BOUNDS FOR PERMANENTS OF NON-NEGATIVE MATRICES[1]

by HENRYK MINC

(Received 26th April 1968)

1. Introduction

Let $v = (a_1, \ldots, a_n)$ be a real n-tuple and $a_1^* \geqq a_2^* \geqq \ldots \geqq a_n^*$ be the numbers $a_1, \ldots, a_n$ arranged in decreasing order. Let $\Sigma^{(m)}v$ denote the sum of m greatest components of v and $\Sigma_{(m)}v$ the sum of m smallest components of v, i.e.,

$$\Sigma^{(m)}v = \sum_{j=1}^{m} a_j^*, \text{ and } \Sigma_{(m)}v = \sum_{j=1}^{m} a_{n-j+1}^*.$$

If A is an n-square matrix, let $A_{(i)}$ denote the ith row of A and $r_i(A)$ (or simply r_i) the ith row sum of A, i.e., $r_i = \sum_{j=1}^{n} a_{ij}$. The permanent of A is defined by

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation on the right-hand side is over all permutations σ of the symmetric group S_n. A matrix is said to be non-negative (positive) if all its entries are non-negative (positive).

Many inequalities for permanents of various classes of non-negative matrices have been obtained recently (see bibliographies in (**3**) and (**5**)), such as doubly stochastic matrices (**4**) and (0, 1)-matrices (see bibliography in (**6**)). Jurkat and Ryser (**1**) obtained the following bounds for the permanent of a non-negative matrix A:

$$\prod_{i=1}^{n} \Sigma_{(i)}A_{(i)} \leqq \text{per}(A) \leqq \prod_{i=1}^{n} \Sigma^{(i)}A_{(i)}. \tag{1}$$

Rather surprisingly these are the only known non-trivial bounds for a general non-negative matrix.

In the present paper the bounds in (1) are presented in a somewhat more general guise (Theorem 1) and are proved by a method substantially simpler than that in (**1**). The cases of equality for both bounds are also discussed. In Theorem 2, we give a lower and an upper bound for the permanent of a non-negative matrix that are better than the bounds in (1).

2. Results

If A is an n-square matrix and i and j are positive integers, $1 \leqq i, j \leqq n$,

[1] This research was supported by the U.S. Air Force Office of Scientific Research under Grant AFOSR 698-67.

then $A(i\,|\,j)$ denotes the submatrix obtained from A by deleting the ith row and the jth column of A.

Theorem 1. *Let* $A = (a_{ij})$ *be a non-negative n-square matrix. Then*

$$\max_{\sigma \in S_n} \prod_{i=1}^{n} \Sigma_{(i)} A_{(\sigma(i))} \leqq \operatorname{per}(A) \leqq \min_{\sigma \in S_n} \prod_{i=1}^{n} \Sigma^{(i)} A_{(\sigma(i))}. \tag{2}$$

If A is positive, then equality can occur in (2) *if and only if A contains an* $(n-1) \times n$ *submatrix all of whose rows are multiples of* $(1, 1, \ldots, 1)$.

Proof. Let $B = (b_{ij}) = PA$ where P is a permutation matrix. We prove by induction that

$$\operatorname{per}(B) \leqq \prod_{i=1}^{n} \Sigma^{(i)} B_{(i)}, \tag{3}$$

and that if B happens to be positive then equality can hold in (3) if and only if

$$b_{i1} = b_{i2} = \ldots = b_{in}, \quad i = 1, \ldots, n-1. \tag{4}$$

It is easily seen that (3) and the assertion about equality hold in case $n = 2$. Assume that they hold for all non-negative $(n-1)$-square matrices. Expand the permanent of B by the last row and apply the induction hypothesis:

$$\operatorname{per}(B) = \sum_{j=1}^{n} b_{nj} \operatorname{per}(B(n\,|\,j)) \leqq \sum_{j=1}^{n} b_{nj} \prod_{i=1}^{n-1} \Sigma^{(i)}(B(n\,|\,j))_{(i)}. \tag{5}$$

Obviously

$$\Sigma^{(i)}(B(n\,|\,j))_{(i)} \leqq \Sigma^{(i)} B_{(i)}, \tag{6}$$

$= 1, \ldots, n-1, j = 1, \ldots, n$, and therefore

$$\operatorname{per}(B) \leqq \sum_{j=1}^{n} b_{nj} \prod_{i=1}^{n-1} \Sigma^{(i)} B_{(i)} = \left(\prod_{i=1}^{n-1} \Sigma^{(i)} B_{(i)}\right)\left(\sum_{j=1}^{n} b_{nj}\right) = \prod_{i=1}^{n} \Sigma^{(i)} B_{(i)}. \tag{7}$$

Thus $\operatorname{per}(A) = \operatorname{per}(PA) \leqq \prod_{i=1}^{n} \Sigma^{(i)}(PA)_{(i)}$ for any permutation matrix P. Hence

$$\operatorname{per}(A) \leqq \min_{P} \prod_{i=1}^{n} \Sigma^{(i)}(PA)_{(i)} = \min_{\sigma \in S_n} \prod_{i=1}^{n} \Sigma^{(i)} A_{(\sigma(i))},$$

which is the upper bound in (2). The lower bound is proved similarly.

Now suppose that A (and thus B) is positive. Then equality can hold on the right-hand side of (2) if and only if (3) is equality for some permutation matrix P. This can occur if and only if both (5) and (7) are equalities. Now, by the induction hypothesis, (5) can be an equality if and only if all the entries in each of the rows $1, \ldots, n-2$ of $B(n\,|\,j)$ are equal for $j = 1, \ldots, n$, and therefore if and only if

$$b_{i1} = b_{i2} = \ldots = b_{in} \tag{8}$$

for $i = 1, \ldots, n-2$. Now, (7) holds with equality if and only if (6) is an equality for $i = 1, \ldots, n-1$, i.e., if and only if (8) holds and

$$\Sigma^{(n-1)}(B(n\,|\,j))_{(n-1)} = \Sigma^{(n-1)} B_{(n-1)}, \tag{9}$$

$j = 1, \ldots, n$. Let $b_{n-1,t} = \max_j (b_{n-1,j})$ and $b_{n-1,s} = \min_j (b_{n-1,j})$. Then for $j = t$, the equality (9) reads

$$\Sigma^{(n-1)}(B(n \mid t))_{(n-1)} = \Sigma^{(n-1)}B_{(n-1)},$$

i.e.,

$$b_{n-1,1}+\ldots+b_{n-1,t-1}+b_{n-1,t+1}+\ldots+b_{n-1,n}$$
$$= b_{n-1,1}+\ldots+b_{n-1,s-1}+b_{n-1,s+1}+\ldots+b_{n-1,n},$$

i.e., $b_{n-1,t} = b_{n-1,s}$ and therefore $b_{n-1,1} = \ldots = b_{n-1,n}$.

Thus, when B is positive, equality in (3) can hold if and only if (4) is satisfied. Now, if the right inequality in (2) is equality, i.e., if

$$\operatorname{per}(A) = \min_{\sigma \in S_n} \prod_{i=1}^{n} \Sigma^{(i)}A_{(\sigma(i))},$$

then

$$\operatorname{per}(PA) = \prod_{i=1}^{n} \Sigma^{(i)}(PA)_{(i)}$$

for some permutation matrix P. We have just shown that this implies that the first $n-1$ rows of PA are multiples of $(1, 1, \ldots, 1)$ which means that A contains an $(n-1)\times n$ submatrix all of whose rows are multiples of $(1, 1, \ldots, 1)$.

To prove the converse, suppose that

$$A_{(i)} = a_i(1, 1, \ldots, 1) \tag{10}$$

for all i, except possibly one: $i = k$, say. Let σ be the transposition (k, n) if $k \neq n$, and let σ be the identity permutation if $k = n$. Then expanding $\operatorname{per}(A)$ by the kth row we get

$$\operatorname{per}(A) = \sum_{j=1}^{n} a_{kj} \operatorname{per}(A(k \mid j)).$$

Now, by (10),

$$\operatorname{per}(A(k \mid j)) = (n-1)! \prod_{\substack{i=1 \\ i \neq k}}^{n} a_i = (n-1)! \prod_{i=1}^{n-1} a_{\sigma(i)} = \prod_{i=1}^{n-1} \Sigma^{(i)}A_{(\sigma(i))},$$

and therefore

$$\operatorname{per}(A) = \sum_{j=1}^{n} a_{\sigma(n),j} \prod_{i=1}^{n-1} \Sigma^{(i)}A_{(\sigma(i))} = \prod_{i=1}^{n} \Sigma^{(i)}A_{(\sigma(i))}.$$

The case of equality for the lower inequality in (2) is proved similarly.

Note that the condition for equality may not be a necessary condition if A is merely non-negative. For example, if

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

https://doi.org/10.1017/S0013091500012748 Published online by Cambridge University Press

then per $(A) = 18$, and the upper bound in (2) is

$$1\times(1+1)\times(2+1+0)\times(2+1+0+0) = 18,$$

while the lower bound is

$$1\times(1+1)\times(0+0+1)\times(0+0+1+2) = 6.$$

Let a_{is} and a_{it} denote the largest coordinate and the smallest coordinate of $A_{(i)}$, respectively.

Theorem 2. *Let $A = (a_{ij})$ be a non-negative matrix. Then*

$$\text{per}\,(A) \leqq \min_{\sigma\in S_n}\left(\prod_{i=1}^{n}\Sigma^{(i)}A_{(\sigma(i))}-(na_{\sigma(1),s}-r_{\sigma(1)})\sum_{k=2}^{n}\Sigma_{(k-1)}A_{\sigma(k)}\right), \tag{11}$$

$$\text{per}\,(A) \geqq \max_{\sigma\in S_n}\left(\prod_{i=1}^{n}\Sigma_{(i)}A_{(\sigma(i))}+(r_{\sigma(1)}-na_{\sigma(1),t})\prod_{k=2}^{n}\Sigma_{(k-1)}A_{\sigma(k)}\right). \tag{12}$$

Proof. In the proof of Theorem 1 we re-proved the Jurkat-Ryser inequalities (**1**), that is, we showed that

$$\prod_{i=1}^{n}\Sigma_{(i)}X_{(i)} \leqq \text{per}\,(X) \leqq \prod_{i=1}^{n}\Sigma^{(i)}X_{(i)} \tag{13}$$

for any non-negative n-square matrix X. Let $B = PA$, where P is a permutation matrix, and let $b_{1s} = \max_j (b_{1j})$ and $b_{1t} = \min_j (b_{1j})$. First we shall prove (11). Let $C = (c_{ij})$ and $D = (d_{ij})$ be n-square matrices defined by:

$$\begin{cases} c_{1j} = b_{1s}-b_{1j}\ (j = 1, \ldots, n) \\ c_{ij} = b_{ij}, \text{ otherwise;}\end{cases} \quad\text{and}\quad \begin{cases} d_{1j} = b_{1s}\ (j = 1, \ldots, n) \\ d_{ij} = b_{ij}, \text{ otherwise.}\end{cases}$$

Now, the permanent function is a multilinear function of the rows [(**2**); see 2.11.4]. Thus per $(D) =$ per $(B)+$per (C), i.e., per $(B) =$ per $(D)-$per (C) But, by (13),

$$\text{per}\,(D) \leqq \prod_{i=1}^{n}\Sigma^{(i)}D_{(i)} = \prod_{i=1}^{n}\Sigma^{(i)}B_{(i)},$$

and

$$\begin{aligned}\text{per}\,(C) &= \sum_{j=1}^{n}c_{1j}\,\text{per}\,(C(1\,|\,j)) = \sum_{j=1}^{n}(b_{1s}-b_{1j})\,\text{per}\,(B(1\,|\,j)) \\ &\geqq \sum_{j=1}^{n}(b_{1s}-b_{1j})\prod_{k=1}^{n-1}\Sigma_{(k)}(B(1\,|\,j))_{(k)} \geqq \sum_{j=1}^{n}(b_{1s}-b_{1j})\prod_{k=2}^{n}\Sigma_{(k-1)}B_{(k)} \\ &= (nb_{1s}-r_1)\prod_{k=2}^{n}\Sigma_{(k-1)}B_{(k)}.\end{aligned}$$

Thus

$$\text{per}\,(B) = \text{per}\,(D)-\text{per}\,(C) \leqq \prod_{i=1}^{n}\Sigma^{(i)}B_{(i)}-(nb_{1s}-r_1)\prod_{k=2}^{b}\Sigma_{(k-1)}B_{(k)},$$

and

$$\operatorname{per}(A) = \min_P (\operatorname{per}(PA))$$
$$\leqq \min_{\sigma \in S_n} \left(\prod_{i=1}^{n} \Sigma^{(i)} A_{(\sigma(i))} - (na_{\sigma(1),s} - r_{\sigma(1)}) \prod_{k=2}^{n} \Sigma_{(k-1)} A_{(\sigma(k))} \right).$$

To prove the lower bound, let $G = (g_{ij})$, $H = (g_{ij})$ be defined by:

$$\begin{cases} g_{1j} = b_{1j} - b_{1t} \ (j = 1, \ldots, n) \\ g_{ij} \ = b_{ij}, \text{ otherwise,} \end{cases} \quad \text{and} \quad \begin{cases} h_{1j} = b_{1t} \ (j = 1, \ldots, n), \\ h_{ij} \ = b_{ij}, \text{ otherwise.} \end{cases}$$

Then, again by the multilinearity of the permanent function,

$$\operatorname{per}(B) = \operatorname{per}(G) + \operatorname{per}(H).$$

Now, by (13),

$$\operatorname{per}(H) \geqq \prod_{i=1}^{n} \Sigma_{(i)} H_{(i)} = \prod_{i=1}^{n} \Sigma_{(i)} B_{(i)},$$

and

$$\operatorname{per}(G) = \sum_{j=1}^{n} g_{1j} \operatorname{per}(G(1 \mid j)) = \sum_{j=1}^{n} (b_{1j} - b_{1t}) \operatorname{per}(B(1 \mid j))$$
$$\geqq \sum_{j=1}^{n} (b_{1j} - b_{1t}) \prod_{k=1}^{n-1} \Sigma_{(k)} (B(1 \mid j))_{(k)} \geqq \sum_{j=1}^{n} (b_{1j} - b_{1t}) \prod_{k=2}^{n} \Sigma_{(k-1)} B_{(k)}$$
$$= (r_1 - nb_{1t}) \prod_{k=2}^{n} \Sigma_{(k-1)} B_{(k)}.$$

Thus

$$\operatorname{per}(B) \geqq \prod_{i=1}^{n} \Sigma_{(i)} B_{(i)} + (r_1 - nb_{1t}) \prod_{k=2}^{n} \Sigma_{(k-1)} B_{(k)},$$

and (12) follows.

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University of California
Santa Barbara, California