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J. ARCHIBALD, Esq., M.A., President, in the Chair.

On the "*e*" Inequality.

By V. RAMASWAMI AIYAR, M.A.

The substance of this paper is contained in *Chrystal*, chap. XXV., §§13, and 15 to 20, with some applications thereof occurring in chap. XXVI. But it is treated here in a fresh manner which would seem simpler on several points. This mode of presentation was, in the start, suggested by Peano's method given by Prof. Gibson in his "Note on the Fundamental Inequality Theorems Connected with e^x and x^m," in Vol. XVIII. of the *Proceedings*.

1. *If x, y, z be any three positive numbers in descending order of magnitude,*

$$x^{y-z}.\,y^{z-x}.\,z^{x-y}<1.$$

Dem.: Applying the arithmetic and geometric means inequality in the form

$$a^p b^q < \left(\frac{pa+qb}{p+q}\right)^{p+q},$$

we have
$$x^{y-z}.\,z^{x-y} < \left(\frac{x(y-z)+z(x-y)}{y-z+x-y}\right)^{y-z+x-y},$$

That is,
$$x^{y-z}.\,z^{x-y} < y^{x-z}. \qquad (1)$$

Hence,
$$x^{y-z}.\,y^{z-x}.\,z^{x-y}<1. \qquad (2)$$

The form (2) is cyclically symmetrical, but for our applications we shall use form (1).

2. *Theorem*: *If $x>y>0$, then $x^{\frac{1}{x-1}}<y^{\frac{1}{y-1}}$.*

This wonderful little inequality appears to require a name; and we shall term it the "*e*" inequality. The name will serve to fix it well in the mind of the young reader. I have readers of *Chrystal* in view. It is supposed in the statement that neither x, nor y, is equal to 1.

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Dem.: First, let $x>y>1$. Then, by inequality (1) of last article,

$$x^{y-1}.\,1^{x-y}<y^{x-1}$$

$$\therefore\quad x^{\frac{1}{x-1}}<y^{\frac{1}{y-1}}\quad - \quad - \quad - \quad - \quad - \quad (a)$$

Secondly, let $x>1>y$. We have in this case

$$x^{1-y}.\,y^{x-1}<1^{x-y}$$

$$\therefore\quad x^{\frac{1}{x-1}}<y^{\frac{1}{y-1}}\quad - \quad - \quad - \quad - \quad - \quad (b)$$

Thirdly, let $1>x>y$. In this case we have

$$1^{x-y}.\,y^{1-x}<x^{1-y}$$

which gives
$$x^{\frac{1}{x-1}}<y^{\frac{1}{y-1}}\quad - \quad - \quad - \quad - \quad - \quad (c)$$

Thus the theorem holds in every case.

3. *Theorem. As x increases from* 0 *to* ∞, *the function* $x^{\frac{1}{x-1}}$ *constantly decreases from* ∞ *to* 1, *and has a definite finite limiting value*, e, *when* $x=1$.

Dem.: The constancy of decrease follows from the inequality of the last article.

When $x=0$, the reciprocal of the function, that is, $x^{\frac{1}{1-x}}$ becomes zero; and the function therefore becomes infinite.

As x increases from 0 to 1, the function decreases in value but remains always greater than $p^{\frac{1}{p-1}}$ where p is any fixed quantity >1 chosen at pleasure; for example, it remains always greater than 2 ($p=2$). Hence $x^{\frac{1}{x-1}}$ approaches a lower limiting value A as x increases to the limit 1.

Also as x increases from 0 to 1, let y be the reciprocal of x. Then y decreases from ∞ to 1; and therefore $y^{\frac{1}{y-1}}$ constantly increases as y tends to its limit 1; but it remains always less than $q^{\frac{1}{q-1}}$ where q is any fixed positive quantity <1 chosen at pleasure; for example, it remains always less than 4 ($q=\frac{1}{2}$). Hence $y^{\frac{1}{y-1}}$ tends to an upper limit B as y decreases to the limit 1.

Now the limits A and B must be equal; for x and y being reciprocals we have, identically,

$$x^{\frac{1}{x-1}} \equiv y \cdot y^{\frac{1}{y-1}}$$

whence $$\underset{x=1-0}{\mathrm{L}} x^{\frac{1}{x-1}} = \underset{y=1+0}{\mathrm{L}} y^{\frac{1}{y-1}}.$$

Thus we have proved that $x^{\frac{1}{x-1}}$ has a definite finite limiting value when $x=1$. This value is denoted by e.

It now remains to show that when x tends to ∞, $x^{\frac{1}{x-1}}$ tends to 1 as its limit.

We observe that $x^{\frac{1}{x-1}}$ is greater than 1 for every value of x greater than 1; for then it is a positive power of a quantity greater than 1. And it decreases as x increases. Hence when $x=\infty$, $x^{\frac{1}{x-1}}$ must have a limiting value l, which is either 1, or some definite quantity greater than 1. We can show that $l=1$ as follows:—

$$\begin{aligned} l &= \mathrm{L} x^{\frac{1}{x-1}} \quad \text{when } x=\infty\,; \\ &= \mathrm{L}\left(x^2\right)^{\frac{1}{x^2-1}}, \quad \text{changing } x \text{ into } x^2\,; \\ &= \mathrm{L}\left(x^{\frac{1}{x-1}}\right)^{\frac{2}{x+1}} \\ &= \left[\mathrm{L}\, x^{\frac{1}{x-1}}\right]^{\mathrm{L}\frac{2}{x+1}} \\ &= l^0 = 1. \end{aligned}$$

The proposition is thus completely proved; and the young reader will do well to fix the graph of $x^{\frac{1}{x-1}}$ unforgettably in his mind. If this be done, all that follows are simple Corollaries.

4. ***Theorem***: *If x is positive and less than* 1, $x^{\frac{1}{x-1}}>e$*; and if* $x>1$, $x^{\frac{1}{x-1}}<e$.

This follows at once from the graph of $x^{\frac{1}{x-1}}$ explained in the last article.

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Cor. 1. $e > 2$.

Cor. 2. If $x > y$, then $e^x > e^y$; and if $x > y > 0$, $\log x > \log y$.
These follow from the fact that $e > 1$.

Cor. 3, 4, 5. If x is positive,

$$(1+x)^{\frac{1}{x}} < e;\quad e^x > 1 + x;\quad x > \log(1+x).$$

Cor. 6, 7, and 8. If x is positive and less than 1,

$$(1-x)^{-\frac{1}{x}} > e;\ e^{-x} > 1 - x;\quad -x > \log(1-x).$$

5. *Theorem*: *If x and y be positive and $x > y$,*

then
$$\frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}.$$

Dem.: Since $\frac{x}{y}$ is greater than 1,

$$\left(\frac{x}{y}\right)^{\frac{1}{\frac{x}{y}-1}} < e$$

$$\therefore\quad \left(\frac{x}{y}\right)^{\frac{1}{x-y}} < e^{\frac{1}{y}}.$$

Taking logarithms, $$\frac{\log x - \log y}{x - y} < \frac{1}{y}. \quad - \quad - \quad - \quad - \quad (1)$$

Similarly, since $\frac{y}{x}$ is positive and less than 1,

$$e^{\frac{1}{x}} < \left(\frac{y}{x}\right)^{\frac{1}{y-x}}$$

that is, $$< \left(\frac{x}{y}\right)^{\frac{1}{x-y}}$$

therefore $$\frac{1}{x} < \frac{\log x - \log y}{x - y}. \quad - \quad - \quad - \quad - \quad (2)$$

Putting (1) and (2) together, we have

$$\frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}.$$

Cor. 1. $$\frac{d}{dx}\log x = \frac{1}{x}.$$

Cor. 2. $$\underset{x=0}{\mathrm{L}} \frac{\log(1+x)}{x} = 1.$$

Cor. 3. For x, y writing e^x, e^y and taking reciprocals, we have, if $x<y$,

$$e^x > \frac{e^x - e^y}{x-y} > e^y.$$

Cor. 4. If $x \neq y$,

$$e^x(x-y) > e^x - e^y > e^y(x-y).$$

This follows from Cor. 3 if $x>y$. But it is readily seen that if the inequality is true with $x>y$, it is also true with $y>x$.

Cor. 5. In the last corollary, for x, y write $x\log a$, $y\log a$ where a is any positive quantity $\neq 1$. And we get

$$a^x\log a\,.\,(x-y) > a^x - a^y > a^y\log a\,.\,(x-y)$$

provided $x \neq y$.

Cor. 6. Hence, if $x>y$

$$a^x\log a > \frac{a^x - a^y}{x-y} > a^y\log a.$$

Cor. 7. $$\frac{d}{dx}a^x = a^x\log a.$$

Cor. 8. $$\underset{x=0}{\mathrm{L}} \frac{a^x - 1}{x} = \log a.$$

6. From the fact that $\underset{x=1}{\mathrm{L}}\ x^{\frac{1}{x-1}} = e$, we get the following corollaries:

(1) $$\underset{x=0}{\mathrm{L}}\ (1+x)^{\frac{1}{x}} = e.$$

(2) $$\underset{n=\infty}{\mathrm{L}} \left(1+\frac{x}{n}\right)^n = e^x.$$

(3) If nu_n tends to the limit a when $n=\infty$,

$$\underset{n=\infty}{\mathrm{L}}\ (1+u_n)^n = e^a.$$

(4) In particular, if $\mathrm{L}nu_n = 0$ when $n=\infty$,

$$\underset{n=\infty}{\mathrm{L}}\ (1+u_n)^n = 1.$$

7. The fact that $\underset{x=\infty}{\mathrm{L}}\ x^{\frac{1}{x-1}}=1$ leads to a number of important limit theorems to which numerous others are reducible.

(1) $\underset{x=\infty}{\mathrm{L}}\ x^{\frac{1}{x}}=1$. (Standard theorem for the form ∞^0.)

(2) $\underset{x=+0}{\mathrm{L}}\ x^x=1$. (Standard theorem for the form 0^0.) This is deducible from (1) by writing $\dfrac{1}{x}$ for x.

(3) $\underset{x=\infty}{\mathrm{L}}\ \dfrac{\log x}{x}=0$, which is deduced from (1) by taking logarithms. This is a standard theorem, explaining the logarithmic scale.

(4) $\underset{x=\infty}{\mathrm{L}}\ \dfrac{e^x}{x}=\infty$. This follows from (3) by writing e^x for x.

(5) $\underset{x=+0}{\mathrm{L}}\ x\log x=0$. This follows from (2) by taking logarithms.

The Higher Logarithmic Inequalities.

8. Let lx, l^2x, ... denote $\log x$, $\log\log x$, ... Then, if x and y be positive and $x>y$, and y sufficiently great to render l^ry positive,

$$\frac{1}{x\,.\,lx\,.\,l^2x\ldots l^rx}<\frac{l^{r+1}x-l^{r+1}y}{x-y}<\frac{1}{y\,.\,ly\,.\,l^2y\ldots l^ry}.$$

Proof: We have, by Art. 5,

$$\begin{aligned}\frac{1}{x}&<\frac{lx-ly}{x-y}&&<\frac{1}{y}\\ \frac{1}{lx}&<\frac{l^2x-l^2y}{lx-ly}&&<\frac{1}{ly}\\ \frac{1}{l^rx}&<\frac{l^{r+1}x-l^{r+1}y}{l^rx-l^ry}&&<\frac{1}{l^ry}.\end{aligned}$$

Multiplying all these together we have the result given

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9. If x and y be positive and $x > y$, and y sufficiently great to render $l^r y$ positive, and β be any positive quantity

$$\frac{\beta}{x \cdot lx \ldots l^{r-1}x(l^r x)^{1+\beta}} < \frac{(l^r y)^{-\beta} - (l^r x)^{-\beta}}{x - y} < \frac{\beta}{y \cdot ly \ldots l^{r-1}y \cdot (l^r y)^{1+\beta}}.$$

Proof: By the power inequality we have if $X > Y > 0$

$$-\beta X^{-\beta-1} > \frac{X^{-\beta} - Y^{-\beta}}{X - Y} > -\beta \cdot Y^{-\beta-1}.$$

Multiplying this by -1, we have

$$\frac{\beta}{X^{1+\beta}} < \frac{Y^{-\beta} - X^{-\beta}}{X - Y} < \frac{\beta}{Y^{1+\beta}}.$$

Now for X, Y write $l^r x$, $l^r y$; and we have

$$\frac{\beta}{(l^r x)^{1+\beta}} < \frac{(l^r y)^{-\beta} - (l^r x)^{-\beta}}{l^r x - l^r y} < \frac{\beta}{(l^r y)^{1+\beta}}. \qquad \cdot \qquad \cdot \qquad (1)$$

And by the theorem of the last article,

$$\frac{1}{x \cdot lx \ldots l^{r-1}x} < \frac{l^r x - l^r y}{x - y} < \frac{1}{y \cdot ly \ldots l^{r-1}y}. \qquad \cdot \qquad \cdot \qquad (2)$$

Multiplying (1) and (2) we have the theorem given.

Application to Infinite Series.

10. By the power inequality, which, it may be remarked, is an equivalent of the "e" inequality, we have, if β be any positive quantity,

$$\frac{\beta}{(n+1)^{1+\beta}} < n^{-\beta} - (n+1)^{-\beta} < \frac{\beta}{n^{1+\beta}}.$$

From this we can readily deduce that the series $\sum_1^\infty \frac{1}{n^{1+\beta}}$ is convergent, $(\beta > 0)$.

The power inequality also gives, if β be positive and less than 1,

$$\frac{\beta}{(n+1)^{1-\beta}} < (n+1)^\beta - n^\beta < \frac{\beta}{n^{1-\beta}}.$$

From this it is deducible that the series $\sum_1^\infty \frac{1}{n^{1-\beta}}$ is divergent, (β positive and < 1).

If β be not less than 1, the divergency of $\sum_1^\infty \frac{1}{n^{1-\beta}}$ is at once apparent.

The divergency of $\sum_1^\infty \frac{1}{n}$ can be similarly deduced from the logarithmic inequality of Art. 5.

It can be similarly proved from the inequality of Art. 8, that the series $\sum_p^\infty \frac{1}{n\,.\,ln\,.\,l^2n\,\ldots\,l^rn}$ is divergent.

It follows that the series

$$\sum_p^\infty \frac{1}{n\,.\,ln\,\ldots\,l^{r-1}n(l^rn)^{1-\beta}},$$

where β is any positive quantity is divergent.

The inequality of Art. 9 similarly gives us that the series

$$\sum_p^\infty \frac{1}{n\,.\,ln\,\ldots\,l^{r-1}n(l^rn)^{1+\beta}}$$

is convergent, if β is any positive quantity.

11. These are the standard theorems for comparison in determining the absolute convergence, or otherwise, of series whose ratio of convergence tends to the limit 1. They are here shown to be deducible without Cauchy's Condensation Test. By the method here indicated we could, further, assign upper and lower limits to the remainder after n terms in all cases where the series is convergent. Even where the series diverges, we derive valuable information regarding the nature of the divergency. Thus, in the case of $\Sigma \frac{1}{n}$ we can see by the inequality of Art. 5, that

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n$$

tends to a definite finite limit, the Eulerian Constant γ, when n becomes infinite.

Equivalents of the "e" Inequality.

12. We remarked that the power inequality was an equivalent of the "*e*" inequality. More fully, it is worth noting that the following *six* inequalities are equivalent to one another.

I. $$\frac{pa+qb}{p+q} > (a^p b^q)^{\frac{1}{p+q}}$$

if a, b, p, q are positive and $a \neq b$.

II. $$a^x(y-z) + a^y(z-x) + a^z(x-y) > 0$$

if x, y, z are in descending order of magnitude and a is any positive quantity $\neq 1$.

III. $$\frac{a^x - 1}{x} > \frac{a^y - 1}{y},$$

if $x > y$ and a is any positive quantity $\neq 1$.

II′. If $$x > y > z > 0,$$

then $$x^{y-z} . y^{z-x} . z^{x-y} < 1.$$

III′. The "e" inequality.

IV. The power inequality :—If a is positive and not equal to 1.

$$a^m - 1 \gtreqless m(a-1) \text{ according as } m(m-1) \gtreqless 0.$$

Arranging these six inequalities in circular order, we can from each inequality deduce the one following it, proceeding in the clockwise order, or counter clockwise order, at pleasure. I shall leave the details to the reader; but shall here indicate how the power inequality can be deduced from the "e" inequality, by proving IV. in one case, that is, when $a > 1$, and $m > 1$. In this case we have

$$a < 1 + m(a-1).$$

Consequently, by the "e" inequality

$$a^{\frac{1}{a-1}} > [1 + m(a-1)]^{\frac{1}{m(a-1)}}.$$

Raising both sides to the power $m(a-1)$ which is positive, and transposing 1,

$$a^m - 1 > m(a-1).$$

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