Bull. Aust. Math. Soc. **89** (2014), 92–96
doi:10.1017/S0004972713000269

A SOLVABILITY CRITERION FOR FINITE LOOPS

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(Received 13 November 2012; accepted 7 February 2013; first published online 30 April 2013)

Abstract

We show that a finite loop, whose inner mapping group is the direct product of a dihedral 2-group and a nonabelian group of order pq (p and q are distinct odd prime numbers), is solvable.

2010 *Mathematics subject classification*: primary 20D10; secondary 20N05.
Keywords and phrases: inner mapping group, solvable loop, solvable group.

1. Introduction

If Q is a loop, then we have two permutations L_a and R_a on Q defined by $L_a(x) = ax$ and $R_a(x) = xa$ for each $a \in Q$. These permutations generate a permutation group $M(Q)$ which is said to be the multiplication group of Q. The stabiliser of the neutral element of Q is denoted by $I(Q)$ and we say that this subgroup of $M(Q)$ is the inner mapping group of Q. A loop Q is solvable if it has a series $1 = Q_0 \subseteq \cdots \subseteq Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and Q_i/Q_{i-1} is an abelian group for each i.

It is well known that the structure of Q depends strongly on the structure of $M(Q)$. One of the most important results in this direction was proved by Vesanen [9] in 1996: if Q is a finite loop and $M(Q)$ is a solvable group, then Q is a solvable loop. In light of this we are naturally interested in those properties of $I(Q)$ which force $M(Q)$ to be a solvable group and Q to be a solvable loop. The result due to Mazur [6] in 2007 indicates that if Q is a finite loop and $I(Q)$ is a nilpotent group, then $M(Q)$ is solvable. (In fact, it was shown in [7] that Q is then a centrally nilpotent loop.) In 2002, Drápal [1, Corollary 4.7] considered the situation that $I(Q)$ is a nonabelian group of order pq, where p and q are distinct prime numbers. He showed that $M(Q)$ is a solvable group and Q a solvable loop. In [5] the authors showed that a finite loop, whose inner mapping group is the direct product of a dihedral group and an abelian group, is also solvable.

The purpose of this paper is to further investigate the situation where $I(Q)$ is a nonabelian group. Our main result is the following theorem.

Theorem 1.1. *Let Q be a finite loop. If $I(Q) = D \times S$, where D is a dihedral 2-group and S is a nonabelian group of order pq, where p and q are distinct odd prime numbers, then $M(Q)$ is a solvable group and Q is a solvable loop.*

The basic tool for our reasoning is the notion of connected transversals in group theory. In Section 2 we introduce this notion and those results on connected transversals which are later needed in the proof of our main theorem. We also explain the connection between loops, their multiplication and inner mapping groups and connected transversals. In Section 3 we give the proof of our main group-theoretical result. Theorem 1.1 then follows as a loop-theoretical interpretation of this result.

2. Connected transversals

Let H be a subgroup of G and let A and B be two left transversals to H in G. If $[a, b] = a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$, then we say that A and B are H-connected transversals in G. The core of H in G (the largest normal subgroup of G contained in H) will be denoted by H_G. If Q is a loop, then the permutations of Q mentioned in the Introduction form two sets, $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$, which are $I(Q)$-connected transversals in $M(Q)$. Furthermore, the core of $I(Q)$ in $M(Q)$ is trivial. The link between loops and connected transversals is given by the following theorem.

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H and H-connected transversals A and B such that $H_G = 1$ and $G = \langle A, B\rangle$.*

For the proof, see [8, Theorem 4.1].

We start with a technical lemma from [8, Lemma 2.5].

Lemma 2.2. *Assume that A and B are H-connected transversals in G. If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

Theorem 2.3. *Assume that A and B are H-connected transversals in G. If G is a finite group and $H \leq G$ is nilpotent, then G is solvable.*

Mazur [6] proved this theorem by using the classification of finite simple groups. A proof which does not use the classification can be found in [7].

By combining Theorem 2.1 and the result of Drápal [1, Corollary 4.7] mentioned in the Introduction, we get the following lemma.

Lemma 2.4. *Let G be a finite group, $H \leq G$ nonabelian and $|H| = pq$, where p and q are distinct prime numbers. If there exist H-connected transversals A and B in G, then G is solvable.*

A short proof of this lemma can be found in [5, Theorem 2.7].

3. Main results

In the proof of our main results, we need the following two classical theorems. The first one is by Wielandt (see [4, Satz 5.8, p. 285]) and the second one by Gorenstein and Walter [3, Theorem 1].

THEOREM 3.1. *Let G be a finite group and let G contain a nilpotent Hall π-subgroup H. Then every π-subgroup of G is contained in a conjugate of H.*

THEOREM 3.2. *Let G be a finite group with dihedral Sylow 2-subgroups. Let $O(G)$ denote the maximal normal subgroup of odd order. Then G satisfies one of the following conditions.*

(1) *$G/O(G)$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $\mathrm{PSL}(2, q)$, q odd.*
(2) *$G/O(G)$ is isomorphic to the alternating group A_7.*
(3) *$G/O(G)$ is isomorphic to a Sylow 2-subgroup of G.*

We shall now prove the following lemma.

LEMMA 3.3. *Let G be a finite group, $H \leq G$ and $H = F \times S$, where F is abelian and S is a nonabelian group of order pq, where p and q are distinct prime numbers. If $\gcd(|F|, |S|) = 1$ and there exist H-connected transversals A and B in G, then G is solvable.*

PROOF. Let G be a minimal counterexample. If $H_G > 1$, then we consider G/H_G and its subgroup H/H_G and by using induction, Theorem 2.3 or Lemma 2.4, it follows that G/H_G is solvable, and hence G is solvable.

Thus we may assume that $H_G = 1$. If H is not maximal in G, then there exists a subgroup T such that $H < T < G$. By Lemma 2.2, $T_G > 1$ and we may consider G/T_G and its subgroup $HT_G/T_G = T/T_G$. It follows that G/T_G is solvable. Since T is solvable by induction, we conclude that G is solvable.

We thus assume that H is a maximal subgroup of G. Let R be a Sylow r-subgroup of F for a prime number r. As $H_G = 1$, we conclude that R is a Sylow r-subgroup of G. From this it follows that F is a Hall subgroup of G. Clearly, $N_G(R) = H = C_G(R)$ and, by using the Burnside normal complement theorem, there exists a normal r-complement in G for each r that divides $|F|$. Clearly, this means that $G = FK$, where K is normal in G and $\gcd(|F|, |K|) = 1$.

If $1 \neq a \in A$, then $a = yx$, where $y \in F$ and $x \in K$. Then $aK = yK$ and $(aK)^d = K$, where d divides $|F|$. Thus $a^d \in K$, hence $(a^d)^t = 1$, where t divides $|K|$. It follows that $(a^t)^d = 1$, hence $|a^t|$ divides d. Since F is an abelian Hall subgroup of G, we may apply Theorem 3.1 and it follows that $a^t \in F^g$ for some $g \in G$. As F is abelian, $\langle a^t \rangle$ is normal in $\langle a, H^g \rangle = G$. As $H_G = 1$, we conclude that $a^t = 1$. Now there exist integers m and n such that $md + nt = 1$. Thus $a = a^{md+nt} = (a^d)^m (a^t)^n \in K$.

We may conclude that $A \cup B \subseteq K$. Clearly, $S \leq K$ and thus $K = AS = BS$. By Lemma 2.4, K is a solvable group. As $G = FK$, it follows that G is solvable, too. □

We are now ready to prove our group-theoretical result.

https://doi.org/10.1017/S0004972713000269 Published online by Cambridge University Press

THEOREM 3.4. *Let G be a finite group, $H \leq G$ and $H = D \times S$, where D is a dihedral group of order 2^t ($t \geq 2$) and S is a nonabelian group of order pq, where p and q are distinct odd prime numbers with $p > q$. If there exist H-connected transversals in G, then G is solvable.*

PROOF. Assume that G is a minimal counterexample. If $t = 2$, then D is abelian and we can use Lemma 3.3. We thus assume that $t \geq 3$. By induction or by using Theorem 2.3 or Lemmas 2.4 and 3.3, we may conclude that $H_G = 1$ and H is maximal in G (see the first part of the proof of Lemma 3.3). It follows that D is a Sylow 2-subgroup of G. If $O(G) > 1$ is the maximal normal subgroup of odd order, then $G = O(G)H$ and G is solvable by the Feit–Thompson theorem [2]. Thus we may assume that $O(G) = 1$.

Now G is a finite group with a dihedral Sylow 2-subgroup and we may use Theorem 3.2. Since pq divides $|G|$, it follows that G cannot be isomorphic to a Sylow 2-subgroup of G. If $G \cong A_7$, we have a contradiction, as then $G \geq H = D \times S$ is not possible. Thus we are left with the case where G is isomorphic to a subgroup of $P\Gamma L(2, r)$ containing $PSL(2, r)$ (here $r = s^n$ for an odd prime s). Since $P\Gamma L(2, r) \cong \mathrm{PGL}(2, r) \rtimes C_n$, it follows that $G = NH$, where N is normal in G, $N \cong PSL(2, r)$ and G/N is abelian.

Now $N \cap D$ is normal in D, and as $N \cap D$ is a Sylow 2-subgroup of N, we may conclude that $N \cap D$ is dihedral of order 2^{t-1} or of order 2^t. Furthermore, $N \cap S$ is normal in S, and if $N \cap S > 1$, then $N \cap S \geq P$, where P is the Sylow p-subgroup of S. But then $N \geq (N \cap D) \times P$, which is not possible as $N \cong PSL(2, r)$. Thus $N \cap H = N \cap D$ and $G/N = HN/N \cong H/N \cap H \cong S$ or $G/N = HN/N \cong H/N \cap H \cong C_2 \times S$. Since S is not abelian, we have reached our final contradiction and the proof is complete. □

The preceding theorem combined with Theorem 2.1 and the theorem by Vesanen [9] implies the loop-theoretical result in Theorem 1.1.

REMARK 3.5. It would be interesting to know if the results of Theorems 3.4 and 1.1 also hold in the case where D is a dihedral group without restrictions on the order of D.

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