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A comprehensive perturbation theorem for estimating magnitudes of roots of polynomials

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Abstract

A comprehensive new perturbation theorem is posed and proven to estimate the magnitudes of roots of polynomials. The theorem successfully determines the magnitudes of roots for arbitrary degree of polynomial equations with no restrictions on the coefficients. In the previous papers 'Pakdemirli and Elmas, *Appl. Math. Comput.* 216 (2010) 1645–1651' and 'Pakdemirli and Yurtsever, *Appl. Math. Comput.* 188 (2007) 2025–2028', the given theorems were valid only for some restricted coefficients. The given theorem in this work is a generalization and unification of the past theorems and valid for arbitrary coefficients. Numerical applications of the theorem are presented as examples. It is shown that the theorem produces good estimates for the magnitudes of roots of polynomial equations of arbitrary order and unrestricted coefficients.

1. *Introduction*

Concepts of perturbation theory [**2, 4**] have been successfully applied to polynomial equations to estimate the magnitudes of roots without solving the equations exactly. Pakdemirli and Yurtsever [**9**] presented two theorems to estimate roots of polynomials with arbitrary degrees. In the first theorem, it is proven that the magnitudes of roots are always order 1 ($O(1)$) if all coefficients are of the same order of magnitude. The second theorem is for a polynomial equation having one relatively large coefficient with all other coefficients being of $O(1)$. Pakdemirli and Elmas [**8**] posed two additional theorems for estimating magnitudes of roots. One of the theorems is for a polynomial equation with one relatively small coefficient with all other coefficients being of the same order of magnitude. The other theorem determines the magnitudes of roots for a polynomial equation with two relatively large coefficients.

As can be seen from the past work, all theorems posed are valid under some restrictions of the coefficients. A unification and generalization of the past theorems is achieved in this study and the new theorem is valid for arbitrary coefficients. The theorem is first proven and then tested via numerical examples. It is found that the theorem predicts well the magnitudes of roots approximately. This theorem can be integrated to the root finding algorithms as an initial step since those algorithms need a good initial guess for convergence to a root.

In a related group of study, the link between perturbation theorems and root finding algorithms were also exploited recently [**5–7**]. Using perturbation theory and Taylor series expansions, well known root finding algorithms as well as new ones were systematically derived. A discussion of root finding algorithms and review of the vast literature published is beyond the scope of this work. For preliminaries of root finding algorithms, see [**1, 3, 10**] for example. The algorithm developed in this work may be integrated to the root finding algorithms presented in [**1, 3, 5–7, 10**].

2. *Previous theorems*

In perturbation theory, the magnitudes of terms are ordered with respect to a small parameter usually expressed as ε, ε being a much smaller quantity than 1 ($\varepsilon \ll 1$). Therefore a term

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of order $1/\varepsilon$, denoted by $O(1/\varepsilon)$ is much bigger than 1. Depending on the magnitudes of coefficients of polynomials, four special theorems were posed and proven previously [**8, 9**].

2.1. *Polynomial with all coefficients the same order of magnitude*

The theorem below was given in [**9**].

THEOREM 2.1. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 = 0 \tag{2.1}$$

if all coefficients a_i $(i = 0 \ldots n)$ are of the same order of magnitude, then the magnitudes of roots are of $O(1)$.

Proof. See [**9**] for details. □

2.2. *Polynomial with one relatively large coefficient*

For a polynomial equation in which one coefficient is substantially larger than the others with all the remaining coefficients being of order 1, the theorem below was given in [**9**].

THEOREM 2.2. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_m x^m + \ldots + a_1 x + a_0 = 0 \tag{2.2}$$

if $a_m \sim O(1/\varepsilon^k)$ $(k > 0)$ with all other coefficients being of $O(1)$ then the possible roots may be of either $O(\varepsilon^{k/m})$ $(m \neq 0$ case$)$ or $O(1/\varepsilon^{k/(n-m)})$ $(m \neq n$ case$)$.

Proof. See [**9**] for details. □

2.3. *Polynomial with one relatively small coefficient*

If one of the coefficients of a polynomial equation is much smaller than the others which are of the same order, the following theorem is stated.

THEOREM 2.3. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_m x^m + \ldots + a_1 x + a_0 = 0 \tag{2.3}$$

if all coefficients are $O(1)$ except a_m which is much smaller, that is, $a_m \sim O(\varepsilon^k)$, $\varepsilon \ll 1$, $k > 0$ then the magnitudes of roots are:
(i) *$x \sim O(1)$ for m arbitrary;*
(ii) *$x \sim O(1/\varepsilon^k)$ if $m = n$;*
(iii) *$x \sim O(\varepsilon^k)$ if $m = 0$.*

Proof. See [**8**] for details. □

2.4. *Polynomial with two relatively large coefficients*

If two of the coefficients of a polynomial equation are much bigger than the others which are of the same order, the following theorem is stated.

THEOREM 2.4. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_m x^m + \ldots + a_p x^p + \ldots + a_1 x + a_0 = 0 \tag{2.4}$$

if $a_m \sim O(1/\varepsilon^{k_1})$, $a_p \sim O(1/\varepsilon^{k_2})$, $\varepsilon \ll 1$, $m > p$, $k_1 > 0$, $k_2 > 0$ and all remaining coefficients are $O(1)$, then the magnitudes of roots are:

(i) $x \sim O(1)$ *if* $k_1 = k_2$*;*
(ii) $x \sim O(\varepsilon^{k_2/p})$ *if* $k_2/p \geqslant (k_1 - k_2)/(m-p),\ a_0 \neq 0,\ p \neq 0$*;*
(iii) $x \sim O(\varepsilon^{k_1/m})$ *if* $k_1/m \leqslant (k_1 - k_2)/(m-p),\ a_0 \neq 0,\ m \neq 0$*;*
(iv) $x \sim O(\varepsilon^{(k_1-k_2)/(m-p)})$ *if* $k_1/m > (k_1 - k_2)/(m-p),\ \ k_2/p > (k_1-k_2)/(m-p),\ k_1 > k_2$*;*
(v) $x \sim O(1/\varepsilon^{k_1/(n-m)})$ *if* $k_1/(n-m) \geqslant (k_2 - k_1)/(m-p),\ m \neq n$*;*
(vi) $x \sim O(1/\varepsilon^{k_2/(n-p)})$ *if* $k_2/(n-p) \leqslant (k_2 - k_1)/(m-p),\ \ p \neq n$*;*
(vii) $x \sim O(1/\varepsilon^{(k_2-k_1)/(m-p)})$ *if* $k_1/(n-m) > (k_2 - k_1)/(m-p)$, $k_2/(n-p) > (k_2-k_1)/(m-p),\ k_2 > k_1$.

Proof. See [8] for details. □

3. *A comprehensive theorem*

Theorems previously developed all contain some restrictions on the coefficients. A theorem which can be applied to any arbitrary polynomial equation is needed for unification of the results.

THEOREM 3.1. *For the polynomial equation*

$$x^n + a_{n-1}x^{n-1} + \ldots + a_m x^m + \ldots + a_p x^p + \ldots + a_1 x + a_0 = 0 \tag{3.1}$$

assume $a_k \sim O(1/\varepsilon^{m_k})$ *where* m_k *may be a positive or a negative real number (note that* $m_n = 0$*). Then the magnitudes of roots are:*
(i) $x \sim O(1)$ *if at least two* m_k *and* m_p *are equal and positive with all remaining* $m_i < m_k$*;*
(ii) $x \sim O(1)$ *if at least one* $m_k = 0$ *and all other* $m_i < 0$*;*
(iii) $x \sim O(\varepsilon^r)$ $r = (m_k - m_p)/(k-p)$ *where the kth and pth terms are selected such that* $rk - m_k = rp - m_p$ *and* $rk - m_k$*,* $rp - m_p$ *are smallest compared to any other* $ri - m_i$*.*

Proof. Two distinct cases for magnitudes of roots should be investigated separately. Roots may be either of $O(1)$ or $O(\varepsilon^r)$ where r may be positive or negative. The steps of proof will also outline the algorithm for such problems.

(i) The case when $x \sim O(1)$. If x is of order 1, then each term in equation (3.1) can be written in its order of magnitude

$$O(1) + O(1/\varepsilon^{m_{n-1}})O(1) + O(1/\varepsilon^{m_{n-2}})O(1) + \ldots + O(1/\varepsilon^{m_1})O(1) + O(1/\varepsilon^{m_0}) = 0.$$

If all m_i are negative, all terms are small compared to the leading term and hence balancing of the leading term is impossible and the root cannot be $O(1)$. If only one of the m_i is positive, again this term cannot be balanced. If some of the $m_i > 0$ and no $m_k = m_p$ for arbitrary k and p, balancing of the terms is again impossible. If at least for two terms $m_k = m_p$ where $m_k > 0$, $m_p > 0$ and all other m_i are smaller than m_k, balancing is always possible and we have a root of $O(1)$. This is *case* (i) in the theorem. If at least one $m_k = 0$ $(k \neq n)$, and all remaining terms $m_i < 0$, then we can balance the leading term with the kth term or terms, all other terms being neglected compared to the balanced terms. This is *case* (ii) in the theorem. Note that theorem 2.1 is a very special case where after dividing the equation with the leading coefficient, all coefficients are $O(1)$ and hence all $m_i = 0$.

(ii) The case when $x \sim O(\varepsilon^r)$. If the root is not $O(1)$, then it may be a large $(r < 0)$ or small $(r > 0)$ quantity. One then substitutes for magnitude of the root as $O(\varepsilon^r)$ to equation (3.1), the order of terms can be written as

$$O(\varepsilon^{r\,n}) + O(\varepsilon^{r(n-1)-m_{n-1}}) + \ldots + O(\varepsilon^{ri-m_i}) + \ldots + O(\varepsilon^{r-m_1}) + O(\varepsilon^{-m_0}) = 0.$$

Balancing needs at least two large terms compared to other terms. Assume that for any arbitrary k and p terms, $rk - m_k$ is equal to $rp - m_p$ and $rk - m_k = rp - m_p$ is smaller than

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all remaining $ri - m_i$, then they are the largest terms and others are neglected compared to these terms. Solving r from the equality yields $r = (m_k - m_p)/(k - p)$. It may happen that more than two terms hold a similar equality, but this would not change the result. Hence, the root is of $O(\varepsilon^r)$ where r may be a negative or positive real number ($r = (m_k - m_p)/(k - p)$). This is *case* (iii) in the theorem and hence the proof is completed

□

4. *Numerical examples*

In this section several numerical examples for Theorem 3.1 will be given. The algorithm starts with calculating the orders of magnitudes of coefficients. If

$$a_k = \frac{1}{\varepsilon^{m_k}} \tag{4.1}$$

then solving m_k yields

$$m_k = \frac{\ln |1/a_k|}{\ln \varepsilon}. \tag{4.2}$$

The small parameter ε is assumed to be 0.1. Other numerical values can be assigned as long as $\varepsilon \ll 1$. The same numerical value of ε should be used for consistency. For $\varepsilon = 0.1$ a term of order $1/\varepsilon$ is expected to be near 10 and that of order $1/\varepsilon^2$ to be near 100. There are intermediate orderings also such as $\sqrt{\varepsilon} \approx 0.3162$ or $1/\sqrt{\varepsilon} \approx 3.162$. Note that the theorems are universal and independent of the selection of the parameter ε as long as this parameter is much smaller than 1.

Theorem 3.1 will be outlined in two worked examples in detail and then the final results will be given for many other polynomials in Table 1.

4.1. *Sample problem 1*

Consider the polynomial equation

$$x^3 - 305x^2 + 200x - 1 = 0.$$

First, from the coefficients, m_k are calculated using equation (4.2)

$$m_0 = 0, \quad m_1 \cong 2.3, \quad m_2 \cong 2.5, \quad m_3 = 0.$$

Note that in calculating m_k, only one decimal is retained for simplicity. The first and second cases of Theorem 3.1 are not met for this specific problem, so one concludes that an $O(1)$ root is impossible. Then one seeks an $O(\varepsilon^r)$ root. The orders of each term are

$$O(\varepsilon^{3r}) + O(\varepsilon^{2r-2.5}) + O(\varepsilon^{r-2.3}) + O(1) = 0.$$

Balancing the first and second terms yields $r = -2.5$ for which the above ordering becomes

$$O(\varepsilon^{-7.5}) + O(\varepsilon^{-7.5}) + O(\varepsilon^{-4.8}) + O(1) = 0.$$

The first two terms can be balanced with all other terms neglected, hence this choice is an admissible choice. Balancing the first and third term yields $r = -1.15$ for which the orders of terms are

$$O(\varepsilon^{-3.45}) + O(\varepsilon^{-4.8}) + O(\varepsilon^{-3.45}) + O(1) = 0.$$

The second term is the largest which cannot be balanced with others, so this choice is discarded. Balancing the first and last term yields $r = 0$ which makes the second term largest and this term cannot be balanced with others, so this case is also discarded. Now balancing the second and third term yields $r = 0.2$ for which the orders of terms are

$$O(\varepsilon^{0.6}) + O(\varepsilon^{-2.1}) + O(\varepsilon^{-2.1}) + O(1) = 0.$$

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Neglecting the first and last terms, the middle terms can be balanced hence this is an admissible choice. Balancing the second and last term yields $r = 1.25$ for which

$$O(\varepsilon^{3.75}) + O(1) + O(\varepsilon^{-1.05}) + O(1) = 0.$$

The largest term is the third term which cannot be balanced with other terms and this case is also discarded. Finally balancing the third and last term yields $r = 2.3$ for which

$$O(\varepsilon^{6.9}) + O(\varepsilon^{2.1}) + O(1) + O(1) = 0.$$

The largest terms are the last two terms which can be balanced, hence this is also an admissible choice. In conclusion, one expects three roots of orders $O(1/\varepsilon^{2.5})$, $O(\varepsilon^{0.2})$, $O(\varepsilon^{2.3})$. For $\varepsilon = 0.1$, the magnitudes of roots are estimated to be 316.227, 0.63 and 0.0050. On the other hand, the real roots are 304.3429, 0.6521 and 0.0050. Hence, a good estimate of the magnitudes can be achieved using the theorem.

4.2. *Sample problem 2*

Consider the polynomial equation

$$x^3 + 0.1x^2 - 106x + 0.01 = 0.$$

First, from the coefficients, m_k are calculated using equation (4.2)

$$m_0 = -2.0, \quad m_1 = 2.0, \quad m_2 = -1.0.$$

Note that in calculating m_k, only one decimal is retained for simplicity. The first and second cases of Theorem 3.1 are not met for this specific problem, so one concludes that an $O(1)$ root is impossible. Then one seeks an $O(\varepsilon^r)$ root. The orders of each term are

$$O(\varepsilon^{3r}) + O(\varepsilon^{2r+1}) + O(\varepsilon^{r-2}) + O(\varepsilon^2) = 0.$$

Balancing the first and second terms yields $r = 1$ for which the above ordering becomes

$$O(\varepsilon^3) + O(\varepsilon^3) + O(\varepsilon^{-1}) + O(\varepsilon^2) = 0.$$

The third term is the largest which cannot be balanced with others, so this choice is discarded. Balancing the first and third terms yields $r = -1$ for which the orders of terms are

$$O(\varepsilon^{-3}) + O(\varepsilon^{-1}) + O(\varepsilon^{-3}) + O(\varepsilon^2) = 0.$$

The first and third terms can be balanced with all other terms neglected, hence this choice is an admissible choice. Balancing the first and the last term yields $r = 2/3$ for which the orders of terms are

$$O(\varepsilon^2) + O(\varepsilon^{7/3}) + O(\varepsilon^{-4/3}) + O(\varepsilon^2) = 0.$$

The third term is the largest which cannot be balanced with others, so this choice is discarded. Balancing the second and third term yields $r = -3$ for which the orders of terms are

$$O(\varepsilon^{-9}) + O(\varepsilon^{-5}) + O(\varepsilon^{-5}) + O(\varepsilon^2) = 0.$$

The first term is the largest which cannot be balanced with others, so this choice is discarded. Balancing the second and the last term yields $r = 1/2$ for which the orders of terms are

$$O(\varepsilon^{3/2}) + O(\varepsilon^2) + O(\varepsilon^{-3/2}) + O(\varepsilon^2) = 0.$$

The third term is the largest which cannot be balanced with others, so this choice is discarded. Balancing the third and the last term yields $r = 4$ for which the orders of terms are

$$O(\varepsilon^{12}) + O(\varepsilon^9) + O(\varepsilon^2) + O(\varepsilon^2) = 0.$$

TABLE 1. *Comparison of magnitudes of roots and estimated magnitudes of roots by Theorem 3.1* ($\varepsilon = 0.1$).

Polynomial equation	Roots	Magnitudes of roots	Estimated magnitudes of roots
$x^3 - 305x^2 + 200x - 1 = 0$	304.3429	304.3429	316.227
	0.6521	0.6521	0.63
	0.0050	0.0050	0.0050
$x^3 + 0.1x^2 - 106x + 0.01 = 0$	−10.3458	10.3458	10
	10.2457	10.2457	0.0001
	0.0001	0.0001	
$x^3 - 0.02x^2 + 0.4x - 0.8 = 0$	−0.3858 + 0.9284i	1.0054	0.9261
	−0.3858 − 0.9284i	1.0054	
	0.7915	0.7915	
$x^3 + 0.07x^2 - 0.001x + 0.2 = 0$	−0.6027	0.6027	0.5847
	0.2698 + 0.5052i	0.5727	
	0.2698 − 0.5052i	0.5727	
$x^5 + 50x^4 + 200x^3 - 100x^2 + 120x + 100 = 0$	−45.5608	45.5608	48.977
	−4.9780	4.9780	3.98
	−0.4796	0.4796	0.7984
	0.5092 + 0.8124i	0.9587	
	0.5092 − 0.8124i	0.9587	
$x^5 + 0.1x^4 - 200x^3 + 0.01x^2 + 0.05x + 0.2 = 0$	−14.1922	14.1922	14.125
	14.0922	14.0922	0.1
	0.1009	0.1009	
	−0.0504 + 0.859i	0.8605	
	−0.0504 − 0.859i	0.8605	
$x^5 + 0.08x^4 - 0.001x^3 + 0.01x^2 + 0.02x + 0.5 = 0$	−0.8839	0.8839	0.87
	−0.2879 + 0.8220i	0.8709	
	−0.2879 − 0.8220i	0.8709	
	0.6899 + 0.5193i	0.8635	
	0.6899 − 0.5193i	0.8635	
$x^8 + 11x^7 - 5x^6 - 0.001x^5 + 1062x^4 + 10x^3 + 0.8x^2 - 0.9x + 3.4 = 0$	−10.5757	10.5757	10
	−5.4843	5.4843	4.6415
	2.5347 + 3.4476i	4.2791	0.237
	2.5347 − 3.4476i	4.2791	
	−0.1700 + 0.1725i	0.2422	
	−0.1700 − 0.1725i	0.2422	
	0.1654 + 0.1650i	0.2336	
	0.1654 − 0.1650i	0.2336	
$x^9 + 1040x^8 - 8x^7 + 0.01x^6 - 102x^5 + 25x^4 - 0.2x^3 + 11x^2 + 2x - 0.0001 = 0$	−1040	1040	1000
	0.5 + 0.2i	0.5385	0.4641
	0.5 − 0.2i	0.5385	0.1820
	−0.4 + 0.4i	0.5656	0.1925
	−0.4 − 0.4i	0.5656	0
	0.4i	0.4	
	−0.4i	0.4	
	−0.2	0.2	
	0	0	
$x^9 - 0.5x^8 + 1224x^6 - 80x^5 - 5x^4 + 0.0003x^2 + 120 = 0$	5.4947 + 9.2615i	10.7688	10.7895
	5.4947 − 9.2615i	10.7688	0.6823
	−10.5548	10.5548	
	0.6003 + 0.3387i	0.6892	
	0.6003 − 0.3387i	0.6892	
	0.0108 + 0.6776i	0.6776	
	0.0108 − 0.6776i	0.6776	
	−0.5784 + 0.3389i	0.6704	
	−0.5784 − 0.3389i	0.6704	

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TABLE 1. (*Continued.*)

Polynomial equation	Roots	Magnitudes of roots	Estimated magnitudes of roots
$x^8 + 4428x^7 - 42x^6 + 0.05x^4$	−4428	4428	5011.87
$- 2x^3 + 0.0002x - 100 = 0$	0.6	0.6	0.5717
	0.4 + 0.5i	0.64	
	0.4 − 0.5i	0.64	
	−0.1 + 0.6i	0.608	
	−0.1 − 0.6i	0.608	
	−0.5 + 0.3i	0.583	
	−0.5 − 0.3i	0.583	
$x^6 + 85x^5 - 0.003x^4 - 1000x^3$	−84.8612	84.8612	79.43
$+ 8x^2 + x - 0.1 = 0$	−3.5068	3.5068	3.5481
	3.36	3.36	0.0464
	−0.0507	0.0507	
	0.0293 + 0.0334i	0.0444	
	0.0293 − 0.0334i	0.0444	
$x^6 + 0.002x^5 - 1025x^3$	10.0683	10.0683	10
$+ 42x^2 + 3 = 0$	−5.0556 + 8.7319i	10.089	0.1469
	−5.0556 − 8.7319i	10.089	
	0.1581	0.1581	
	−0.0586 + 0.1228i	0.136	
	−0.0586 − 0.1228i	0.136	
$x^7 + 81x^6 - 886x^5 + 0.02x^4$	−90.7487	90.7487	79.4328
$+ 10842x^3 - 100x^2 + 47x$	8.0861	8.0861	10
$- 0.0001 = 0$	4.7591	4.7591	3.1623
	−3.1057	3.1057	0.0708
	0.0046 + 0.0657i	0.06586	0
	0.0046 − 0.0657i	0.06586	
	0	0	
$x^7 + 42x^5 - 8x^4 + 0.002x^3$	0.1841 + 6.5045i	6.5071	6.31
$- 1002x^2 + x - 0.3 = 0$	0.1841 − 6.5045i	6.5071	2.9282
	−1.5726 + 2.46i	2.9197	2.8183
	−1.5726 − 2.46i	2.9197	0.0178
	2.776	2.776	
	0.0005 + 0.0173i	0.0173	
	0.0005 − 0.0173i	0.0173	
$x^7 + 105x^6 - 0.001x^5 + 0.2x^4$	−105	105	104.6574
$+ 2x^3 + 0.8x^2 + 20x - 5 = 0$	−0.76	0.76	0.7244
	−0.26 + 0.69i	0.7374	0.2512
	−0.26 − 0.69i	0.7347	
	0.52 + 0.44i	0.6812	
	0.52 − 0.44i	0.6812	
	0.24	0.24	
$x^6 + 412x^5 - 0.1x^4 + 0.005x^3$	−412	412	412.0026
$+ 8x^2 - 55x - 7 = 0$	0.62	0.62	0.6045
	0.04 + 0.61i	0.6113	0.1273
	0.04 − 0.61i	0.6113	
	−0.58	0.58	
	−0.13	0.13	

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Neglecting the first and second terms, the last two terms can be balanced hence this is an admissible choice. In conclusion, one expects three roots of orders $O(1/\varepsilon)$, $O(\varepsilon^4)$. For $\varepsilon = 0.1$, the magnitudes of roots are estimated to be 10, 0.0001. On the other hand, the real roots are −10.3458, 10.2457, 0.0001. Hence, a good estimate of the magnitudes can be achieved using the theorem.

In Table 1, many worked examples are given. As can be seen, the equations are of arbitrary order with no restrictions on the coefficients. From the worked examples, one may conclude that Theorem 3.1 estimates reasonably well the magnitudes of roots of polynomials of arbitrary order and coefficients.

5. *Concluding remarks*

Based on this work and on the previous work [**8, 9**], the following conclusions can be made.

(1) Present analysis (Theorem 3.1) can successfully estimate magnitudes of roots for arbitrary orders and coefficients of the polynomials.

(2) Previous theorems (Theorems 2.1–2.4 given in [**8, 9**]) are all special cases of Theorem 3.1.

(3) Results of Theorem 3.1 can be used as initial guesses for root finding algorithms, since it is well know that if the initial estimate is not good enough, algorithms may diverge from the root.

(4) Theorems may be employed for rough checking of the outputs of numerical results.

(5) Rough estimates are possible even just by inspecting the coefficients of the polynomial equation.

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