JOINS AND DIRECT PRODUCTS OF EQUATIONAL CLASSES

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Let K_0 and K_1 be equational classes of algebras of the same type[2]. The smallest equational class K containing K_0 and K_1 is the join of K_0 and K_1; in notation, $K = K_0 \vee K_1$. The direct product $K_0 \times K_1$ is the class of all algebras $\mathfrak{A}$ which are isomorphic to an algebra of the form $\mathfrak{A}_0 \times \mathfrak{A}_1$, $\mathfrak{A}_0 \in K_0$, $\mathfrak{A}_1 \in K_1$. Naturally, $K_0 \times K_1 \subseteq K_0 \vee K_1$. Our first theorem states a very simple condition under which $K_0 \times K_1 = K_0 \vee K_1$, and an additional condition under which the representation $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1$ is unique.

Let us call K_0 and K_1 independent if there exists a binary polynomial symbol $\underline{p}$ such that the identity $\underline{p} = x_i$ holds in K_i, $i = 0, 1$.

THEOREM 1. Let K_0 and K_1 be independent. Then $K_0 \times K_1 = K_0 \vee K_1$. If, in addition, each algebra $\mathfrak{A} \in K_0 \vee K_1$ has a modular congruence lattice, then each $\mathfrak{A} \in K_0 \vee K_1$ has, up to isomorphism, a unique representation $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1$, $\mathfrak{A}_0 \in K_0$, $\mathfrak{A}_1 \in K_1$.

Remark. Many special cases of this theorem can be found in the literature; for example, see A.L. Foster [4] and A. Astromoff [1]; a special case of the first statement of this theorem was observed independently by P. Kelenson [7].

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2. For the concepts and notations see [5].

As an illustration of independence, we present an example quite different from those in the literature. The equational classes K_0 and K_1 are of type $\langle 2, 2\rangle$. Let K_0 consist of all algebras $\langle G; f_0, f_1\rangle$ where G is a group, $f_0(x, y) = xy$, and $f_1(x, y) = xy^{-1}$. Let K_1 consist of all algebras $\langle L; f_0, f_1\rangle$ where L is a lattice, $f_0(x, y) = x \vee y$, and $f_1(x, y) = x \wedge y$. The polynomial symbol $\underline{p} = \underline{f}_1(\underline{f}_0(\underline{x}_0, \underline{x}_1), \underline{x}_1)$ establishes the independence of K_0 and K_1.

Proof of Theorem 1. Let $\mathfrak{A} \in K_0 \vee K_1$, and let Θ_i denote the smallest congruence relation on $\mathfrak{A}$ such that $\mathfrak{A}/\Theta_i \in K_i$, $i = 0, 1$. Then $\mathfrak{A}/\Theta_0 \vee \Theta_1 \in K_0 \wedge K_1$, and so satisfies $\underline{x}_0 = \underline{p} = \underline{x}_1$; hence $\Theta_0 \vee \Theta_1 = \iota$.

We claim that[3] $a_0 \equiv a_1(\Theta_0)$ if and only if $p(a_0, a_1) = a_1$. Indeed, if $p(a_0, a_1) = a_1$ then $[a_0]\Theta_0 = p([a_0]\Theta_0, [a_1]\Theta_0) = [a_1]\Theta_0$; hence $a_0 \equiv a_1(\Theta_0)$. Let Φ_0 be the relation defined by $a_0 \equiv a_1(\phi_0)$ if and only if $p(a_0, a_1) = a_1$. To show that $\Theta_0 = \phi_0$ it suffices to show that ϕ_0 is a congruence relation. Reflexivity, symmetry, transitivity, and the substitution property for the operation f follow from the identities:

$$\begin{aligned}
\underline{p}(\underline{x}, \underline{x}) &= \underline{x}, \\
\underline{p}(\underline{p}(\underline{x}, \underline{y}), \underline{x}) &= \underline{x}, \\
\underline{p}(\underline{x}, \underline{p}(\underline{y}, \underline{z})) &= \underline{p}(\underline{p}(\underline{x}, \underline{y}), \underline{z}), \\
\underline{p}(\underline{f}(\underline{x}_0, \underline{x}_1, \ldots), \underline{f}(\underline{y}_0, \underline{y}_1, \ldots)) &= \underline{f}(\underline{p}(\underline{x}_0, \underline{y}_0), \underline{p}(\underline{x}_1, \underline{y}_1), \ldots).
\end{aligned}$$

Since these identities clearly hold in K_0 and K_1, they hold in $K_0 \vee K_1$; thus $\Theta_0 = \Phi_0$. Similarly, $a_0 \equiv a_1(\Theta_1)$ if and only if $p(a_0, a_1) = a_0$.

Consequently, if $a_0 \equiv a_1(\Theta_0 \wedge \Theta_1)$ then $a_0 \equiv a_1(\Theta_i)$; hence $p(a_0, a_1) = a_i$, and so $a_0 = a_1$, establishing $\Theta_0 \wedge \Theta_1 = \omega$. Now let $a \equiv b(\Theta_0)$, $b \equiv c(\Theta_1)$; then $a \equiv p(c, a)\ (\Theta_1)$, $p(c, a) \equiv c(\Theta_0)$, and so Θ_0 and Θ_1 permute. Thus (see e.g. [5, Theorem 19.3]) $\mathfrak{A} \cong \mathfrak{A}/\Theta_0 \times \mathfrak{A}/\Theta_1$, $\mathfrak{A}/\Theta_0 \in K_0$, $\mathfrak{A}/\Theta_1 \in K_1$, verifying the first statement of the theorem.

3. This idea can be traced to N. Kimura [8], [9], see also C. C. Chang, B. Jónsson, and A. Tarski [2].

Now let $\mathfrak{A}$ have a modular congruence lattice, $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1$, $\mathfrak{A}_0 \in K_0$, $\mathfrak{A}_1 \in K_1$. Then $\mathfrak{A}_0 \cong \mathfrak{A}/\Phi_0$, $\mathfrak{A}_1 \cong \mathfrak{A}/\Phi_1$, where $\Phi_0 \wedge \Phi_1 = \omega$, $\Phi_0 \vee \Phi_1 = \iota$, and Φ_0, Φ_1 permute. Because of the minimal property of Θ_i, $\Phi_i \geq \Theta_i$, $i = 0, 1$, and so by modularity $\Phi_0 = \Phi_0 \wedge (\Theta_0 \vee \Theta_1)$ $= \Theta_0 \vee (\Phi_0 \wedge \Theta_1) = \Theta_0$, and $\Phi_1 = \Theta_1$, completing the proof of the theorem.

Does $K_0 \vee K_1 = K_0 \times K_1$ imply that K_0 and K_1 are independent? Trivial examples show that this is not the case. Let C_p denote the equational class of Abelian groups satisfying $px = 0$. Set $K_0 = C_2 \vee C_3$, $K_1 = C_3 \vee C_5$. Then $K_0 \vee K_1 = K_0 \times K_1$; but K_0 and K_1 are not independent, because the meet $K_0 \wedge K_1$ of two independent classes can contain one-element algebras only, while $K_0 \wedge K_1$ in this example is C_3. However, we can prove the following theorem.

THEOREM 2. Let $K_0 \wedge K_1$ consist of one-element algebras only and let every $\mathfrak{A} \in K_0 \vee K_1$ have a modular congruence lattice. Then $K_0 \vee K_1 = K_0 \times K_1$ if and only if K_0 and K_1 are independent.

Proof. Theorem 1 contains the "if" part. Now let $K_0 \vee K_1 = K_0 \times K_1$. Let $\mathfrak{F}$ be the free algebra over $K_0 \vee K_1$ with two generators x_0 and x_1. It follows from the assumptions that $\mathfrak{F} \cong \mathfrak{F}/\Phi_0 \times \mathfrak{F}/\Phi_1$, where $\mathfrak{F}/\Phi_i \in K_i$, $i = 0, 1$. Now let Θ_0 and Θ_1 be defined as in the proof of Theorem 1. Then $\Theta_0 \leq \Phi_0$, $\Theta_1 \leq \Phi_1$, and $\Theta_0 \vee \Theta_1 = \iota$ as before. Now take $\mathfrak{F}/\Theta_0 \wedge \Theta_1$; since every homomorphism of $\mathfrak{F}$ to an $\mathfrak{A}_i \in K_i$ factors through $\mathfrak{F}/\Theta_0 \wedge \Theta_1$, and every algebra in $K_0 \vee K_1$ is isomorphic to an algebra of the form $\mathfrak{A}_0 \times \mathfrak{A}_1$ ($\mathfrak{A}_i \in K_i$, $i = 0, 1$), we conclude that $\mathfrak{F}/\Theta_0 \wedge \Theta_1$ also is free over K on two generators. Hence $\Theta_0 \wedge \Theta_1 = \omega$, and $\Theta_i = \Phi_i$ follows by modularity. Thus $\mathfrak{F} \cong \mathfrak{F}/\Theta_0 \times \mathfrak{F}/\Theta_1$, and $\mathfrak{F}/\Theta_i$ is the free algebra over K_i generated by, say, x_0^i, x_1^i $(i = 0, 1)$, where x_j corresponds to $\langle x_j^0, x_j^1 \rangle$ under this isomorphism $(j = 0, 1)$. Let $\underline{p}$ be a polynomial symbol that represents an element of $\mathfrak{F}$ corresponding to $\langle x_0^0, x_1^1 \rangle$ under the above isomorphism. Then $p(x_0, x_1) \equiv x_i(\Theta_i)$, $i = 0, 1$; hence $\underline{p}$ establishes the independence of K_0 and K_1, completing the proof of Theorem 2.

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It should be noted that the independence of K_0 and K_1 means that the polynomials of K_0 and K_1 can be arbitrarily "paired". In other words, if p_i is a polynomial on K_i, $i = 0, 1$, then there is a polynomial p on $K_0 \vee K_1$ acting as p_i on K_i $(i = 0, 1)$. This implies that every "Mal'cev type condition" (see [6]) shared by K_0 and K_1 holds for $K_0 \vee K_1$, provided K_0 and K_1 are independent. By A. Day [3], modularity of congruence lattices is of Mal'cev type. Hence in the second statement of Theorem 1 the condition "every $\mathfrak{A} \in K_0 \vee K_1$ has a modular congruence lattice" can be replaced by "every $\mathfrak{A}$ in K_0 or K_1 has a modular congruence lattice".

REFERENCES

1. A. Astromoff, Some structure theorems for primal and categorical algebras. Math. Z. 87 (1965) 365-377.

2. C. C. Chang, B. Jónsson, and A. Tarski, Refinement properties for relational structures. Fund. Math. 55 (1964) 249-281.

3. A. Day, A characterization of modularity for congruence lattices of algebras. Canad. Math. Bull. 12 (1969) 167-173.

4. A. L. Foster, The identities of - and unique subdirect factorization within - classes of universal algebras. Math. Z. 62 (1955) 171-188.

5. G. Grätzer, Universal algebra. (The University Series in Higher Mathematics, D. Van Nostrand Co. Inc., Princeton, N. J., 1968).

6. G. Grätzer, Mal'cev type conditions. J. Comb. Theory (to appear).

7. P. Kelenson, Thesis. (Berkeley, 1969).

8. N. Kimura, Note on idempotent semigroups, III. Proc. Japan Acad. 34 (1958) 113-114.

9. N. Kimura, The structure of idempotent semigroups, I. Pacific J. Math. 8 (1958) 257-275.

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