuous real-valued function on S. Then f can be extended to a real-valued continuous function $\bar{f}$ on $\upsilon S = \gamma S = \text{cl}_{\gamma X} S$. Then $\bar{f}|\gamma S \cap Y$ is a continuous function on a closed subspace of the normal space Y. Therefore there exists $g \in C(Y)$ such that $g|\gamma S \cap Y = \bar{f}$. Let $h = g|X$. Then $h \in C(X)$ and $h|S = f$. Therefore S is C-embedded in X.

Remark. The following examples show that the requirement that X be a collectionwise normal space is independent of the requirement that υX be collectionwise normal (see the hypothesis of Theorem 5.4). For, if X is the Tychonoff plank (**11**, 8.20), then X is not a collectionwise normal space, but υX is collectionwise normal. Moreover, if X is a Σ-product of an uncountable number of complete metric spaces that are not compact (**6**), then X is a collectionwise normal space, but υX is not collectionwise normal.

We can, however, show

5.6. THEOREM. *Suppose that X is completely regular, that $|X|$ is non-measurable, and that there exists a collectionwise normal space Y such that $X \subset Y \subset \upsilon X$. If X is normal, then X is collectionwise normal.*

Proof. If S is a closed subset of X, then S is C-embedded in X, and hence, by 5.4, S is P-embedded in X. Therefore, by 5.2, X is collectionwise normal.

Our final result shows that P-embedding can be used to give a new proof that a paracompact space is collectionwise normal.

5.7. THEOREM (Bing, **4**). *If X is a paracompact topological space, then X is collectionwise normal.*

Proof. First note that X is normal (**7**, Théorème 1). Now suppose that S is a non-empty closed subset of X and let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ be a locally finite normal cozero-set cover of S. For each $\alpha \in I$, let V_α be an open subset of X such that $V_\alpha \cap S = U_\alpha$. Then $X - \bigcup_{\alpha \in I} V_\alpha$ and S are disjoint closed subsets of the normal space X, so there exists an open subset G of X such that $S \cap G = \emptyset$ and $X - \bigcup_{\alpha \in I} V_\alpha \subset G$. Choose an arbitrary $\beta \in I$ and define $\mathfrak{W} = (W_\alpha)_{\alpha \in I}$ as follows: set $W_\beta = V_\beta \cup G$; and if $\alpha \neq \beta$, let $W_\alpha = V_\alpha$. Then $\mathfrak{W}$ is an open cover of X, so there exists a locally finite open cover $\mathfrak{A}$ of X such that $\mathfrak{A}$ refines $\mathfrak{W}$. Since X is normal, $\mathfrak{A}$ is normal (4.5). Moreover, $\mathfrak{A}|S$ refines $\mathfrak{U}$, so it follows that S is P-embedded in X (2.1 and 2.8). Thus, by 5.2, X is collectionwise normal.

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