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TREE THEORY: INTERPRETABILITY BETWEEN WEAK FIRST-ORDER THEORIES OF TREES

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Abstract. Elementary first-order theories of trees allowing at most, exactly m, and any finite number of immediate descendants are introduced and proved mutually interpretable among themselves and with Robinson arithmetic, Adjunctive Set Theory with Extensionality and other well-known weak theories of numbers, sets, and strings.

§1. Introduction. In logic, trees are almost everywhere. Algorithmic tests for basic syntactic and semantic properties such as well-formedness, satisfiability, or validity typically involve construction of tree-like arrays. Elementary model theory of first-order logic is often presented as revolving around properties of a certain kind of tree—witness König's Lemma—and basic objects of proof theory, such as proofs, derivations, and sequents, are most directly represented in tree-like form. One does get the feeling that trees might be as fundamental to logical theory as are numbers and sets. Yet they seem not have to been systematically and explicitly investigated as subject of a specific theory in axiomatic form.

An important step in this direction was taken recently in the work of Kristiansen and Murwanashyaka [4] who introduced an elementary theory T of *full binary trees*, finite trees in which every non-terminal node has exactly two immediate descendants. (We shall call such trees *dyadic*.) They showed that the basic arithmetical operations of addition and multiplication are definable in T, and more specifically, that Robinson arithmetic, Q, is formally interpretable in T. In [3], we showed that T, on the other hand, is also interpretable in Q, and hence mutually interpretable with several other well-known theories of numbers, sets, and strings. The argument given in [3] hinges on a formalized representation of dyadic trees by binary strings in elementary concatenation theory QT^+.

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Here we extend and generalize this approach to finite trees in general. We take a planar tree to be a connected and acyclical planar graph with one node singled out as the "root" of the tree. We consider the vocabulary $\mathcal{L}_{T^*} = \{0, \tau_1, \tau_2, \dots, \sqsubseteq\}$ with a single individual constant 0 and infinitely many function symbols $\tau_1, \tau_2, \dots$ where τ_n has arity *n* for each $n \geq 1$, and a 2-place relation symbol $\sqsubseteq$, along with the (universal closures) of the following formulae as axioms:

$(T1_n)$ $\neg\tau_n(x_1, \dots, x_n) = 0$ for each $n \geq 1$.

$(T2_{m,n})$ $\neg\tau_m(x_1, \dots, x_m) = \tau_n(y_1, \dots, y_n)$ for each m, n where $m \neq n$.

$(T3_n)$ $\tau_n(x_1, \dots, x_n) = \tau_n(y_1, \dots, y_n) \rightarrow \wedge_{1 \leq i \leq n} x_i = y_i$ for each $n \geq 1$.

$(T4_n)$ $x \sqsubseteq \tau_n(y_1, \dots, y_n) \leftrightarrow x = \tau_n(y_1, \dots, y_n) \vee \vee_{1 \leq i \leq n} x \sqsubseteq y_i$ for each $n \geq 1$.

(T5) $x \sqsubseteq x$.

(T6) $0 \sqsubseteq x$.

(T7) $x \sqsubseteq y \;\&\; y \sqsubseteq x \rightarrow x = y$.

(T8) $x \sqsubseteq y \;\&\; y \sqsubseteq z \rightarrow x \sqsubseteq z$.

We call the resulting first-order theory with identity T^*. The domain of the intended interpretation consists of finite planar trees whose nodes may have any number, including single, immediate descendants. (Alternatively, we may think of the domain as consisting of all variable-free $\mathcal{L}_{T^*}$-terms.) Here we are thinking of finite trees as constructed from 0 by finitely many applications of any combination of tree building operations associated with $\tau_1, \tau_2, \dots$. The individual constant 0 refers to a trivial, single node tree, and, for each $n \geq 1$, the n-ary operation associated with τ_n applied to trees $T_1, \dots, T_n$ yields the tree $\tau_n(T_1, \dots, T_n)$ whose root node has as its immediate descendants the root nodes of $T_1, \dots, T_n$, respectively. We think of these trees as linearly ordered in that the immediate subtrees $T_1, \dots, T_n$ of $\tau_n(T_1, \dots, T_n)$ are ordered, say, from left to right. However, when considering nodes with a single immediate descendant (singleton), we do not distinguish between "branching to the right" and "branching to the left." (This we will do in Section 9 with appropriate modifications to the corresponding theories.) The relational symbol $\sqsubseteq$ is meant to express the subtree relation between trees, where subtrees are defined so that for a given tree T, any of its nodes x determines a subtree T_x consisting of all and only the descendants of x in T including itself.

By including only the appropriate instances of each of the schemas $(T1_n)$, $(T2_{m,n})$, $(T3_n)$, and $(T4_n)$ along with (T5)–(T8) we obtain theories T_m of trees of fixed arity or theories $T_{\leq m}$ of $\leq$ m-trees with at most m branchings, for fixed m. The theory T of dyadic trees formulated by Kristiansen and Murwanashyaka and further investigated in [3] is a subtheory of T_2, with

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$x \sqsubseteq 0 \leftrightarrow x = 0$ in place of (T5)–(T8). The latter formula is easily seen to be implied by (T5)–(T7).

In Section 2 we identify characteristic traces of tree structure in binary strings. On that basis in Section 3 we set up a formal coding apparatus for finite trees, which we call *Catalan coding*. In Section 4 we establish that, for $m \geq 2$, theories $T_{\leq m}$ (and T_m) are each formally interpretable in $T_{\leq 2}$, and in Section 5 that, in turn, $T_{\leq 2}$ is interpretable in the concatenation theory QT^+. We use this in Section 6 to establish mutual interpretability of all of the finitely axiomatized theories $T_{\leq m}$ and T_m. In Section 7 we interpret the theory of all finite trees T^* in QT^+. In Section 8 we consider some theories of dyadic trees that extend T and establish their mutual interpretability, showing, among other things, that T_2 is also interpretable in T. In Section 9 we introduce theories T_e, $T_{\leq n,e}$, and ${T^*}_e$ that distinguish between left and right single branchings and prove their mutual interpretability with T, $T_{\leq n}$, and T^*, respectively. Finally, in Section 10 we prove that all these theories are mutually interpretable among themselves and with Q, Adjunctive Set Theory with or without Extensionality, various concatenation theories, etc.

§2. Almost even strings. We first turn to concatenation theory. The language $\mathcal{L}_C = \{a, b, {}^*\}$ of concatenation theory has two individual constants a and b, and a single binary operation symbol *. Its intended interpretation Σ^* has as its domain the set of all non-empty finite strings of a's and b's, the constants "a" and "b," respectively, stand for the digits a and b (or 0 and 1, resp.), and, for given strings x and y from the domain of Σ^*, we let x^*y be the string obtained by concatenation (i.e., juxtaposition) of the successive digits of y to the right of the end digit of x. For the moment we reason informally in the first-order theory $\mathrm{Th}(\Sigma^*)$ consisting of all true sentences of $\mathcal{L}_C$ in Σ^*. (Later, in Section 5, we introduce a finitely axiomatized subtheory QT^+ of $\mathrm{Th}(\Sigma^*)$ that will play a key role in our formal argument.)

We shall pay particular attention to the number of occurrences of a's and b's in a given string. The functions $\alpha(x)$ and $\beta(x)$ that count the number of a's and b's, respectively, in a string x are defined by a straightforward recursion on strings:

$$\begin{array}{ll}
\alpha(a) = 1 & \beta(a) = 0. \\
\alpha(b) = 0 & \beta(b) = 1. \\
\alpha(x^*a) = \alpha(x) + 1 & \beta(x^*a) = \beta(x). \\
\alpha(x^*b) = \alpha(x) & \beta(x^*b) = \beta(x) + 1.
\end{array}$$

An equally straightforward induction on strings shows that the functions α and β are additive with respect to concatenation operation * on strings:

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$$\alpha(x^*y) = \alpha(x) + \alpha(y) \quad \text{and} \quad \beta(x^*y) = \beta(x) + \beta(y),$$

and that for each string x, $\alpha(x) + \beta(x) > 0$.

On account of associativity of the binary concatenation operation we sometimes omit writing the symbol * and parentheses, simply juxtaposing concatenated strings one next to another. We let

$$\begin{gathered} xBy \equiv \exists z\, x^*z = y \;\text{ and } xEy \equiv \exists z\, z^*x = y \text{ and also} \\ x \subseteq_p y \equiv x = y \vee xBy \vee xEy \vee \exists y_1 \exists y_2 y = y_1^*(x^*y_2). \end{gathered}$$

We define $\mathrm{Tally}_b(x) \equiv \forall y \subseteq_p x(\mathrm{Digit}(y) \to y = b)$ where $\mathrm{Digit}(x) \equiv x = a \vee x = b$. In $\mathcal{L}_C$, the b-tallies b, bb, bbb, ... may be identified with the formal numerals $\underline{0}$, $\underline{1}$, $\underline{2}$, ..., so we sometimes write $\underline{n}$ for the string b^{n+1} of $n + 1$ consecutive *b*'s.

One of the principal results of [3] is that (the graphs of) the functions $\alpha(x)$ and $\beta(x)$ are expressible by $\mathcal{L}_C$ formulae $A^\#(x, y)$ and $B^\#(x, y)$, respectively, along with (the graph of) the binary operation AddTally on strings that simulates addition of non-negative integers by concatenation of b-tallies. With that in mind, we shall write, e.g., "$\alpha(x) = \beta(y) + \underline{1}$" and "$\alpha(x) + \beta(x) = \underline{n}$" with the understanding that these expressions abbreviate appropriately chosen $\mathcal{L}_C$ formulas such as "$\exists x_1, y_1(A^\#(x, x_1)\ \&\ B^\#(y, y_1)\ \&\ x_1 = by_1)$" and "$\exists x_1 y_1 \big(A^\#(x, x_1)\ \&\ B^\#(x, y_1)\ \&\ \mathrm{AddTally}(x_1, y_1, b^{n+1})$," respectively. We then have:

2.1 For each $n \geq 1$,

$$\Sigma^* \models \forall x\, [\alpha(x) + \beta(x) = \underline{n} \leftrightarrow \exists x_1 \ldots x_n(\wedge_{1 \leq i \leq n} \mathrm{Digit}(x_i)\ \&\ x = x_1 \ldots x_n)].$$

We omit the proof.

We call a string x *almost even*, writing Æ(x), if (ci) $\alpha(x) = \beta(x) + 1$, ("the +1 property") and (cii) for each proper initial segment u of x, $\alpha(u) \leq \beta(u)$.

In other words, the almost even strings are the shortest initial segments of themselves in which the number of *a*'s strictly exceeds the number of *b*'s. We let Æ(x) abbreviate the $\mathcal{L}_C$ formula

$$\begin{gathered} \exists y, z\, (A^\#(x, y)\ \&\ B^\#(x, z)\ \&\ y = z^*b)\ \& \\ \&\ \forall u, v, w\, (uBx\ \&\ A^\#(u, v)\ \&\ B^\#(u, w) \to v \leq w). \end{gathered}$$

(See [3] for details.)

We note the following:

2.2 $\Sigma^* \models$ Æ$(x) \to x = a \vee (bBx\ \&\ aaEx)$.
For the proof see [3, 2.1(a)].

2.3 (a) $\Sigma^* \models$ Æ(x) & Æ$(y) \to \neg xBy\ \&\ \neg yBx$.
(b) $\Sigma^* \models$ Æ(x) & $x_2Ex \to \alpha(x_2) \geq \beta(x_2) + \underline{1}$.

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(c) $\Sigma^* \models Æ(x)\ \&\ Æ(y) \rightarrow \neg\exists z(zEx\ \&\ zBy)$.
(d) $\Sigma^* \models Æ(x) \rightarrow \neg\exists z(zEx\ \&\ zBx)$.

PROOF. (a) is immediate from (ci) and (cii). For (b), see the proof in [3, 2.1(b)]; (c) and (d) follow from (b). ⊣

2.4 For any $n \geq 1$,

$$\Sigma^* \models \mathrm{Tally_b}(s)\ \&\ \mathrm{Tally_b}(t)\ \&\ x = sax_1 \ldots x_n\ \&\ y = tay_1 \ldots y_n\ \& \\ \&\ \wedge_{1\leq i\leq n} Æ(x_i)\ \&\ \wedge_{1\leq i\leq n} Æ(y_i) \rightarrow (x = y \rightarrow s = t\ \&\ \wedge_{1\leq i\leq n} x_i = y_i).$$

PROOF. Assume that $\Sigma^* \models x = sax_1 \ldots x_n\ \&\ y = tay_1 \ldots y_n$, where $\Sigma^* \models \mathrm{Tally_b}(s)\ \&\ \mathrm{Tally_b}(t)$ and $\Sigma^* \models \wedge_{1\leq i\leq n} Æ(x_i)\ \&\ \wedge_{1\leq i\leq n} Æ(y_i)$. Suppose $\Sigma^* \models x = y$. Then $\Sigma^* \models sax_1 \ldots x_n = tay_1 \ldots y_n$, whence $\Sigma^* \models sBx\ \&\ tBx$. In that case, we have that $\Sigma^* \models sBt \vee s = t \vee tBs$. Suppose, for a reductio, that $\Sigma^* \models sBt$. Then $\Sigma^* \models \exists u\ (\mathrm{Tally_b}(u)\ \&\ t = su)$, whence $\Sigma^* \models sax_1 \ldots x_n = suay_1 \ldots y_n$. But then $\Sigma^* \models ax_1 \ldots x_n = uay_1 \ldots y_n$, which is a contradiction because $\Sigma^* \models \mathrm{Tally_b}(u)$. Hence $\Sigma^* \models \neg sBt$. Exactly analogously, $\Sigma^* \models \neg tBs$. It follows that $\Sigma^* \models s = t$.

Thus we obtain that $\Sigma^* \models sax_1 \ldots x_n = say_1 \ldots y_n$, whence $\Sigma^* \models x_1 \ldots x_n = z = y_1 \ldots y_n$. So $\Sigma^* \models x_1Bz\ \&\ y_1Bz$. Then $\Sigma^* \models x_1By_1 \vee x_1 = y_1 \vee y_1Bx_1$. But from $\Sigma^* \models Æ(x_1)\ \&\ Æ(y_1)$, by 2.3(a), we have $\Sigma^* \models \neg x_1By_1\ \&\ \neg y_1Bx_1$. Therefore, $\Sigma^* \models x_1 = y_1$. Assume now that $\Sigma^* \models \wedge_{1\leq i\leq k} x_i = y_i$ for $k < n$. Then from $\Sigma^* \models s = t$ and hypothesis $\Sigma^* \models sax_1 \ldots x_n = tay_1 \ldots y_n$ we obtain

$$\Sigma^* \models sax_1 \ldots x_k(x_{k+1} \ldots x_n) = sax_1 \ldots x_k(y_{k+1} \ldots y_n),$$

whence $\Sigma^* \models x_{k+1} \ldots x_n = y_{k+1} \ldots y_n$. Then the same argument as above shows that $\Sigma^* \models x_{k+1} = y_{k+1}$. Hence for all j, $1 \leq j \leq n$, $\Sigma^* \models x_j = y_j$. ⊣

2.5 (a) $\Sigma^* \models Æ(x) \rightarrow x = a \vee \exists t(\mathrm{Tally_b}(t)\ \&\ taBx)$.
(b) $\Sigma^* \models Æ(x)\ \&\ x = bay \rightarrow Æ(y)$.

PROOF. For (a), assume $\Sigma^* \models Æ(x)\ \&\ x \neq a$. By 2.2, $\Sigma^* \models bBx\ \&\ aaEx$. Hence $\Sigma^* \models \exists x_1 x = x_1aa$. So x has at least one proper initial segment ($= x_1a$) that ends with a. Let s_0 be the shortest such initial segment. Then $\Sigma^* \models aEs_0$, so $\Sigma^* \models \exists t\ s_0 = ta$. But then $\Sigma^* \models s_0Bx\ \&\ bBs_0$, so we must have $\Sigma^* \models \mathrm{Tally_b}(t)$ by the choice of s_0.

For (b), assume $\Sigma^* \models x = bay$ where $\Sigma^* \models Æ(x)$. Then

$$\Sigma^* \models \alpha(y) = \alpha(bay)–\underline{1} = (\beta(bay) + \underline{1})–\underline{1} = \beta(bay) = \beta(y) + \underline{1}.$$

Suppose $\Sigma^* \models zBy$. Then $\Sigma^* \models (baz)B(bay)$. Hence from hypothesis $\Sigma^* \models Æ(x)$ we have $\Sigma^* \models \alpha(z) = \alpha(baz)–\underline{1} \leq \beta(baz)–\underline{1} = (\beta(z) + \underline{1})–\underline{1} = \beta(z)$. Hence also $\Sigma^* \models \forall z\ (zBy \rightarrow \alpha(z) \leq \beta(z))$. Therefore, $\Sigma^* \models Æ(y)$. ⊣

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2.6 Let u be any closed $\mathcal{L}_C$ term. If $\Sigma^* \models \text{Æ}(u)\ \&\ u \neq a$, then for some $n \geq 1$, $\Sigma^* \models \exists u_1 \dots u_n u = b^n a u_1 \dots u_n$.

Proof. Assume $\Sigma^* \models \text{Æ}(u)\ \&\ u \neq a$. Then, by 2.5(a), $\Sigma^* \models \exists y\ u = tay$ where $\Sigma^* \models \mathrm{Tally}_b(t)$. Let $t = b^n$. If $n = 1$, we are done, for we can let $\Sigma^* \models u_1 = y$.

Suppose $n > 1$. We claim that if $\Sigma^* \models u = b^n ay$, then $\Sigma^* \models \alpha(y) + \beta(y) \geq \underline{n}$.

We have that

$$\Sigma^* \models \alpha(u) = \alpha(b^n ay) = \underline{1} + \alpha(y) \text{ and } \Sigma^* \models \beta(u) = \beta(b^n ay) = \underline{n} + \beta(y).$$

From $\Sigma^* \models \text{Æ}(u)$, we have $\Sigma^* \models \alpha(u) = \beta(u) + \underline{1}$.

Then $\Sigma^* \models \underline{1} + \alpha(y) = n + \beta(y) + 1$. So $\Sigma^* \models \alpha(y) = \beta(y) + \underline{n} \geq \underline{n}$. But then $\Sigma^* \models \alpha(y) + \beta(y) \geq \underline{n} + \beta(y) \geq \underline{n}$, which proves the claim.

Now let $\Sigma^* \models \alpha(y) + \beta(y) = \underline{k} \geq \underline{n}$. By 2.1 we have that

$$\Sigma^* \models \exists y_1 \dots y_k (\wedge_{1 \leq i \leq k} \mathrm{Digit}(y_i)\ \&\ y = y_1 \dots y_k).$$

If $k = n$, we are done. If $k > n$, then $\Sigma^* \models \exists z\ y = y_1 \dots y_n z = y_1 \dots y_{n-1}(y_n z)$.

But then $\Sigma^* \models u = b^n ay = b^n au_1 \dots u_n$ where $\Sigma^* \models \wedge_{1 \leq i < n}\ u_i = y_i\ \&\ u_n = y_n z$. ⊣

2.7 For any $n \geq 1$, $\Sigma^* \models \wedge_{1 \leq i \leq n} \text{Æ}(x_i)\ \&\ y = b^n a x_1 \dots x_n \rightarrow \text{Æ}(y)$.

Proof. Assume $\Sigma^* \models \wedge_{1 \leq i \leq n} \text{Æ}(x_i)$, and let $\Sigma^* \models y = b^n a x_1 \dots x_n$. Then

$$\begin{aligned}\Sigma^* \models \alpha(y) &= \alpha(b^n a x_1 \dots x_n) = \underline{1} + \textstyle\sum_{1 \leq i \leq n} \alpha(x_i) = \underline{1} + \sum_{1 \leq i \leq n} (\beta(x_i) + \underline{1}) = \\ &= \underline{1} + \underline{n} + \textstyle\sum_{1 \leq i \leq n} \beta(x_i) = \beta(b^n a x_1 \dots x_n) + \underline{1} = \beta(y) + \underline{1}.\end{aligned}$$

Hence (ci) holds. Suppose now that $\Sigma^* \models wBy$. We have that

$$\begin{aligned}\Sigma^* \models y = b^n a x_1 \dots x_n\ \&\ wBy \rightarrow wBb^n \vee w = b^n \vee w = b^n a \vee \\ \vee (\exists z_1 (z_1 B x_1\ \&\ w = b^n a z_1) \vee w = b^n a x_1) \vee \\ \vee \textstyle\bigvee_{1 < i < n} (\exists z_i (z_i B x_i\ \&\ w = b^n a x_1 \dots x_{i-1} z_i) \vee w = b^n a x_1 \dots x_i) \vee \\ \vee \exists z_n (z_n B x_n\ \&\ w = b^n a x_1 \dots x_{n-1} z_n).\end{aligned}$$

We consider one of the cases as an illustration of the proof. Suppose

$$\Sigma^* \models \exists z_i (z_i B x_i\ \&\ w = b^n a x_1 \dots x_{i-1} z_i) \text{ where } 1 < i < n.$$

From $\Sigma^* \models \wedge_{1 \leq j < i} \text{Æ}(x_j)$ we have $\Sigma^* \models \wedge_{1 \leq j < i} \alpha(x_j) = \beta(x_j) + \underline{1}$. On the other hand, from $\Sigma^* \models zBx_i\ \&\ \text{Æ}(x_i)$, we have $\Sigma^* \models \alpha(z_i) \leq \beta(z_i)$. Then

$$\begin{aligned}\Sigma^* \models \alpha(w) &= \alpha(b^n a x_1 \dots x_{i-1} z_i) = \underline{1} + \textstyle\sum_{1 \leq j < i} \alpha(x_j) + \alpha(z_i) = \\ &= \underline{1} + \textstyle\sum_{1 \leq j < i} (\beta(x_j) + \underline{1}) + \alpha(z_i) \leq \sum_{1 \leq j < i} \beta(x_j) + \underline{i} + \beta(z_i) \leq \\ &\leq \underline{n} + \textstyle\sum_{1 \leq j < i} \beta(x_j) + \beta(z_i) = \beta(b^n a x_1 \dots x_{i-1} z_i) = \beta(w),\end{aligned}$$

as required. Detailed consideration of the cases shows that $\Sigma^* \models \forall w\,(wBy \to \alpha(w) \le \beta(w))$. This establishes (cii), and so $\Sigma^* \models Æ(y)$. ⊣

It will be useful to have a sharpened form of this result. For $m \ge 1$, let

$$Æ_{\le m}(x) \equiv Æ(x)\ \&\ \forall u\,(u \subseteq_p x\ \&\ \mathrm{Tally}_b(u) \to \vee_{1\le i\le m} u = b^i),$$

that is, the almost even string x contains no b-tallies of length $> m$. Clearly, we have:

2.8 (a) For any $m \ge 1$, $\Sigma^* \models Æ_{\le m}(x)\ \&\ Æ(y)\ \&\ y \subseteq_p x \to Æ_{\le m}(y)$.
(b) For each $m \ge 1$, $\Sigma^* \models Æ_{\le m}(a)$.
(c) If $m \le n$, then $\Sigma^* \models Æ_{\le m}(x) \to Æ_{\le n}(x)$.

We can then restate 2.7 as follows:

2.9 For any $m(1), \dots, m(n) \ge 1$, $\Sigma^* \models \wedge_{1\le i\le n} Æ_{\le m(i)}(x_i)\ \&\ y = b^n a x_1 \dots x_n \to Æ_{\le k}(y)$, where $k = \max(n, m(1), \dots, m(n))$.
We omit the proof, which is straightforward.

2.10 For each $n \ge 1$, $\Sigma^* \models b^n a y_1 \dots y_n = x_1 aa \to x_1 = b^n \text{ v } b^n B x_1$.

PROOF. We have from $\Sigma^* \models b^n a y_1 \dots y_n = z = x_1 aa$ that $\Sigma^* \models b^n Bz\ \&\ x_1 Bz$. Then $\Sigma^* \models b^n B x_1 \text{ v } b^n = x_1 \text{ v } x_1 B b^n$. Suppose, for a reductio, that $\Sigma^* \models x_1 B b^n$. Then $\Sigma^* \models \exists z_1 b^n = x_1 z_1$ where $\Sigma^* \models \mathrm{Tally}_b(z_1)$. But then $\Sigma^* \models (x_1 z_1) a y_1 \dots y_n = x_1 aa$, that is, $\Sigma^* \models x_1(z_1 a y_1 \dots y_n) = x_1 aa$, whence $\Sigma^* \models z_1 a y_1 \dots y_n = aa$, which is a contradiction because $\Sigma^* \models \mathrm{Tally}_b(z_1)$. Therefore, $\Sigma^* \models b^n B x_1 \text{ v } b^n = x_1$. ⊣

Now, 2.7 tells us that Æ strings are closed under the operation of prefixing a juxtaposition of n such strings with a b-tally of length n followed by a single occurrence of digit a. On the other hand, 2.4 tells us that the juxtaposed Æ strings are uniquely recoverable from the resulting string. Along with 2.6, all this suggests that the Æ strings may be inductively characterized as the smallest set of strings that contains the single digit *a* and is closed under the n-ary juxtaposition operations of the type just described, and that each Æ string has a unique Æ decomposition. That is exactly what we'll do now.

THEOREM 2.11 (*Unique decomposition of Æ strings*). *Let* u *be any closed* $\mathcal{L}_C$*-term. If* $\Sigma^* \models Æ(u)\ \&\ u \ne a$, *then for some* $n \ge 1$,

$$\Sigma^* \models \exists!\, u_1 \dots \exists!\, u_n (\wedge_{1\le i\le n} Æ(u_i)\ \&\ u = b^n a u_1 \dots u_n).$$

PROOF. Assume $\Sigma^* \models Æ(u)\ \&\ u \ne a$. By 2.6 we have that $\Sigma^* \models \exists y_1 \dots y_n u = b^n a y_1 \dots y_n$. We need to show that there are unique Æ strings $u_1, \dots, u_n$ such that $\Sigma^* \models u_1 \dots u_n = y_1 \dots y_n$.

If $n = 1$, then $\Sigma^* \models u = bay$. From hypothesis $\Sigma^* \models Æ(u)$ we have by 2.5(b) that $\Sigma^* \models Æ(y)$. So we may let $u_1 = y$. Uniqueness is immediate.

So we may assume that $n \ge 2$. By 2.2, from $\Sigma^* \models Æ(u)\ \&\ u \ne a$ we have that $\Sigma^* \models aaEu$. Hence $\Sigma^* \models \exists x_1 u = x_1 aa$. By 2.10, we have that

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$\Sigma^* \models x_1 = b^n \vee b^nBx_1$. Now, we cannot have $\Sigma^* \models x_1 = b^n$, for then $\Sigma^* \models b^nay_1 \dots y_n = b^naa$, whence $\Sigma^* \models y_1 \dots y_n = a$, which is impossible given that $n \geq 2$. Therefore, $\Sigma^* \models b^nBx_1$. Then $\Sigma^* \models \exists x_2 x_1 = b^n x_2$, whence

$$\Sigma^* \models b^nay_1 \dots y_n = u = x_1aa = (b^nx_2)aa = b^n(x_2aa).$$

Then $\Sigma^* \models ay_1 \dots y_n = x_2aa$, and further, $\Sigma^* \models x_2 = a \vee aBx_2$.

If $\Sigma^* \models x_2 = a$, then $\Sigma^* \models ay_1 \dots y_n = aaa$, so $\Sigma^* \models y_1 \dots y_n = aa$. But then $n = 2$ and $\Sigma^* \models y_1 = y_2 = a$. If we let $u_1 = u_2 = a$ we have that $\Sigma^* \models$ Æ(u_1) & Æ(u_2) and uniqueness is immediate. In this case we are done. So we may assume that $\Sigma^* \models aBx_2$. Then $\Sigma^* \models \exists x_3 x_2 = ax_3$, that is, $\Sigma^* \models b^nay_1 \dots y_n = u = b^nax_3aa$. Thus $\Sigma^* \models x_3aaEu$, and from $\Sigma^* \models$ Æ(u), by 2.3(b), we have that $\Sigma^* \models \alpha(x_3aa) \geq \beta(x_3aa) + \underline{1}$. Then x_3aa has at least one initial segment v, namely itself, such that $\Sigma^* \models \alpha(v) \geq \beta(v) + \underline{1}$. Let u_1 be the *shortest* such initial segment of x_3aa.

Claim 1. $\Sigma^* \models$ Æ(u_1).

By choice of u_1 we have that $\Sigma^* \models \alpha(u_1) \geq \beta(u_1) + 1$. We may assume that $\Sigma^* \models u_1 \neq a$; otherwise, we are done. Suppose, for a reductio, that $\Sigma^* \models \alpha(u_1) > \beta(u_1) + \underline{1}$. Then $\Sigma^* \models u_1 \neq b$.

Suppose that $\Sigma^* \models aEu_1$. Then $\Sigma^* \models \exists z\, u_1 = za$. Then from hypothesis $\Sigma^* \models \alpha(u_1) > \beta(u_1) + \underline{1}$, we obtain $\Sigma^* \models \alpha(z) = \alpha(u_1) – \underline{1} > \beta(u_1) = \beta(z)$, so $\Sigma^* \models \alpha(z) \geq \beta(z) + \underline{1}$. Since $\Sigma^* \models zBu_1$ & $(u_1Bx_3aa \vee u_1 = x_3aa)$, we have that $\Sigma^* \models zBx_3aa$. But this contradicts the choice of u_1. Hence $\Sigma^* \models \neg aEu_1$.

Suppose that $\Sigma^* \models bEu_1$. Then $\Sigma^* \models \exists z\, u_1 = zb$; hence

$$\Sigma^* \models \alpha(z) = \alpha(zb) = \alpha(u_1) \geq \beta(u_1) + \underline{1} = \beta(zb) + \underline{1} > \beta(z) + \underline{1},$$

where $\Sigma^* \models zBu_1$ & $(u_1Bx_3aa \vee u_1 = x_3aa)$. So $\Sigma^* \models zBx_3aa$, again contradicting the choice of u_1. Hence also $\Sigma^* \models \neg bEu_1$.

But then we have that $\Sigma^* \models u_1 \neq a$ & $u_1 \neq b$ & $\neg aEu_1$ & $\neg bEu_1$, which is impossible. Therefore $\Sigma^* \models \alpha(u_1) = \beta(u_1) + \underline{1}$. By choice of u_1 it follows also that $\Sigma^* \models \forall w\, (wBu_1 \to \alpha(w) \leq \beta(w))$. This completes the proof of Claim 1.

We have that $\Sigma^* \models u_1Bx_3aa \vee u_1 = x_3aa$.

If $\Sigma^* \models u_1 = x_3aa$, then $\Sigma^* \models b^nau_1 = b^nax_3aa = u$. Since $\Sigma^* \models$ Æ(u_1), we then have $\Sigma^* \models \alpha(u) = \alpha(b^nau_1) = \underline{1} + \alpha(u_1) = \underline{1} + (\beta(u_1) + \underline{1}) = \beta(u_1) + \underline{2}$, whereas $\Sigma^* \models \beta(u) = \beta(b^nau_1) = \underline{n} + \beta(u_1)$. But from hypothesis $\Sigma^* \models$ Æ(u) we have $\Sigma^* \models \alpha(u) = \beta(u) + 1$. Hence $\Sigma^* \models \beta(u_1) + \underline{2} = \beta(u_1) + \underline{n} + \underline{1}$, contradicting hypothesis $n \geq 2$. Therefore $\Sigma^* \models u_1 \neq x_3aa$. But then it follows that $\Sigma^* \models u_1Bx_3aa$. So $\Sigma^* \models \exists v_1 x_3aa = u_1v_1$, whence $\Sigma^* \models b^nau_1v_1 = b^nax_3aa = u$, which means that $\Sigma^* \models b^nau_1Bu$.

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CLAIM 2. *For each* i, $1 \leq \mathrm{i} < \mathrm{n}$,

$\Sigma^* \models \forall \mathrm{v}_1 \ldots \mathrm{v}_\mathrm{i}(\wedge_{1\leq \mathrm{j} \leq \mathrm{i}} \text{Æ}(\mathrm{v}_\mathrm{j})\ \&\ (\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i})\mathrm{Bu} \rightarrow \exists \mathrm{v}_{\mathrm{i}+1}(\text{Æ}(\mathrm{v}_{\mathrm{i}+1})\ \&$
$\&\ ((\underline{\mathrm{i}+1} < \underline{\mathrm{n}}\ \&\ (\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{v}_{\mathrm{i}+1})\mathrm{Bu})\ \mathrm{v}\ (\underline{\mathrm{i}+1} = \underline{\mathrm{n}}\ \&\ \mathrm{u} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{v}_{\mathrm{i}+1})))).$

Assume that $\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{i}$ have been picked such that

$$\Sigma^* \models \wedge_{1\leq \mathrm{j}\leq \mathrm{i}} \text{Æ}(\mathrm{v}_\mathrm{j})\ \&\ (\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i})\mathrm{Bu}.$$

Then $\Sigma^* \models \exists \mathrm{w}_\mathrm{i}\ \mathrm{u} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i}$; hence $\Sigma^* \models \mathrm{w}_\mathrm{i}\mathrm{Eu}$. From hypothesis $\Sigma^* \models \text{Æ}(\mathrm{u})$ by 2.3(b) we then have that $\Sigma^* \models \alpha(\mathrm{w}_\mathrm{i}) \geq \beta(\mathrm{w}_\mathrm{i}) + \underline{1}$. Then we have

$$\begin{aligned}\Sigma^* \models \alpha(\mathrm{u}) &= \alpha(\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i}) = \underline{1} + \alpha(\mathrm{v}_1 \ldots \mathrm{v}_\mathrm{i}) + \alpha(\mathrm{w}_\mathrm{i}) = \\ &= \underline{1} + \textstyle\sum_{1\leq \mathrm{j}\leq \mathrm{i}}\alpha(\mathrm{v}_\mathrm{j}) + \alpha(\mathrm{w}_\mathrm{i}) = \underline{1} + \sum_{1\leq \mathrm{j}\leq \mathrm{i}}(\beta(\mathrm{v}_\mathrm{j}) + \underline{1}) + \alpha(\mathrm{w}_\mathrm{i}) = \\ &= \underline{1} + \underline{\mathrm{i}} + \textstyle\sum_{1\leq \mathrm{j}\leq \mathrm{i}}\beta(\mathrm{v}_\mathrm{j}) + \alpha(\mathrm{w}_\mathrm{i}),\end{aligned}$$

whereas $$\begin{aligned}\Sigma^* \models \beta(\mathrm{u}) = \beta(\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i}) &= \underline{\mathrm{n}} + \beta(\mathrm{v}_1 \ldots \mathrm{v}_\mathrm{i}) + \beta(\mathrm{w}_\mathrm{i}) = \\ &= \underline{\mathrm{n}} + \textstyle\sum_{1\leq \mathrm{j}\leq \mathrm{i}}\beta(\mathrm{v}_\mathrm{j}) + \beta(\mathrm{w}_\mathrm{i}).\end{aligned}$$

But from hypothesis $\Sigma^* \models \text{Æ}(\mathrm{u})$ we have $\Sigma^* \models \alpha(\mathrm{u}) = \beta(\mathrm{u}) + \underline{1}$. Then $\Sigma^* \models \underline{1} + \underline{\mathrm{i}} + \sum_{1\leq \mathrm{j}\leq \mathrm{i}}\beta(\mathrm{v}_\mathrm{j}) + \alpha(\mathrm{w}_\mathrm{i}) = \underline{\mathrm{n}} + \sum_{1\leq \mathrm{j}\leq \mathrm{i}}\beta(\mathrm{v}_\mathrm{j}) + \beta(\mathrm{w}_\mathrm{i}) + \underline{1}$, whence

$$\Sigma^* \models \underline{\mathrm{i}} + \alpha(\mathrm{w}_\mathrm{i}) = \underline{\mathrm{n}} + \beta(\mathrm{w}_\mathrm{i}). \tag{\#}$$

We pick $\mathrm{v}_{\mathrm{i}+1}$ as follows: if $\Sigma^* \models \mathrm{w}_\mathrm{i} = \mathrm{a}$, we let $\mathrm{v}_{\mathrm{i}+1} = \mathrm{a}$; if $\Sigma^* \models \mathrm{w}_\mathrm{i} \neq \mathrm{a}$, let $\mathrm{v}_{\mathrm{i}+1}$ be the *shortest* initial segment v of w_i such that $\Sigma^* \models \alpha(\mathrm{v}) \geq \beta(\mathrm{v}) + \underline{1}$.

If $\Sigma^* \models \mathrm{w}_\mathrm{i} = \mathrm{a}$, then $\Sigma^* \models \mathrm{u} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{a}$, and we of course have that $\Sigma^* \models \text{Æ}(\mathrm{v}_{\mathrm{i}+1})$. Since $\Sigma^* \models \mathrm{w}_\mathrm{i} = \mathrm{a}$, we have $\Sigma^* \models \alpha(\mathrm{w}_\mathrm{i}) = \underline{1}\ \&\ \beta(\mathrm{w}_\mathrm{i}) = \underline{0}$, so from (#) we obtain $\Sigma^* \models \underline{\mathrm{i}} + \underline{1} = \underline{\mathrm{n}}$.

If $\Sigma^* \models \mathrm{w}_\mathrm{i} \neq \mathrm{a}$, then the same argument as in Claim 1 with $\mathrm{v}_{\mathrm{i}+1}$ in place of u_1 and w_i in place of $\mathrm{x}_3\mathrm{aa}$ shows that $\Sigma^* \models \alpha(\mathrm{v}_{\mathrm{i}+1}) = \beta(\mathrm{v}_{\mathrm{i}+1}) + \underline{1}$, and that $\Sigma^* \models \text{Æ}(\mathrm{v}_{\mathrm{i}+1})$. By choice of $\mathrm{v}_{\mathrm{i}+1}$ we have that $\Sigma^* \models \mathrm{v}_{\mathrm{i}+1} = \mathrm{w}_\mathrm{i}\ \mathrm{v}\ \mathrm{v}_{\mathrm{i}+1}\mathrm{Bw}_\mathrm{i}$.

If $\Sigma^* \models \mathrm{v}_{\mathrm{i}+1} = \mathrm{w}_\mathrm{i}$, then $\Sigma^* \models \mathrm{u} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{v}_{\mathrm{i}+1}$. Then $\Sigma^* \models \alpha(\mathrm{w}_\mathrm{i}) = \beta(\mathrm{w}_\mathrm{i}) + \underline{1}$, so from (#) we obtain

$\Sigma^* \models \underline{\mathrm{i}} + \beta(\mathrm{w}_\mathrm{i}) + \underline{1} = \underline{\mathrm{n}} + \beta(\mathrm{w}_\mathrm{i})$, whence $\Sigma^* \models \underline{\mathrm{i}} + \underline{1} = \underline{\mathrm{n}}$.

If $\Sigma^* \models \mathrm{v}_{\mathrm{i}+1}\mathrm{Bw}_\mathrm{i}$, then $\Sigma^* \models \exists \mathrm{w}_{\mathrm{i}+1}\mathrm{w}_\mathrm{i} = \mathrm{v}_{\mathrm{i}+1}\mathrm{w}_{\mathrm{i}+1}$. So

$$\Sigma^* \models \mathrm{u} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{w}_\mathrm{i} = \mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}(\mathrm{v}_{\mathrm{i}+1}\mathrm{w}_{\mathrm{i}+1}).$$

Then we have that

$$\Sigma^* \models \alpha(\mathrm{w}_\mathrm{i}) = \alpha(\mathrm{v}_{\mathrm{i}+1}\mathrm{w}_{\mathrm{i}+1}) = \alpha(\mathrm{v}_{\mathrm{i}+1}) + \alpha(\mathrm{w}_{\mathrm{i}+1}) = \beta(\mathrm{v}_{\mathrm{i}+1}) + \underline{1} + \alpha(\mathrm{w}_{\mathrm{i}+1})$$

and $\beta(\mathrm{w}_\mathrm{i}) = \beta(\mathrm{v}_{\mathrm{i}+1}\mathrm{w}_{\mathrm{i}+1}) = \beta(\mathrm{v}_{\mathrm{i}+1}) + \beta(\mathrm{w}_{\mathrm{i}+1})$. Hence from (#) we obtain $\Sigma^* \models \underline{\mathrm{i}} + \beta(\mathrm{v}_{\mathrm{i}+1}) + \underline{1} + \alpha(\mathrm{w}_{\mathrm{i}+1}) = \underline{\mathrm{n}} + \beta(\mathrm{v}_{\mathrm{i}+1}) + \beta(\mathrm{w}_{\mathrm{i}+1})$. But $\Sigma^* \models \mathrm{w}_{\mathrm{i}+1}\mathrm{Eu}$, so from $\Sigma^* \models \text{Æ}(\mathrm{u})$ by 2.3(b), we have $\Sigma^* \models \alpha(\mathrm{w}_{\mathrm{i}+1}) \geq \beta(\mathrm{w}_{\mathrm{i}+1}) + \underline{1} > \beta(\mathrm{w}_{\mathrm{i}+1})$. But then $\Sigma^* \models \underline{\mathrm{i}} + \underline{1} < \mathrm{n}\ \&\ (\mathrm{b}^\mathrm{n}\mathrm{av}_1 \ldots \mathrm{v}_\mathrm{i}\mathrm{v}_{\mathrm{i}+1})\mathrm{Bu}$.

This completes the proof of Claim 2.

From Claim 1 and n–1 applications of Claim 2 we obtain strings $u_1, \dots, u_n$ such that $\Sigma^* \models \wedge_{1\leq i\leq n} Æ(u_i)$ & $b^n ay_1 \dots y_n = u = b^n au_1 \dots u_n$. Hence $\Sigma^* \models y_1 \dots y_n = u_1 \dots u_n$. Then uniqueness follows from 2.4. ⊣

THEOREM 2.12. *Let* u *be any closed* $\mathcal{L}_C$*-term.*

For each $m \geq 1$, if $\Sigma^* \models Æ_{\leq m}(u)$ & $u \neq a$, then for some n, $1 \leq n \leq m$,

$$\Sigma^* \models \exists!\, u_1 \dots \exists!\, u_n(\wedge_{1\leq i\leq n} Æ_{\leq m}(u_i) \,\&\, u = b^n au_1 \dots u_n).$$

PROOF. Assume $\Sigma^* \models Æ_{\leq m}(u)$. Then $\Sigma^* \models Æ(u)$. We proceed exactly as in the proof of Theorem 2.11. If $\Sigma^* \models u = b^n ay_1 \dots y_n$ we have that $n \leq m$. If $n = 1$ and $\Sigma^* \models u_1 = a$, then $\Sigma^* \models Æ_{\leq 1}(u_1)$, and if $n = 2$ and $\Sigma^* \models u_1 = u_2 = a$, then $\Sigma^* \models Æ_{\leq 1}(u_1)$ & $Æ_{\leq 1}(u_2)$ and so $\Sigma^* \models Æ_{\leq m}(u_1)$ & $Æ_{\leq m}(u_2)$. In the general case, once we reach the end of the proof of Theorem 2.11, since $\Sigma^* \models y_1 \dots y_n = u_1 \dots u_n$ we have that $\Sigma^* \models \wedge_{1\leq i\leq n} u_i \subseteq_p u$. But then from the principal hypothesis $\Sigma^* \models Æ_{\leq m}(u)$ it follows that $\Sigma^* \models \wedge_{1\leq i\leq n} Æ_{\leq m}(u_i)$ as claimed. ⊣

REMARK. If we let

$$Æ_m(x) \equiv Æ(x) \,\&\, \forall t\, (\text{Tally}_b(t) \,\&\, (taBx \text{ v } ata \subseteq_p x) \to t = b^m),$$

we have:

2.13 (a) $\Sigma^* \models \forall x, y\, (Æ_m(x) \,\&\, Æ(y) \,\&\, y \subseteq_p x \to Æ_m(x))$.
(b) For each $m \geq 1$, $\Sigma^* \models Æ_m(a)$.
(c) If $m \neq n$, then $\Sigma^* \models \forall x\, (x \neq a \,\&\, Æ_m(x) \to \neg Æ_n(x))$.
We omit the proof.

2.14 Let u be any closed $\mathcal{L}_C$-term. For each $m \geq 1$, if $\Sigma^* \models Æ_m(u)$ & $u \neq a$, then $\Sigma^* \models \exists x_1 \dots x_m u = b^m ax_1 \dots x_m$.

PROOF. Assume $\Sigma^* \models Æ_m(u)$ & $u \neq a$. Then $\Sigma^* \models Æ(u)$, and by 2.6 we have that, for some $n \geq 1$, $\Sigma^* \models \exists u_1 \dots u_n u = b^n au_1 \dots u_n$. Then $\Sigma^* \models \text{Tally}_b(b^n)$ & $b^n aBu$, so from $\Sigma^* \models Æ_m(u)$ we have $\Sigma^* \models b^n = b^m$, whence $n = m$. But then we may take $x_1, \dots, x_m$ to be $u_1, \dots, u_n$. ⊣

2.15 (a) For each $m \geq 1$,

$$\Sigma^* \models \forall x_1 \dots x_m \forall y\, (\wedge_{1\leq i\leq m} Æ_m(x_i) \,\&\, y = b^m ax_1 \dots x_m \to Æ_m(y)).$$

(b) Let u be any closed $\mathcal{L}_C$-term. For each $m \geq 1$, if $\Sigma^* \models Æ_m(u)$ & $u \neq a$, then $\Sigma^* \models \exists!\, u_1 \dots \exists!\, u_m(u = b^m au_1 \dots u_m \,\&\, \wedge_{1\leq i\leq m} Æ_m(u_i)))$.
These are proved analogously to 2.9 and Theorem 2.12 with appropriate modifications.

The inductive characterization of $Æ_{\leq n}$ strings given in 2.9 and Theorem 2.12 opens up the possibility of defining operations by recursion with respect to the corresponding generating relations for $Æ_{\leq n}$ strings. In Section 4 we will need to employ operations f of that type, satisfying, e.g., the schema

$$f(a) = p \quad f(b^k ax_1 \dots x_k) = g_k(f(x_1), \dots, f(x_k)) \text{ for any } k,\, 1 \leq k \leq n,$$

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where p is a fixed string and g_k some given k-ary operations on strings taking values from $Æ_{\leq m}$ for some fixed m.

THEOREM 2.16 (*Recursion on* $Æ_{\leq n}$ *strings*). *Let* $n \geq 1$, *let* p *be a closed* $\mathcal{L}_C$*-term, and let* $G_k(y_1, \dots, y_k, y)$ *be* $\mathcal{L}_C$*-formulae for* $1 \leq k \leq n$. *Let* $m \geq 1$. *Suppose that* (*a*) $\Sigma^* \models Æ_{\leq m}(p)$, *and*
(*b*) $\Sigma^* \models \forall y_1 \dots y_k(\wedge_{1\leq i\leq k}Æ_{\leq m}(y_i) \rightarrow \exists!\, y\ (Æ_{\leq m}(y)\ \&\ G_k(y_1, \dots, y_k, y)))$.

Then there is an $\mathcal{L}_C$*-formula* $F(x, y)$ *such that*:

(i) $\Sigma^* \models \forall x\ \big(Æ_{\leq n}(x) \rightarrow \exists!\, y\ (Æ_{\leq m}(y)\ \&\ F(x, y)\big)$.
(ii) $\Sigma^* \models \forall y\ (F(a, y) \leftrightarrow y = p)$.
(iii) *For each* k, $1 \leq k \leq n$,
$$\Sigma^* \models \forall x_1 \dots x_k \forall y_1 \dots y_k\big(\wedge_{1\leq i\leq k}F(x_i, y_i) \rightarrow \\ \rightarrow \forall x, z\ (x = b^k a x_1 \dots x_k \rightarrow (F(x, z) \leftrightarrow G_k(y_1, \dots, y_k, z)))\big).$$

PROOF. We rely on the coding scheme for sequences of strings explained in [3, Section 4], and follow the notation used there. In particular, we have that $\Sigma^* \models \mathrm{Pair}[x, y, z]$ iff string z codes the pair of strings x and y, $\Sigma^* \models \mathrm{Set}(x)$ iff x is a set code, and $\Sigma^* \models x\ \varepsilon\ y$ just in case string x is a member of the set coded by string y. Let $R_k(u, x)$ abbreviate the $\mathcal{L}_C$-formula

$$\forall z_1{\subseteq_p}x \dots \forall z_k{\subseteq_p}x \forall y_1 \dots y_k \forall u_1 \dots u_k \forall y, w\ \big(\wedge_{1\leq i\leq k}Æ_{\leq k}(z_i)\ \&\ \wedge_{1\leq i\leq k}\mathrm{Pair}[z_i, y_i, u_i]\ \& \\ \&\ G_k(y_1, \dots, y_k, y)\ \&\ w = b^k a z_1 \dots z_k \rightarrow \exists u'{\subseteq_p}u\ (\mathrm{Pair}[w, y, u']\ \&\ u'\varepsilon\ u)\big).$$

Let $\mathrm{Comp}_{\leq n}(u, x)$ abbreviate

$$\mathrm{Set}(u)\ \&\ Æ_{\leq n}(x)\ \&\ \exists w\ (\mathrm{Pair}[a, p, w]\ \&\ w\ \varepsilon\ u)\ \&\ \wedge_{1\leq k\leq n}R_k(u, x)\ \& \\ \&\ \forall z \forall v_1, v_2, w_1, w_2\big(\mathrm{Pair}[z, v_1, w_1]\ \&\ \mathrm{Pair}[z, v_2, w_2]\ \&\ w_1\ \varepsilon\ u\ \&\ w_2\ \varepsilon\ u \rightarrow \\ \rightarrow v_1 = v_2\ \&\ w_1 = w_2\big).$$

Let $\mathrm{MinComp}_{\leq n}(u, x)$ abbreviate

$$\mathrm{Comp}_{\leq n}(u, x)\ \&\ \forall u'(\mathrm{Comp}_{\leq n}(u', x) \rightarrow \forall y\ (y\ \varepsilon\ u \rightarrow y\ \varepsilon\ u'))\ \& \\ \&\ \forall z, y, w\ (\mathrm{Pair}[z, y, w]\ \&\ w\ \varepsilon\ u \rightarrow z{\subseteq_p}x\ \&\ Æ_{\leq n}(z)).$$

We then let $F(x, y) \equiv \exists u, w(\mathrm{MinComp}_{\leq n}(u, x)\ \&\ \mathrm{Pair}[x, y, w]\ \&\ w\ \varepsilon\ u)$.

Now, suppose (a) and (b) hold. For (i), assume that $\Sigma^* \models Æ_{\leq n}(x)$. We argue by induction on the generating relation of $Æ_{\leq n}$ strings. If $\Sigma^* \models x = a$, let w_0 be the string such that $\Sigma^* \models \mathrm{Pair}[a, p, w_0]$, and let u_0 be the string that codes the singleton sequence of w_0. Then the desired claim follows immediately from (a) and the definition of $F(x, y)$. Assume $\Sigma^* \models x = b^k a x_1 \dots x_k$ where the claim holds for $x_1, \dots, x_k$ with $k \leq n$. Then

$$\Sigma^* \models \exists!\, y_1 \dots \exists!\, y_k(\wedge_{1\leq i\leq k}Æ_{\leq m}(y_i)\ \&\ \wedge_{1\leq i\leq k}F(x_i, y_i)).$$

Let $w_1, \dots, w_k$ be the strings such that $\Sigma^* \models \wedge_{1\leq i\leq k} \mathrm{Pair}[x_i, y_i, w_i]$, and let u be the string that codes the sequence $w_1, \dots, w_k$. From (b) we have that $\Sigma^* \models \exists!\, y\ (Æ_{\leq m}(y)\ \&\ G_k(y_1, \dots, y_k, y))$. Pick the string w such that

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$\Sigma^* \models \mathrm{Pair}[\mathrm{b}^k \mathrm{a} \mathrm{x}_1 \dots \mathrm{x}_k, \mathrm{y}, \mathrm{w}]$, and let u' be the string that codes the sequence $\mathrm{w}_1, \dots, \mathrm{w}_k, \mathrm{w}$. Then the claim follows immediately from the definition of $\mathrm{F}(\mathrm{x}, \mathrm{y})$. (ii) and (iii) follow straightforwardly from the definition of $\mathrm{F}(\mathrm{x}, \mathrm{y})$ and (i). ⊣

We say that the function whose graph is expressed by the formula $\mathrm{F}(\mathrm{x}, \mathrm{y})$ is defined by $Æ_{\leq n} \to Æ_{\leq m}$ recursion. If in the proof of Theorem 2.16 the formulae $Æ_{\leq k}(\mathrm{x})$ and $\mathrm{G}_k(\mathrm{y}_1, \dots, \mathrm{y}_k, \mathrm{y})$ for $1 \leq k \leq n$ and $Æ_{\leq n}(\mathrm{x})$, respectively, are replaced by $Æ_n(\mathrm{x})$ and $\mathrm{G}(\mathrm{y}_1, \dots, \mathrm{y}_k, \mathrm{y})$ for $1 \leq k \leq n$ and $Æ_n(\mathrm{x})$, respectively, we obtain:

THEOREM 2.17 (Recursion on $Æ_n$ strings). *Let* $n \geq 1$, *let* p *be a closed* $\mathcal{L}_C$*-term, and let* $\mathrm{G}(\mathrm{y}_1, \dots, \mathrm{y}_k, \mathrm{y})$ *be an* $\mathcal{L}_C$*-formula. Let* $m \geq 1$. *Suppose that*
(a) $\Sigma^* \models Æ_{\leq m}(\mathrm{p})$, *and*
(b) $\Sigma^* \models \forall \mathrm{y}_1 \dots \mathrm{y}_n \big(\wedge_{1 \leq i \leq n} Æ_n(\mathrm{y}_i) \to \exists! \, \mathrm{y} \, (Æ_{\leq m}(\mathrm{y}) \,\&\, \mathrm{G}(\mathrm{y}_1, \dots, \mathrm{y}_k, \mathrm{y})\big)$.
Then there is an $\mathcal{L}_C$*-formula* $\mathrm{F}(\mathrm{x}, \mathrm{y})$ *such that*:

(i) $\Sigma^* \models \forall \mathrm{x} \, (Æ_n(\mathrm{x}) \to \exists! \, \mathrm{y} \, (Æ_{\leq m}(\mathrm{y}) \,\&\, \mathrm{F}(\mathrm{x}, \mathrm{y}))$.
(ii) $\Sigma^* \models \forall \mathrm{y} \, (\mathrm{F}(\mathrm{a}, \mathrm{y}) \leftrightarrow \mathrm{y} = \mathrm{p})$.
(iii) $\Sigma^* \models \forall \mathrm{x}_1 \dots \mathrm{x}_n \forall \mathrm{y}_1 \dots \mathrm{y}_n \big(\wedge_{1 \leq i \leq n} \mathrm{F}(\mathrm{x}_i, \mathrm{y}_i) \to$
$\to \forall \mathrm{x}, \mathrm{z} \, (\mathrm{x} = \mathrm{b}^n \mathrm{a} \mathrm{x}_1 \dots \mathrm{x}_n \to (\mathrm{F}(\mathrm{x}, \mathrm{z}) \leftrightarrow \mathrm{G}(\mathrm{y}_1, \dots, \mathrm{y}_k, \mathrm{z})))\big)$.

§3. Catalan coding of trees. Theorem 2.11 yields an algorithm—"peel off the prefix $\mathrm{b}^n\mathrm{a}$ and search for the shortest initial segment of the remainder having the $+1$ property"—which, when applied to a given Æ string x and then repeatedly to the resulting component Æ strings, eventually terminates in a unique tree-like arrangement T_x of substrings of x in which every node is an Æ string and all of the endnodes = a. And, conversely, given any finite tree T, by labelling each endnode with a, we can obtain, by repeatedly applying 2.7, a unique Æ string c_T that we may think of as a code for T, providing an explicit formal representation in linear form of the characteristic structure of the tree T.

There are other ways to directly display the structure of planar trees in linear form, e.g., via symmetric bracketing. Because in concatenation theory parentheses are used associatively, and the symmetric bracketing notation is not associative—x(xx) does not represent the same tree as (xx)x—for our purposes we need to be able to rely exclusively on juxtaposition along with some indicators of arity as the sole means of expressing the tree structure by concatenation of binary strings. A simplified variant of this approach was used in [3] (see [3, 2.2]) to obtain a result analogous to Theorem 2.12, which served there as a basis for a coding of full binary, or dyadic, finite trees, where every non-terminal node has exactly two immediate descendants. Even though here we are allowing, in principle, any finite number of immediate

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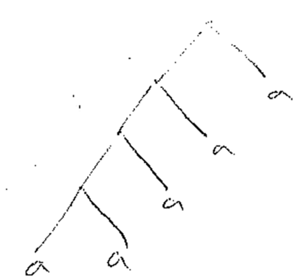
descendants, remarkably, the *very same strings*, the Æ strings, function as codes for trees in both cases.

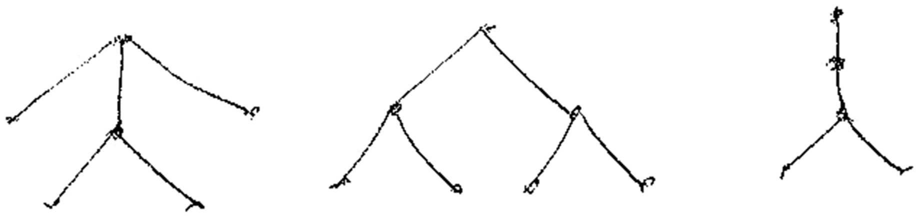
We call the coding of finite trees given here the *Catalan coding*. Eugene Charles Catalan (1814–1894) identified the numbers, now named after him, that count different ways to fully (symmetrically) parenthesize a string of given lengths. As we defined them, Æ strings are finite words in the alphabet $\{a, b\}$ of the form wa where w is either empty or has exactly as many a's as b's and has no initial segment in which a's strictly outnumber b's. Words with this property have been extensively studied in combinatorics where they are called *Catalan words* (see, e.g., [5]).

The dyadic coding given in [3] and the Catalan coding parse Æ strings differently. For example, bbbbaaaaa $(= \mathrm{b}^4\mathrm{a}^5)$ produces, via the dyadic coding, the full binary tree on the left,



and via the Catalan coding the 4-ary tree on the right. Both coding schemes group codes of immediate subtrees analogously to the way in which argument terms of a given binary, and, respectively, m-ary operation are successively listed when written in the so-called Polish notation. But the tree branchings are recorded in different ways: in the dyadic code, each b stands for a dyadic branching node, in the Catalan coding each block of consecutive b's followed by a single a signals a branching node, with the number of consecutive b's in the same block indicating the number of that node's immediate descendants. Consequently, in the Catalan coding, the a's in a given tree code count the nodes and the b's the edges of the tree. Nonetheless, some basic information about the tree coded by the string—the number of branching nodes (= the number of blocks of consecutive b's) and the number of terminal nodes (= the number of a's in the dyadic case, and = the number of a's minus the number of blocks of consecutive b's in the present scheme)—and, consequently, the total number of nodes in the tree (= the total number of digits and = the total number of a's, resp.) are read off the encoding Æ string essentially in the same way in both codings.

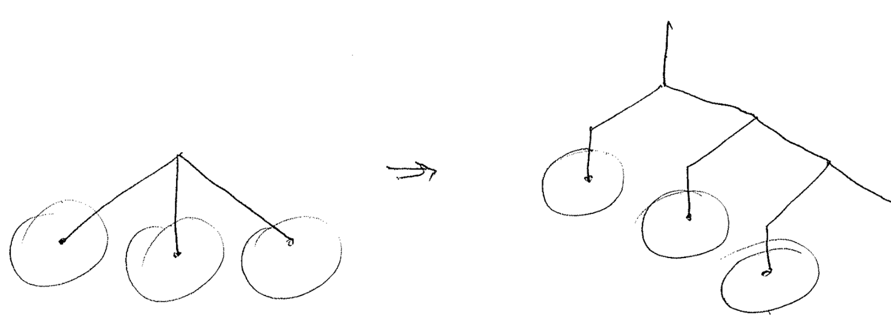
The Catalan coding offers a practical and easy way to communicate descriptions of the immense variety of finite planar trees, allowing any number, including single, immediate descendants: for example,



are coded by $b^3a^2b^2a^4$, $b^2ab^2a^3b^2a^3$, and $babab^2a^3$, respectively. (Always ignore the first *a* after a block of consecutive *b*'s.) Restricting attention to $Æ_{\leq m}$ and $Æ_m$ strings allows us to consider in particular $\leq$m- and m-, respectively, trees with branchings allowing no more than m and exactly m, respectively, immediate descendants, $m \geq 1$.

The fact that the Æ strings work as codes just as well for the more general class of trees of arbitrary arity as they do for dyadic trees suggests that in the specific setting of concatenation theory there may be a way to systematically interpret statements about the former in a domain that consists exclusively of binary trees or their formal simulacra. We now proceed to develop this idea.

§4. Interpreting $T_{\leq m}$ **in** $T_{\leq 2}$**.** In contrast to T_2, the theory $T_{\leq 2}$ explicitly allows for trees having nodes with single immediate descendants. We exploit this feature to show that such $\leq$2-trees can encode information about multiple branchings. Consider, for example, the simple triple-branching tree b^3a^4 on the left, and the tree



$bab^2a(ba^2)b^2a(ba^2)b^2a(ba^2)a$ on the right. The latter has three binary branchings, and in each of these the left branch has a single node ba^2 which in turn has a single terminal node as its sole immediate descendant. This is meant to capture the ternary connections of the root node of b^3a^4. The right branchings of the $\leq$2-tree serve as the supporting "spine" to express this structure. So we are replacing an m-branching by m binary branchings with singletons branching to the left. Here is a slightly more complicated example:

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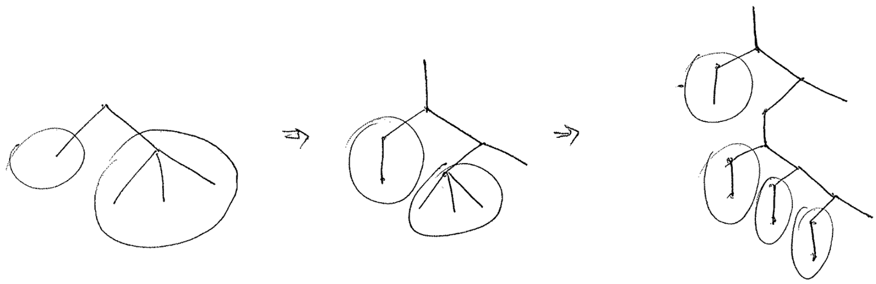

$$b^2a^2b^3a^4 \Rightarrow bab^2a((ba^2)b^2a(b^3a^4))a \Rightarrow$$
$$\Rightarrow bab^2a(ba^2)b^2a(bab^2a(ba^2)b^2a(ba^2)b^2a(ba^2)a)a.$$

(The parentheses serve to highlight the replacements—they are not essential to reading the tree code.). For this to work generally, the replacement procedure has to operate inductively, starting with the root of the $\leq$ m-tree systematically transforming it step-by-step by working down its branches until all of its branchings—single, binary, or multiple—are appropriately converted. The ultimate result will be a $\leq$2-tree resembling a funhouse chandelier heavily tilting to the left, with single drop crystals at all (and only) the left terminal nodes.

More formally, for fixed m, we define a map $\theta(x)$ on $Æ_{\leq m}$ strings such that $\theta(a) = baa$ and $\theta(b^k ax_1 \dots x_k) = bab^2a\theta(x_1)b^2a\theta(x_2)\dots b^2a\theta(x_k)a$ for $k \leq m$. The operation θ is defined by $Æ_{\leq m} \to Æ_{\leq 2}$ recursion, by letting $p = baa$ and $G_k(y_1, \dots, y_k, y)$ be the $\mathcal{L}_C$-formulae

$$\exists z_1 \dots z_k (z_1 = b^2ay_k \;\&\; \wedge_{1\leq j<k} z_{j+1} = b^2ay_{k-j}z_j \;\&\; y = baz_ka).$$

Hence by Theorem 2.16 its graph is expressible in $\mathcal{L}_C$. Then

$$\Sigma^* \models \forall x \,(Æ_{\leq m}(x) \to Æ_{\leq 2}(\theta(x))).$$

We say that the $\leq$ 2-tree $\theta(x)$ is an SLS-code (singleton/left singleton) code of the $\leq$m-tree x.

To show that $T_{\leq m}$ is formally interpretable in $T_{\leq 2}$ we need to reformulate $T_{\leq m}$ in relational vocabulary $\{0, T_1, \dots, T_m, \sqsubseteq\}$ where T_k for $1 \leq k \leq m$ are $(k+1)$-place relational symbols. As axioms we take the universal closures of:

($T1_n$) $\neg T_n(x_1, \dots, x_n, 0)$ for each n, $1 \leq n \leq m$,

($T2_{k,n}$) $T_k(x_1, \dots, x_k, x) \;\&\; T_n(y_1, \dots, y_n, y) \to x \neq y$ for each k, n, $1 \leq k < n \leq m$,

($T3_n$) $T_n(x_1, \dots, x_n, x) \;\&\; T_n(y_1, \dots, y_n, y) \;\&\; x = y \to \wedge_{1\leq i\leq n} x_i = y_i$ for each n, $1 \leq n \leq m$.

($T4_n$) $T_n(y_1, \dots, y_n, y) \to (x \sqsubseteq y \leftrightarrow x = y \vee \vee_{1\leq i\leq n} x \sqsubseteq y_i)$ for each n, $1 \leq n \leq m$,

(T5) – (T8),

($T9_n$) $\exists x T_n(x_1, \dots, x_n, x)$ for each n, $1 \leq n \leq m$,

($T10_n$) $T_n(x_1, \dots, x_n, x) \;\&\; T_n(x_1, \dots, x_n, y) \to x = y$ for each n, $1 \leq n \leq m$.

We define the domain of the formal interpretation so that the SSL codes of $\leq$ m-trees are guaranteed to be included in it:

$$\mathrm{I}(\mathrm{x}) \equiv \exists \mathrm{y}\ \mathrm{x} = \tau_1(\mathrm{y}).$$

Let $0^\tau =: \tau_1(0)$.

For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, let $(\mathrm{T_n}(\mathrm{y_1}, \dots, \mathrm{y_n}, \mathrm{y}))^\tau$ be the formula

$$\exists \mathrm{u_1} \dots \mathrm{u_n}(\mathrm{u_1} = \tau_2(\mathrm{y_n}, 0)\ \&\ \wedge_{1 \leq \mathrm{i} < \mathrm{n}} \mathrm{u_{i+1}} = \tau_2(\mathrm{y_{n-i}}, \mathrm{u_i})\ \&\ \mathrm{y} = \tau_1(\mathrm{u_n})),$$

and let $(\mathrm{x} \sqsubseteq \mathrm{y})^\tau \equiv \mathrm{x} \sqsubseteq \mathrm{y}$. Throughout the rest of this section, we let M be an arbitrary model of $\mathrm{T}_{\leq 2}$.

4.1 (a) For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T1_n)^\tau$.
(b) For each k and n, $1 \leq \mathrm{k} < \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T2_{k,n})^\tau$.
(c) For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T3_n)^\tau$.
(d) For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T4_n)^\tau$.
(e) For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T9_n)^\tau$.
(f) For each n, $1 \leq \mathrm{n} \leq \mathrm{m}$, $T_{\leq 2} \vdash (T10_n)^\tau$.

Proof. (a) Assume $\mathrm{M} \models (\mathrm{T_n}(\mathrm{y_1}, \dots, \mathrm{y_n}, 0))^\tau$ where $\mathrm{M} \models \wedge_{1 \leq \mathrm{i} \leq \mathrm{n}} \mathrm{I}(\mathrm{y_i})$. Then

$$\mathrm{M} \models \exists \mathrm{u_1} \dots \mathrm{u_n}(\mathrm{u_1} = \tau_2(\mathrm{y_n}, 0)\ \&\ \wedge_{1 \leq \mathrm{i} < \mathrm{n}} \mathrm{u_{i+1}} = \tau_2(\mathrm{y_{n-i}}, \mathrm{u_i})\ \&\ \tau_1(0) = \tau_1(\mathrm{u_n})).$$

Suppose $\mathrm{n} > 1$. Then $\mathrm{M} \models \tau_1(0) = \tau_1(\mathrm{u_n})\ \&\ \mathrm{u_n} = \tau_2(\mathrm{y_1}, \mathrm{u_{n-1}})$. But then, by $(\mathrm{T3_1})$, we have $\mathrm{M} \models 0 = \tau_2(\mathrm{y_1}, \mathrm{u_{n-1}})$, contradicting $(\mathrm{T1_2})$. If $\mathrm{n} = 1$, then $\mathrm{M} \models \exists \mathrm{u_1}(\mathrm{u_1} = \tau_2(\mathrm{y_1}, 0)\ \&\ \tau_1(0) = \tau_1(\mathrm{u_1}))$. Then, by $(\mathrm{T3_1})$, $\mathrm{M} \models 0 = \tau_2(\mathrm{y_1}, 0)$, contradicting $(\mathrm{T1_2})$. Hence $\mathrm{M} \models \neg(\mathrm{T_n}(\mathrm{y_1}, \dots, \mathrm{y_n}, 0))^\tau$.

(b) Assume $\mathrm{M} \models (\mathrm{T_k}(\mathrm{x_1}, \dots, \mathrm{x_k}, \mathrm{x}))^\tau\ \&\ (\mathrm{T_n}(\mathrm{y_1}, \dots, \mathrm{y_n}, \mathrm{y}))^\tau\ \&\ \mathrm{x} = \mathrm{y}$ where $\mathrm{M} \models \wedge_{1 \leq \mathrm{i} \leq \mathrm{k}} \mathrm{I}(\mathrm{x_i})\ \&\ \wedge_{1 \leq \mathrm{j} \leq \mathrm{n}} \mathrm{I}(\mathrm{y_j})\ \&\ \mathrm{I}(\mathrm{x})\ \&\ \mathrm{I}(\mathrm{y})$. Then $\mathrm{M} \models \exists \mathrm{u_1} \dots \mathrm{u_k}(\mathrm{u_1} = \tau_2(\mathrm{x_k}, 0)\ \&\ \wedge_{1 \leq \mathrm{i} < \mathrm{k}} \mathrm{u_{i+1}} = \tau_2(\mathrm{x_{k-i}}, \mathrm{u_i})\ \&\ \mathrm{x} = \tau_1(\mathrm{u_k}))$ and $\mathrm{M} \models \exists \mathrm{v_1} \dots \mathrm{v_n}(\mathrm{v_1} = \tau_2(\mathrm{y_n}, 0)\ \&\ \wedge_{1 \leq \mathrm{j} < \mathrm{n}} \mathrm{v_{j+1}} = \tau_2(\mathrm{y_{n-j}}, \mathrm{v_j})\ \&\ \mathrm{y} = \tau_1(\mathrm{v_n}))$, whence $\mathrm{M} \models \tau_1(\mathrm{u_k}) = \mathrm{x} = \mathrm{y} = \tau_1(\mathrm{v_n})$, and further, by $(\mathrm{T3_1})$, $\mathrm{M} \models \mathrm{u_k} = \mathrm{v_n}$.

Let $\mathrm{x_0} = \mathrm{x}$ and $\mathrm{y_0} = \mathrm{y}$. By hypothesis, we have that

$$\mathrm{M} \models \mathrm{x_0} = \mathrm{y_0}. \tag{1}$$

Assume now that $\mathrm{M} \models \mathrm{x_j} = \mathrm{y_j}\ \&\ \mathrm{u_{k-j}} = \mathrm{v_{n-j}}$, where $0 \leq \mathrm{j} < \mathrm{k}$. Then $\mathrm{M} \models \tau_2(\mathrm{x_{j+1}}, \mathrm{u_{k-(j+1)}}) = \mathrm{u_{k-j}} = \mathrm{v_{n-j}} = \tau_2(\mathrm{y_{j+1}}, \mathrm{v_{n-(j+1)}})$, whence, by $(\mathrm{T3_2})$, $\mathrm{M} \models \mathrm{x_{j+1}} = \mathrm{y_{j+1}}\ \&\ \mathrm{u_{k-(j+1)}} = \mathrm{v_{n-(j+1)}}$. Thus, we have

$$\mathrm{M} \models \mathrm{x_j} = \mathrm{y_j}\ \&\ \mathrm{u_{k-j}} = \mathrm{v_{n-j}} \rightarrow \mathrm{x_{j+1}} = \mathrm{y_{j+1}}\ \&\ \mathrm{u_{k-(j+1)}} = \mathrm{v_{n-(j+1)}}. \tag{2}$$

From (1) after k applications of (2) we obtain

$$\mathrm{M} \models \mathrm{x_k} = \mathrm{y_k}\ \&\ \mathrm{u_1} = \mathrm{u_{k-(k-1)}} = \mathrm{v_{n-(k-1)}} = \mathrm{v_{n-k+1}}.$$

But $\mathrm{M} \models \mathrm{u_1} = \tau_2(\mathrm{x_k}, 0)$ and $\mathrm{M} \models \mathrm{v_{n-k+1}} = \tau_2(\mathrm{y_k}, \mathrm{v_{n-k}})$ where $\mathrm{n-k} \neq 0$. Hence $\mathrm{M} \models \tau_2(\mathrm{x_k}, 0) = \tau_2(\mathrm{y_k}, \mathrm{v_{n-k}})$. But then, by $(\mathrm{T3_2})$, $\mathrm{M} \models 0 = \mathrm{v_{n-k}} =$

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$\tau_2(y_{k+1}, v_{n-(k+1)})$, contradicting $(T1_2)$. Hence $M \models x \neq y$, and we obtain $M \models (T2_{k,n})^\tau$.

(c) Assume $M \models (T_n(x_1, \ldots, x_n, x))^\tau$ & $(T_n(y_1, \ldots, y_n, y))^\tau$ & $x = y$ where $M \models \wedge_{1\leq i\leq n} I(x_i)$ & $\wedge_{1\leq j\leq n} I(y_j)$ & $I(x)$ & $I(y)$. Then $M \models \exists u_1 \ldots u_k(u_1 = \tau_2(x_n, 0)$ & $\wedge_{1\leq i<n} u_{i+1} = \tau_2(x_{n-i}, u_i)$ & $x = \tau_1(u_n))$ and $M \models \exists v_1 \ldots v_n(v_1 = \tau_2(y_n, 0)$ & $\wedge_{1\leq j<n} v_{j+1} = \tau_2(y_{n-j}, v_j)$ & $y = \tau_1(v_n))$, whence $M \models \tau_1(u_n) = x = y = \tau_1(v_n)$, and further, by $(T3_1)$, $M \models u_n = v_n$. We now show, for $0 \leq j < n$, that

$$M \models x_j = y_j \;\&\; u_{n-j} = v_{n-j} \to x_{j+1} = y_{j+1} \;\&\; u_{n-(j+1)} = v_{n-(j+1)}. \qquad (*)$$

Assume $M \models x_j = y_j$ & $u_{n-j} = v_{n-j}$. Then $M \models \tau_2(x_{j+1}, u_{n-(j+1)}) = u_{n-j} = v_{n-j} = \tau_2(y_{j+1}, v_{n-(j+1)})$, whence, by $(T3_2)$, $M \models x_{j+1} = y_{j+1}$ & $u_{n-(j+1)} = v_{n-(j+1)}$, as claimed.

Letting $x_0 = x$ and $y_0 = y$, after n applications of $(*)$ we obtain $M \models x_n = y_n$. Hence $M \models \wedge_{1\leq i\leq n} x_i = y_i$ as needed.

(d) Assume $M \models (T_n(y_1, \ldots, y_n, y))^\tau$ where $M \models \wedge_{1\leq i\leq n} I(y_i)$. Then

$$M \models \exists u_1 \ldots u_n(u_1 = \tau_2(y_n, 0) \;\&\; \wedge_{1\leq i<n} u_{i+1} = \tau_2(y_{n-i}, u_i) \;\&\; y = \tau_1(u_n)).$$

Suppose further that $M \models x \sqsubseteq y$ where $M \models I(x)$. Then $M \models x \sqsubseteq \tau_1(u_n)$, whence, by $(T4_1)$, $M \models x = \tau_1(u_n) \vee x \sqsubseteq u_n$, and further, $M \models x = y \vee x \sqsubseteq u_n$. We now show, for $0 \leq j < n$, that

$$M \models x \sqsubseteq u_{n-j} \to x \sqsubseteq y_{j+1} \vee x \sqsubseteq u_{n-(j+1)}. \qquad (**)$$

Assume that $M \models x \sqsubseteq u_{n-j}$. Then $M \models x \sqsubseteq \tau_2(y_{j+1}, u_{n-(j+1)})$, whence, by $(T4_2)$, $M \models x = u_{n-j} \vee x \sqsubseteq y_{j+1} \vee x \sqsubseteq u_{n-(j+1)}$. Suppose, for a reductio, that $M \models x = u_{n-j}$. Then $M \models x = \tau_2(y_{j+1}, u_{n-(j+1)})$. But from $M \models I(x)$ we have that $M \models \exists z\, x = \tau_1(z)$, whence $M \models \tau_2(y_{j+1}, u_{n-(j+1)}) = \tau_1(z)$, contradicting $(T2_{1,2})$. Hence $M \models x \neq u_{n-j}$. Therefore $M \models x \sqsubseteq y_{j+1} \vee x \sqsubseteq u_{n-(j+1)}$, as claimed. After n–1 applications of $(**)$ we obtain from $M \models x = y \vee x \sqsubseteq u_n$ that $M \models x = y \vee \vee_{1\leq i\leq n-1} x \sqsubseteq y_i \vee x \sqsubseteq u_1$. If $M \models x \sqsubseteq u_1$, then $M \models x \sqsubseteq \tau_2(y_n, 0)$, whence by $(T4_2)$, we have $M \models x = \tau_2(y_n, 0) \vee x \sqsubseteq y_n \vee x \sqsubseteq 0$. But $M \models x = \tau_2(y_n, 0)$ contradicts $M \models I(x)$ by $(T2_{1,2})$. And from $M \models x \sqsubseteq 0$ by (T6) and (T7) we obtain $M \models x = 0$. But then from $M \models I(x)$ we have $M \models \exists z\, \tau_1(z) = x = 0$, contradicting $(T1_1)$. Hence both $M \models x = \tau_2(y_n, 0)$ and $M \models x \sqsubseteq 0$ are ruled out, and we obtain $M \models x = y \vee \vee_{1\leq i\leq n} x \sqsubseteq y_i$. But then

$$M \models x \sqsubseteq y \to x = y \vee \vee_{1\leq i\leq n} x \sqsubseteq y_i.$$

Conversely, assume that $M \models x = y \vee \vee_{1\leq i\leq n} x \sqsubseteq y_i$. Letting $y_0 = y$, we now show that the principal hypothesis $M \models (T_n(y_1, \ldots, y_n, y))^\tau$ implies that

$$M \models \wedge_{1\leq i\leq n} y_i \sqsubseteq y. \qquad (\dagger)$$

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By (T5) we have that $M \models y \sqsubseteq y$. Hence $M \models y_0 \sqsubseteq y$. By (T4$_1$) we have that $M \models u_n \sqsubseteq \tau_1(u_n) = y$. Next, we show, for $0 \leq j < n$, that

$$M \models y_j \sqsubseteq y \ \& \ u_{n-j} \sqsubseteq y \rightarrow y_{j+1} \sqsubseteq y \ \& \ u_{n-(j+1)} \sqsubseteq y. \qquad (***)$$

Assume that $M \models y_j \sqsubseteq y \ \& \ u_{n-j} \sqsubseteq y$. Then, by (T4$_2$), we have that $M \models y_{j+1} \sqsubseteq \tau_2(y_{j+1}, u_{n-(j+1)}) = u_{n-j} \sqsubseteq y$ and $M \models u_{n-(j+1)} \sqsubseteq \tau_2(y_j, u_{n-(j+1)}) = u_{n-j} \sqsubseteq y$.

This proves (***).

After n applications of (***) we then obtain from $M \models y_0 \sqsubseteq y \ \& \ u_n \sqsubseteq y$ that

$$M \models \wedge_{1 \leq i < n} y_i \sqsubseteq y \ \& \ u_1 \sqsubseteq y.$$

But then, by (T4$_2$), $M \models y_n \sqsubseteq \tau_2(y_n, 0) = u_1 \sqsubseteq y$. Hence by (T8) it follows that

$$M \models \wedge_{1 \leq i \leq n} y_i \sqsubseteq y.$$

This establishes (†). But then from hypothesis $M \models x = y \text{ v} \vee_{1 \leq i \leq n} x \sqsubseteq y_i$ and $y_0 = y$, we obtain, by (T8), that $M \models x \sqsubseteq y$. Therefore also

$$M \models x = y \text{ v} \vee_{1 \leq i \leq n} x \sqsubseteq y_i \rightarrow x \sqsubseteq y,$$

as required.

(e) Assume $M \models \wedge_{1 \leq i \leq n} I(x_i)$. We have that $M \models \exists u_1 \, u_1 = \tau_2(x_n, 0)$.

Let $u_0 = 0$. We now argue that, for $0 \leq j - 1 < n$,

$$M \models u_{j+1} = \tau_2(x_{n-j}, u_j) \rightarrow \exists u_{j+2} u_{j+2} = \tau_2(x_{n-(j+1)}, u_{j+1}). \qquad (††)$$

Assume that $M \models u_{j+1} = \tau_2(x_{n-j}, u_j)$. Then, trivially in $T_{\leq 2}$,

$$M \models \exists u_{j+2} u_{j+2} = \tau_2(x_{n-(j+1)}, u_{j+1}).$$

Hence (††) holds. After n–1 applications of (††) we obtain from $M \models \exists u_1 u_1 = \tau_2(x_n, 0)$ that

$$M \models \exists u_1 \dots u_n (u_1 = \tau_2(x_n, 0) \ \& \wedge_{1 \leq i < n} u_{i+1} = \tau_2(x_{n-i}, u_i)).$$

Now, we also have that $M \models \exists x \ x = \tau_1(u_n)$. From the definition of I we have that $M \models \exists x \ (x = \tau_1(u_n) \ \& \ I(x))$. Hence we have proved that $M \models \exists x \ (I(x) \ \& \ (T_n(x_1, \dots, x_n, x))^\tau)$, as required.

(f) Assume $M \models (T_n(x_1, \dots, x_n, x))^\tau \ \& \ (T_n(x_1, \dots, x_n, y))^\tau$ where $M \models \wedge_{1 \leq i \leq n} I(x_i) \ \& \ I(x) \ \& \ I(y)$. Then $M \models \exists u_1 \dots u_n (u_1 = \tau_2(x_n, 0) \ \& \wedge_{1 \leq i < n} u_{i+1} = \tau_2(x_{n-i}, u_i) \ \& \ x = \tau_1(u_n))$ and $M \models \exists v_1 \dots v_n (v_1 = \tau_2(x_n, 0) \ \& \wedge_{1 \leq i < n} v_{i+1} = \tau_2(x_{n-i}, v_i) \ \& \ y = \tau_1(v_n))$. Hence $M \models u_1 = v_1$. Now, for $1 \leq j < n$, we have that

$$M \models u_j = v_j \rightarrow u_{j+1} = v_{j+1}. \qquad (†††)$$

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After n–1 applications of († † †) we obtain from $M \models u_1 = v_1$ that $M \models u_n = v_n$. But then $M \models x = \tau_1(u_n) = \tau_1(v_n) = y$. So $M \models x = y$, as required. ⊣

From 4.1(a)–(f) we conclude:

THEOREM 4.2. *For each* $m \geq 1$, $T_{\leq m}$ *is interpretable in* $T_{\leq 2}$.

Now, for fixed $n \geq 1$, the (relational form of) theory T_n of full n-trees is formulated in the reduced vocabulary $\{0, T_n, \sqsubseteq\}$ with $(T1_n)$, $(T3_n)$, $(T4_n)$, (T5) – (T8), $(T9_n)$, and $(T10_n)$ as axioms. So the argument for Theorem 4.2 also establishes:

THEOREM 4.3. *For each* $n \geq 1$, T_n *is interpretable in* $T_{\leq 2}$.

§5. Interpreting $T_{\leq 2}$ **in** QT^+**.** In [3] we have shown that the theory T_2 of dyadic trees is interpretable in formal concatenation theory QT^+, a first-order theory formulated in vocabulary $\mathcal{L}_C$, with the (universal closures) of the following formulae as axioms:

$$(x^*y)^*z = x^*(y^*z). \tag{QT1}$$

$$\neg(x^*y = a) \;\&\; \neg(x^*y = b). \tag{QT2}$$

$$\begin{aligned}&(x^*a = y^*a \to x = y) \;\&\; (x^*b = y^*b \to x = y) \;\&\\ &\& \;(a^*x = a^*y \to x = y) \;\&\; (b^*x = b^*y \to x = y).\end{aligned} \tag{QT3}$$

$$\neg(a^*x = b^*y) \;\&\; \neg(x^*a = y^*b). \tag{QT4}$$

$$x = a \vee x = b \vee (\exists y(a^*y = x \vee b^*y = x) \;\&\; \exists z(z^*a = x \vee z^*b = x)). \tag{QT5}$$

It is convenient to have a function symbol for a successor operation on strings:

$$Sx = y \leftrightarrow ((x = a \;\&\; y = b) \vee (\neg x = a \;\&\; x^*b = y)). \tag{QT6}$$

Since the last axiom is basically a definition, adding it to the other five results in an inessential (i.e., conservative) extension.

The proof given in [3] of interpretability of T_2 in QT^+ relies on the binary representation of dyadic trees by Æ strings but uses the coding scheme given there. Here we adapt the argument to the Catalan coding described in Section 3 to establish interpretability of theory $T_{\leq 2}$ of ≤ 2-trees in QT^+. To define the domain of the interpretation we use the formula $I^*(x)$ in the language of concatenation theory obtained in [3, Section 6], which defines the set of Æ strings in QT^+. Let

$$I_{\leq 2}(x) \equiv I^*(x) \;\&\; \forall t\,(\mathrm{Tally}_b(t) \;\&\; t \subseteq_p x \to t = b \vee t = bb).$$

We interpret 0 by the digit a, the function symbols τ_1 and τ_2 by setting

$$\tau_1(x) =: bax \quad \tau_2(x, y) =: bbaxy,$$

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and let $x \sqsubseteq y$ be interpreted by the formula $x = y \vee ax{\subseteq_p}y$. Let A^κ be the corresponding $\mathcal{L}_C$-translation of a formula A of $T_{\leq 2}$. Note that here, unlike in Section 2, we reason formally within the concatenation theory QT^+. Throughout this section we let M be an arbitrary model of QT^+.

First, some preliminaries.

5.1 (a) $QT^+ \vdash \forall x \in I^*\forall t, y, w, z\ (Tally_b(t)\ \&\ Tally_a(y)\ \&\ x = wyz\ \&\ t \subseteq_p x \to t \subseteq_p w \vee t \subseteq_p z)$.

(b) $QT^+ \vdash I^*(x)\ \&\ I^*(y)\ \&\ z = bxy \to I^*(z)$.

(c) $QT^+ \vdash I^*(x)\ \&\ I^*(y) \to \neg xBy\ \&\ \neg yBx$.

(d) $QT^+ \vdash I^*(x)\ \&\ I^*(y) \to \neg\exists z(zBx\ \&\ zEy)$.

(e) $QT^+ \vdash I^*(y)\ \&\ I^*(z) \to \forall v, w(v \subseteq_p z\ \&\ wE(bbayv) \to wEv \vee w = v \vee vEw)$.

(f) $QT^+ \vdash I^*(z) \to \forall u, v, w\ (w \subseteq_p z\ \&\ uw = uw \to u = w)$.

(g) $QT^+ \vdash I^*(x) \to \neg(ax \subseteq_p x)$.

Proof. For (a), see [2, 4.17(b)]; in [3] the formula $I^*(x)$ is selected so as to ensure this property. Part (b) is proved as (I1) in [3, Section 6.1(c)]. Parts (c) and (d) are straightforwardly obtained from the definition of I^* in [3] (cf. 2.3(a) and (c)). For (e), assume $M \models I^*(y)\ \&\ I^*(z)$. Then $M \models J^*(y)\ \&\ J^*(z)$, where J^* is the string form selected in [3, Section 6] that is closed under * and downward closed under $\subseteq_p$. Hence $M \models J^*(bbayz)$. The claim follows from the fact that J^* was also selected to have the property described in (3.10) of [2]. The same proof for (f) and (g) taking into account (3.6) and (3.12) of [2]. ⊣

5.2 (a) $QT^+ \vdash I_{\leq 2}(x)\ \&\ z = bax \to I_{\leq 2}(z)$.

(b) $QT^+ \vdash I_{\leq 2}(x)\ \&\ I_{\leq 2}(y)\ \&\ z = bbaxy \to I_{\leq 2}(z)$.

Proof. For (a), assume $M \models I_{\leq 2}(x)\ \&\ z = bax$. Then $M \models I^*(x)$. We have that $M \models I^*(a)$, by definition of I^*. By 5.1(b) we then have that $M \models I^*(z)$.

Assume now that $M \models Tally_b(t)\ \&\ t{\subseteq_p}z$. Then $M \models t{\subseteq_p}bax$. But then, by 5.1(a) we have that $M \models t{\subseteq_p}b \vee t{\subseteq_p}x$. Now, from $M \models t{\subseteq_p}b$ we have $M \models t = b$, and from $M \models t{\subseteq_p}x$ and hypothesis $M \models I_{\leq 2}(x)$ we have $M \models t = b \vee t = bb$. Hence $M \models t = b \vee t = bb$, as required.

For (b), assume $M \models I_{\leq 2}(x)\ \&\ I_{\leq 2}(y)\ \&\ z = bbaxy$. Then $M \models I_{\leq 2}(bax)$ by (a), so $M \models I^*(bax)$. From hypothesis $M \models I_{\leq 2}(y)$ we have $M \models I^*(y)$. But then $M \models I^*(z)$ by 5.1(b).

Assume now that $M \models Tally_b(t)\ \&\ t{\subseteq_p}z$. Then $M \models t{\subseteq_p}bbaxy$. By 5.1(a), we have that $M \models t{\subseteq_p}bb \vee t{\subseteq_p}xy$. If $M \models t{\subseteq_p}bb$, then $M \models t = b \vee t = bb$. Suppose now that $M \models t{\subseteq_p}xy$. From $M \models I^*(x)$ we have that $M \models x = a \vee aaEx$. If $M \models x = a$, then from $M \models t{\subseteq_p}ay$ we have $M \models t{\subseteq_p}y$.

If $M \models aaEx$, then $M \models \exists x_1 x = x_1aa$, so $M \models t{\subseteq_p}(x_1aa)y$, whence $M \models t{\subseteq_p}x_1 \vee t{\subseteq_p}y$. So from $M \models t{\subseteq_p}xy$ we have $M \models t{\subseteq_p}x \vee t{\subseteq_p}y$, and thus

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from hypothesis $M \models I_{\leq 2}(x)$ & $I_{\leq 2}(y)$ we obtain $M \models t = b \vee t = bb$, as required. ⊣

5.2 shows that the domain of the formal interpretation is closed under the concatenation operations chosen to interpret the tree-building functions τ_1 and τ_2 in $T_{\leq 2}$. We next verify that the translations of the axioms of $T_{\leq 2}$ are derivable in QT^+. Here we have to make sure that the subtree relation in $T_{\leq 2}$ is adequately represented by the particular variant of subword relation between strings we chose above.

5.3. (a) $QT^+ \vdash (T4_1)^\kappa$.
(b) $QT^+ \vdash (T4_2)^\kappa$.
(c) $QT^+ \vdash (T6)^\kappa$.
(d) $QT^+ \vdash (T7)^\kappa$.
(e) $QT^+ \vdash (T8)^\kappa$.

Proof. (a) We show that
$QT^+ \vdash I_{\leq 2}(x) \,\&\, I_{\leq 2}(y) \to (x = bay \vee ax \subseteq_p bay \leftrightarrow x = bay \vee x = y \vee ax \subseteq_p y)$. Assume $M \models I_{\leq 2}(x)$ & $I_{\leq 2}(y)$. Suppose that $M \models ax \subseteq_p bay$. We can rule out $M \models ax = bay \vee axBbay$ immediately. Then we have $M \models axEbay \vee \exists x_1, x_2\ bay = x_1(ax)x_2$, that is,

$$M \models \exists x_1 bay = x_1(ax) \vee \exists x_1, x_2\ bay = x_1(ax)x_2.$$

Hence $M \models \exists x_1 x_1 Bbay$. Now, we have that

$$QT^+ \vdash zBbaw \to z = b \vee z = ba \vee baBz$$

So $M \models x_1 = b \vee x_1 = ba \vee baBx_1$. We distinguish the cases:

(1) $$M \models x_1 = b.$$

Then $M \models bay = bax \vee bay = b(ax)x_2$, whence $M \models y = x \vee y = xx_2$, so $M \models y = x \vee xBy$. But $M \models xBy$ is ruled out by $M \models I^*(x)$ & $I^*(y)$ and 5.1(c). Hence $M \models x = y$.

(2) $$M \models x_1 = ba.$$

Then $M \models bay = ba(ax) \vee bay = ba(ax)x_2$, whence $M \models y = ax \vee y = axx_2$. But $M \models aBy$ contradicts $M \models I^*(y)$. So this case is ruled out.

(3) $$M \models baBx_1.$$

Then $M \models \exists x_3 x_1 = bax_3$, so $M \models bay = bax_3(ax) \vee bay = bax_3(ax)x_2$, whence $M \models y = x_3(ax) \vee y = x_3(ax)x_2$. But then $M \models ax \subseteq_p y$.

Therefore, $M \models ax \subseteq_p bay \to x = y \vee ax \subseteq_p y$. Conversely, suppose $M \models x = y \vee ax \subseteq_p y$. Then $M \models ax = ay \vee ax \subseteq_p y$, whence $M \models axEbay \vee ax \subseteq_p bay$, which yields $M \models ax \subseteq_p bay$. Therefore we also have $M \models x = y \vee ax \subseteq_p y \to ax \subseteq_p bay$. But then $M \models ax \subseteq_p bay \leftrightarrow x = y \vee ax \subseteq_p y$, and a fortiori, $M \models x = bay \vee ax \subseteq_p bay \leftrightarrow x = bay \vee x = y \vee ax \subseteq_p y$, as needed.

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(b) We show that

$$\begin{aligned}&\mathrm{QT}^{+} \vdash \mathrm{I}_{\leq 2}(\mathrm{x}) \;\&\; \mathrm{I}_{\leq 2}(\mathrm{y}) \;\&\; \mathrm{I}_{\leq 2}(\mathrm{z}) \rightarrow \\ &\rightarrow (\mathrm{x} = \mathrm{bbayz} \vee \mathrm{ax} \subseteq_{\mathrm{p}} \mathrm{bbayz} \leftrightarrow \mathrm{x} = \mathrm{bbayz} \vee \\ &\quad \vee (\mathrm{x} = \mathrm{y} \vee \mathrm{ax} \subseteq_{\mathrm{p}} \mathrm{y}) \vee (\mathrm{x} = \mathrm{z} \vee \mathrm{ax} \subseteq_{\mathrm{p}} \mathrm{z})).\end{aligned}$$

Assume $M \models I_{\leq 2}(x) \;\&\; I_{\leq 2}(y) \;\&\; I_{\leq 2}(z)$. Suppose that $M \models ax \subseteq_p bbayz$. Again we can rule out $M \models ax = bbayz \vee axB(bbayz)$. So we are left with $M \models axE(bbayz) \vee \exists x_1, x_2\ bbayz = x_1(ax)x_2$.

(1) $$M \models axE(bbayz).$$

Then by 5.1(e), $M \models axEz \vee ax = z \vee zEax$. If $M \models axEz \vee ax = z$, then $M \models ax \subseteq_p z$, and we are done. So we may assume that $M \models zEax$. Then $M \models \exists x_1 ax = x_1 z$, whence $M \models x_1 = a \vee aBx_1$. If $M \models x_1 = a$, then $M \models ax = az$, and we obtain $M \models x = z$, as needed. If $M \models aBx_1$, then $M \models \exists x_2\ ax = x_1 z = (ax_2)z$, whence $M \models x = x_2 z$, and $M \models x_2 Bx$.

From hypothesis $M \models axE(bbayz)$, we have $M \models \exists z_1\ bbayz = z_1 ax = z_1 a(x_2 z)$.

By 5.1(f), we obtain $M \models bbay = z_1 ax_2$, whence $M \models z_1 = b \vee bBz_1$. It follows that $M \models bbay = bax_2 \vee \exists z_2\ bbay = (bz_2)ax_2$, hence also $M \models bay = ax_2 \vee bay = z_2 ax_2$. But $M \models bay = ax_2$ is ruled out. Hence $M \models bay = z_2 ax_2$, and so $M \models x_2 Ebay$. Now by 5.1(b), we have that $M \models I^*(bay)$ since $M \models I^*(a) \;\&\; I^*(y)$. But then $M \models x_2 Bx \;\&\; x_2 Ebay$ contradicts $M \models I^*(x) \;\&\; I^*(y)$ by 5.1(d). Hence subcase $M \models aBx_1$ is ruled out.

(2) $$M \models \exists x_1, x_2\ bbayz = x_1(ax)x_2.$$

Then by 5.1(e), $M \models x_2 Ez \vee x_2 = z \vee zEx_2$.

(2a) $$M \models zEx_2 \vee x_2 = z.$$

Then $M \models \exists x_4 x_2 = x_4 z \vee x_2 = z$, so $M \models bbayz = x_1 ax(x_4 z) \vee bbayz = x_1(ax)z$.

By 5.1(f), we have $M \models bbay = x_1 axx_4 \vee bbay = x_1(ax)$, whence $M \models x_1 = b \vee bBx_1$. It follows that

$$M \models bbay = baxx_4 \vee \exists x_3 bbay = (bx_3)axx_4 \vee bbay = b(ax) \vee \exists x_3 bbay = (bx_3)ax,$$

Hence $M \models bay = axx_4 \vee bay = x_3 axx_4 \vee bay = ax \vee bay = x_3 ax$, whence $M \models ax \subseteq_p bay$. Then $M \models x = y \vee ax \subseteq_p y$ follows as in 5.3(a).

(2b) $$M \models x_2 Ez.$$

Then $M \models \exists x_3\ \ z = x_3 x_2$; thus $M \models x_3 Bz$. Also, $M \models bbay(x_3 x_2) = x_1(ax)x_2$. Hence, by 5.1(f), $M \models bbayx_3 = x_1(ax)$. Then, by 5.1(e), $M \models x_3 Ex \vee x_3 = x \vee xEx_3$.

(2bi) $$M \models x_3 Ex \vee x_3 = x.$$

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Then either way from $M \models x_3Bz$ and $M \models I^*(x)$ & $I^*(z)$ we obtain a contradiction by 5.1(c) and 5.1(d). Hence subcase (2bi) is ruled out.

(2bii) $$M \models xEx_3.$$

Then $M \models \exists x_5\, x_3 = x_5x$, whence $M \models bbayx_5x = x_1ax$. By 5.1(f), we have $M \models bbayx_5 = x_1a$. Hence $M \models x_5 = a \vee aEx_5$. Then from $M \models x_3 = x_5x$, we obtain $M \models x_3 = ax \vee axEx_3$. From $M \models x_3Bz$, we obtain $M \models ax{\subseteq_p}z$, as needed.

This proves that $M \models ax{\subseteq_p}bbayz \to (x = y \vee ax{\subseteq_p}y) \vee (x = z \vee ax{\subseteq_p}z)$.

Conversely, suppose that $M \models x = y \vee ax{\subseteq_p}y$. If $M \models x = y$, then $M \models ax = ay$, whence $M \models ax{\subseteq_p}bb(ay)z$. So $M \models x = y \vee ax{\subseteq_p}y \to ax{\subseteq_p}bbayz$. Likewise $M \models x = z \vee ax{\subseteq_p}z \to ax{\subseteq_p}bbayz$. Therefore we also have

$$M \models (x = y \vee ax{\subseteq_p}y) \vee (x = z \vee ax{\subseteq_p}z) \to ax{\subseteq_p}bbayz.$$

But then $M \models ax{\subseteq_p}bbayz \leftrightarrow (x = y \vee ax{\subseteq_p}y) \vee (x = z \vee ax{\subseteq_p}z)$, and finally $M \models x = bbayz \vee ax{\subseteq_p}bbayz \leftrightarrow x = bbayz \vee (x = y \vee ax{\subseteq_p}y) \vee (x = z \vee ax{\subseteq_p}z)$, as required.

(c) That $QT^+ \vdash I_{\leq 2}(x) \to x = a \vee aa \subseteq_p x$ is immediate from the definition of $I^*(x)$.

(d) We show that

$$QT^+ \vdash I_{\leq 2}(x) \,\&\, I_{\leq 2}(y) \to ((x = y \vee ax \subseteq_p y) \,\&\, (y = x \vee ay \subseteq_p x) \to x = y)$$

Assume $M \models I_{\leq 2}(x)$ & $I_{\leq 2}(y)$, and suppose, for a reductio, that $M \models ax{\subseteq_p}y$ & $ay{\subseteq_p}x$. Then $M \models ax{\subseteq_p}y{\subseteq_p}ay{\subseteq_p}x$. But this contradicts 5.1(g) since $M \models I^*(x)$. But then $M \models (x = y \vee ax{\subseteq_p}y) \,\&\, (y = x \vee ay{\subseteq_p}x) \to x = y$ follows.

(e) We show that

$$\begin{aligned} &QT^+ \vdash I_{\leq 2}(x) \,\&\, I_{\leq 2}(y) \,\&\, I_{\leq 2}(z) \to \\ &\quad \to ((x = y \vee ax \subseteq_p y) \,\&\, (y = z \vee ay \subseteq_p z) \to (x = z \vee ax \subseteq_p z)). \end{aligned}$$

Assume $M \models I_{\leq 2}(x) \,\&\, I_{\leq 2}(y) \,\&\, I_{\leq 2}(z)$ and suppose $M \models ax{\subseteq_p}y \,\&\, ay{\subseteq_p}z$. Then $M \models ax{\subseteq_p}y{\subseteq_p}ay{\subseteq_p}z$, so $M \models ax{\subseteq_p}z$. Then $M \models (x = y \vee ax{\subseteq_p}y) \,\&\, (y = z \vee ay{\subseteq_p}z) \to (x = z \vee ax{\subseteq_p}z)$, as needed. ⊣

From 5.2–5.3 we obtain:

THEOREM 5.4. *$T_{\leq 2}$ is interpretable in* QT^+.

§6. Mutual interpretability of finitely axiomatized tree theories. For each $m \geq 2$, we have that T_2 is a subtheory of the theory of $\leq$ m-trees $T_{\leq m}$: so $T_2 {\leq_I} T_{\leq m}$. Now, $T_{\leq m}$ allows any number $\leq$ m of immediate descendants. We

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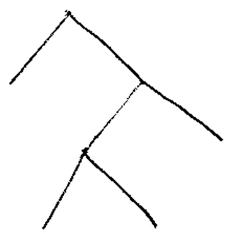
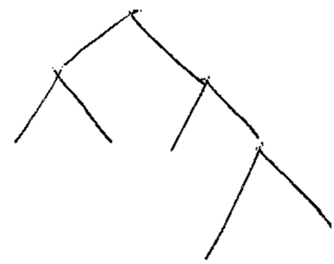
wish to show that $T_2 \leq_I T_n$ for each $n \geq 2$, so that formal simulacra of dyadic trees can be systematically identified from among n-ary trees. Consider, for example, the dyadic trees $b^2a^2b^2ab^2a^4$ and $b^2ab^2a^3b^2a^2b^2a^3$:





If we ignore the middle single-descendant branchings, we see that the ternary trees $b^3a^3b^3ab^3a^6$ and $b^3ab^3a^5b^3a^3b^3a^4$ reproduce their characteristic forms in the latters' left and right branchings:

The same idea works with any $n \geq 3$, with n–2 singleton branchings instead. For fixed $n \geq 2$, we define a map on $Æ_2$ strings such that

$$\xi(a) = a \qquad \xi(b^2ax_1x_2) = b^na\xi(x_1)a^{n-2}\xi(x_2).$$

The operation is defined by $Æ_2 \twoheadrightarrow Æ_{\leq n}$ recursion, by letting $p = a$ and $G(y_1, y_2, y)$ be the $\mathcal{L}_C$-formula $\exists z_1, z_2, z_3(y = b^naz_1 \ \& \ z_1 = y_1z_2 \ \& \ z_2 = a^{n-2}y_2)$. Hence by Theorem 2.17 its graph is expressible in $\mathcal{L}_C$, and we have that

$$\Sigma^* \models \forall x \ (Æ_2(x) \rightarrow Æ_n(\xi(x))).$$

To define the formal interpretation of T_2 in T_n we let the domain be defined by the formula

$$I(x) \equiv \forall y \ (y \sqsubseteq x \rightarrow \forall y_1 \dots y_n(y = \tau_n(y_1, \dots, y_n) \rightarrow \wedge_{1<i<n} \ y_i = 0)),$$

and let 0 be interpreted by 0, the binary operation $\tau_2(x, y)$ of T_2 by $\tau_n(x, 0, \dots, 0, y)$, and the relational symbol $\sqsubseteq$ of T_2 by $\sqsubseteq$ in T_n. First we show that $I(x)$ is closed, provably in T_n, under the n-ary operation that is to serve as the interpretation of τ_2, and next we verify that the translations of the axioms of T_2 are indeed deducible in T_n. Throughout this section we let M be an arbitrary model of T_n, for fixed $n \geq 2$.

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6.1. (a) For each $n \geq 2$, $T_n \vdash I(x)\ \&\ I(y)\ \&\ z = \tau_n(x, 0, \dots, 0, y) \to I(z)$..
(b) For each $n \geq 2$,

$$T_n \vdash I(x)\ \&\ I(y)\ \&\ I(z) \to (z \sqsubseteq \tau_n(x, 0, \dots, 0, y) \leftrightarrow z = \tau_n(x, 0, \dots, 0, y) \vee z \sqsubseteq x \vee z \sqsubseteq y).$$

Proof. (a) Assume that $M \models z = \tau_n(x, 0, \dots, 0, y)$ where $M \models I(x)\ \&\ I(y)$. Suppose $M \models w \sqsubseteq z$ where $M \models w = \tau_n(w_1, w_2, \dots, w_n)$. We need to show that $M \models \wedge_{1<i<n} w_i = 0$. We have that $M \models w \sqsubseteq \tau_n(x, 0, \dots, 0, y)$. By $(T4_n)$, we get

$$M \models w = \tau_n(x, 0, \dots, 0, y) \vee w \sqsubseteq x \vee w \sqsubseteq 0 \vee \dots \vee w \sqsubseteq 0 \vee w \sqsubseteq y).$$

We distinguish the cases:
(i) $M \models w = \tau_n(x, 0, \dots, 0, y)$.
Then from $M \models w = \tau_n(w_1, w_2, \dots, w_n)$, by $(T3_n)$, we have $M \models \wedge_{1<i<n} w_i = 0$.
(ii) $M \models w \sqsubseteq x$.
Then from $M \models I(x)\ \&\ w = \tau_n(w_1, w_2, \dots, w_n)$ we have $M \models \wedge_{1<i<n} w_i = 0$.
(iii) $M \models w \sqsubseteq 0$.
Then from (T6) and (T7), we have $M \models w = 0$. But this contradicts hypothesis $M \models w = \tau_n(w_1, w_2, \dots, w_n)$ by $(T1_n)$. Hence each of these n–2 cases is ruled out.
(iv) $M \models w \sqsubseteq y$.
Exactly analogous to (ii).
With (i)–(iv) we have that

$$M \models \forall w\ (w \sqsubseteq z \to \forall w_1 \dots w_n (w = \tau_n(w_1, \dots, w_n) \to \wedge_{1<i<n} w_i = 0)),$$

which means that $M \models I(z)$.
(b) Assume $M \models I(x)\ \&\ I(y)\ \&\ I(z)$, and let $M \models z \sqsubseteq \tau_n(x, 0, \dots, 0, y)$. Then we have, by $(T4_n)$, $M \models z = \tau_n(x, 0, \dots, 0, y) \vee z \sqsubseteq x \vee z \sqsubseteq 0 \vee \dots \vee z \sqsubseteq 0 \vee z \sqsubseteq y$.
Now, suppose $M \models z \sqsubseteq 0$. Then, by (T6) and (T7), we have $M \models z = 0$. But then, again by (T6), $M \models z \sqsubseteq x$. Hence $M \models z \sqsubseteq 0 \to z \sqsubseteq x$, and so we have, under hypothesis $M \models z \sqsubseteq \tau_n(x, 0, \dots, 0, y)$, that in fact

$$M \models z = \tau_n(x, 0, \dots, 0, y) \vee z \sqsubseteq x \vee z \sqsubseteq y.$$

Therefore, $M \models z \sqsubseteq \tau_n(x, 0, \dots, 0, y) \to z = \tau_n(x, 0, \dots, 0, y) \vee z \sqsubseteq x \vee z \sqsubseteq y$.
The converse follows from $(T4_n)$. ⊣
We then obtain:

Theorem 6.2. *For each* $n \geq 2$, $T_2 \leq_I T_n$.

On the other hand, since T_n is a subtheory of $T_{\leq n}$, we have that $T_n \leq_I T_{\leq n}$. By Theorem 4.2, for each $n \geq 2$, also $T_{\leq n} \leq_I T_{\leq 2}$, and by Theorem 5.4 we have that $T_{\leq 2} \leq_I QT^+$.

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In [3], we have established that the concatenation theory QT^+ is formally interpretable in the theory of dyadic trees, so $QT^+ \leq_I T_2$. Putting all this together, we obtain:

THEOREM 6.3. *For each* $n \geq 2$, $T_{\leq n} \equiv_I T_{\leq 2} \equiv_I T_2 \equiv_I T_n$.

§7. Interpreting T^* **in** QT^+**.** We now let all Æ strings into play and have $I^*(x)$ define the domain of the interpretation. Again, 0 is interpreted by the digit a, and $x \sqsubseteq y$ by the formula $x = y \vee ax \subseteq_p y$. We use the relational formulation of the axioms of T^* given in Section 4 and interpret the $(n+1)$-place relational symbols T_n for $n \geq 1$ by setting $T_n(x_1, \ldots, x_n, x) \equiv b^n a x_1 \ldots x_n = x$. Essentially, we are interpreting, for each fixed n, the function symbol for the n-ary tree-building operation $\tau_n(x_1, \ldots, x_n)$ by the concatenation operation $b^n a x_1 \ldots x_n$. We let M be an arbitrary model of QT^+. Again, we'll need some preliminaries:

7.1. (a) $QT^+ \vdash J^*(z) \rightarrow \forall x, y(xBz \mathbin{\&} yBz \rightarrow xBy \vee x = y \vee ybx)$.
(b) $QT^+ \vdash J^*(z) \rightarrow \forall x, y(xEz \mathbin{\&} yEz \rightarrow xEy \vee x = y \vee yEx)$.
(c) $QT^+ \vdash J^*(y) \mathbin{\&} J^*(z) \rightarrow \forall x(xByz \leftrightarrow xBy \vee x = y \vee \exists w(wBz \mathbin{\&} yw = x))$.
(d) $QT^+ \vdash J^*(x) \mathbin{\&} J^*(y) \rightarrow \forall u(uBb(xy) \rightarrow u = b \vee uBbx \vee u = bx \vee \exists y_1 (y_1By \mathbin{\&} u = bxy_1))$.
(e) For each $n \geq 1$,

$$QT^+ \vdash \wedge_{1 \leq i \leq n} J^*(x_i) \rightarrow \forall w(wBb^n a x_1 \ldots x_n \rightarrow wBb^n \vee w = b^n \vee \\ \vee w = b^n a \vee \exists z_1(z_1 B x_1 \mathbin{\&} w = b^n a z_1) \vee \vee_{1 \leq i \leq n} w = b^n a x_1 \ldots x_i \vee \\ \vee \vee_{1 \leq i \leq n} \exists z_1(z_i B x_i \mathbin{\&} w = b^n a x_1 \ldots x_{i-1} z_i)).$$

PROOF. For (a) and (b), see the proof of 5.1(e)–(g), this time with reference to (3.8) of [2] for (a) and (3.10) for (b). For (c) and (d), see [3, 3.7(b) and (c)]. We focus on (e), arguing by induction on n. Assume that $M \models \wedge_{1 \leq i \leq n} J^*(x_i)$. For $n = 1$, we have from (d) that

$$QT^+ \vdash wBbax_1 \rightarrow w = b \vee wBba \vee = ba \vee \exists z_1(z_1 B x_1 \mathbin{\&} w = baz_1).$$

Now, $QT^+ \vdash wBba \rightarrow w = b$ and $QT^+ \vdash \neg wBba$, so in fact we have that

$$QT^+ \vdash wBbax_1 \rightarrow w = b \vee w = ba \vee \exists z_1(z_1 B x_1 \mathbin{\&} w = baz_1),$$

as needed. Assume now that the claim holds for k. We then have that

$$M \models wBb^k a x_1 \ldots x_k \rightarrow wBb^k \vee w = b^k \vee w = b^k a \vee \exists z_1(z_1 B x_1 \mathbin{\&} w = b^k a z_1) \vee \\ \vee \vee_{1 \leq i < k} w = b^k a x_1 \ldots x_i \vee \vee_{1 \leq i \leq k} \exists z_i(z_i B x_i \mathbin{\&} w = b^k a x_1 \ldots x_{i-1} z_i).$$

Assume $M \models wBb((b^k a x_1 \ldots x_k) x_{k+1})$. By (d),

$$M \models w = b \vee wBb(b^k a x_1 \ldots x_k) \ \vee w = b(b^k a x_1 \ldots x_k) \vee \\ \vee \exists z_{k+1}(z_{k+1} B x_{k+1} \mathbin{\&} w = b(b^k a x_1 \ldots x_k) z_{k+1}).$$

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From $M \models w = b$, we have $M \models wBb(b^k)$, i.e., $M \models wBb^{k+1}$.
Suppose that $M \models wBb(b^k ax_1 \dots x_k)$. By (c), then

$$M \models wBb \vee w = b \vee \exists z(zBb^k ax_1 \dots x_k \;\&\; bz = w).$$

But $QT^+ \vdash \neg wBb$, and we have $M \models w = b \to wBb^{k+1}$. Given the inductive hypothesis, we have from $M \models zBb^k ax_1 \dots x_k \;\&\; bz = w$ that

$$\begin{aligned} &M \models zBb^k \vee z = b^k \vee z = b^k a \vee \exists z_1(z_1 Bx_1 \;\&\; z = b^k az_1) \vee \\ &\vee \textstyle\bigvee_{1\leq i<k} z = b^k ax_1 \dots x_i \vee \bigvee_{1\leq i\leq k} \exists z_i(z_i Bx_i \;\&\; z = b^k ax_1 \dots x_{i-1} z_i). \end{aligned}$$

Now, it is easily seen that $M \models bz = w \;\&\; zBb^k \to wBb^{k+1}$ and $M \models bz = w \;\&\; z = b^k \to w = b^{k+1}$, and also that $M \models bz = w \;\&\; z = b^k a \to w = b^{k+1}a$.

Furthermore, $M \models bz = w \;\&\; \exists z_1(z_1Bx_1 \;\&\; z = b^k az_1) \to \exists z_1(z_1Bx_1 \;\&\; w = b^{k+1}az_1)$ and $M \models bz = w \;\&\; z = b^k ax_1 \to w = b^{k+1}ax_1$, and so on, plus $M \models bz = w \;\&\; \exists z_k(z_kBx_k \;\&\; z = b^k ax_1 \dots x_{k-1}z_k) \to \exists z_k(z_kBx_k \;\&\; w = b^{k+1} ax_1 \dots x_{k-1}z_k)$.

Thus we have

$$\begin{aligned} &M \models wBb(b^k ax_1 \dots x_k) \to wBb^{k+1} \vee w = b^{k+1} \vee w = b^{k+1}a \vee \\ &\vee \exists z_1(z_1Bx_1 \;\&\; w = b^{k+1}az_1) \vee \textstyle\bigvee_{1\leq i<k} w = b^k ax_1 \dots x_i \vee \\ &\vee \textstyle\bigvee_{1\leq i\leq k} \exists z_i(z_iBx_i \;\&\; w = b^k ax_1 \dots x_{i-1}z_i). \end{aligned}$$

But then

$$\begin{aligned} &M \models wBb((b^k ax_1 \dots x_k)x_{k+1}) \to wBb^{k+1} \vee w = b^{k+1} \vee w = b^{k+1}a \vee \\ &\vee \exists z_1(z_1Bx_1 \;\&\; w = b^{k+1}az_1) \vee \textstyle\bigvee_{1\leq i<k+1} w = b^{k+1}ax_1 \dots x_i \vee \\ &\vee \textstyle\bigvee_{1\leq i\leq k+1} \exists z_i(z_iBx_i \;\&\; w = b^{k+1}ax_1 \dots x_{i-1}z_i). \end{aligned}$$

as required. ⊣

We then have:

7.2 (a) $QT^+ \vdash \neg(b^n ax_1 \dots x_n = a)$.
(b) For $1 \leq m < n$, $QT^+ \vdash \neg(b^m ax_1 \dots x_m = b^n ay_1 \dots y_n)$.
(c) $QT^+ \vdash \bigwedge_{1\leq i\leq n} I^*(x_i) \;\&\; \bigwedge_{1\leq i\leq n} I^*(y_1) \to$
$\to (b^n ax_1 \dots x_n = b^n ay_1 \dots y_n \to \bigwedge_{1\leq i\leq n} x_i = y_i)$.
(d) $QT^+ \vdash \bigwedge_{1\leq i\leq n} I^*(x_i) \;\&\; x = b^n ax_1 \dots x_n \to I^*(x)$.
(e) $QT^+ \vdash \bigwedge_{1\leq i\leq n} I^*(x_i) \;\&\; I^*(y) \to (y = b^n ax_1 \dots x_n \vee ay \subseteq_p b^n ax_1 \dots x_n$
$\leftrightarrow y = b^n ax_1 \dots x_n \vee \bigvee_{1\leq i\leq n} (y = x_i \vee ay \subseteq_p x_i))$.

Proof. (b) Suppose, for a reductio, that $M \models b^m ax_1 \dots x_m = b^n ay_1 \dots y_n$ where $m < n$. After m applications of (QT3) we obtain $M \models ax_1 \dots x_m = b^{m-n} ay_1 \dots y_n$ contradicting (QT4).

(c) Assume $M \models b^n ax_1 \dots x_n = b^n ay_1 \dots y_n$ where $M \models \bigwedge_{1\leq i\leq n} I^*(x_i) \;\&\; \bigwedge_{1\leq i\leq n} I^*(y_i)$. We formalize the reasoning of 2.4 in QT^+. After $n + 1$ applications of (QT3) we obtain $M \models x_1 \dots x_n = y_1 \dots y_n$. We have that

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$M \models J^*(x_1 \ldots x_n)$ where $J^*(x)$ is the string form constructed in [3, Section 6], and $M \models x_1B(x_1 \ldots x_n)$ & $y_1B(x_1 \ldots x_n)$. By 7.1(a), $M \models x_1By_1 \vee x_1 = y_1 \vee y_1Bx_1$. By 5.1(c), $M \models \neg x_1By_1$ & $\neg y_1Bx_1$. But then $M \models x_1 = y_1$. The rest of the proof follows the pattern of 2.4.

(d) For this we need to formalize the proof of 2.7 in QT^+. The proof has two parts: first we show in QT^+ that under the hypothesis $\wedge_{1\leq i\leq n}I^*(x_i)$ we have (ci) $\alpha(x) = \beta(x) + 1$, and, secondly, (cii) $\forall w\big(wBx \to \alpha(w) \leq \beta(w)$. In [3, Section 5], we have shown that the graphs of the counting functions α and β are expressible by $\mathcal{L}_C$-formulae $A^\#(x, y)$ and $B^\#(x, y)$, respectively, and that their fundamental properties, including additivity, are provable in QT^+ modulo the method of formula selection explained there. These functions take finite strings as arguments and yield natural numbers as values. In the formal definition of their graphs in [3], the numerical values are represented by b-tallies, and a key role is played by the relation Addtally(x, y, z) between b-tallies that behaves like addition on natural numbers. (See [3, Section 3] for the relevant properties of Addtally and the associated relation $\leq$ between b-tallies.) With this machinery in place, it is a straightforward exercise in proof formalization to show, following the computation in the first part of the proof of 2.7, that for each $n \geq 1$,

$$QT^+ \vdash \wedge_{1\leq i\leq n}I^*(x_i) \& x = b^n ax_1 \ldots x_n \to (A^\#(x, u) \;\&\; B^\#(x, v) \to u = Sv)$$

i.e., that (ci) holds. For (cii), assume that $M \models wBx$ & $A^\#(x, u)$ & $B^\#(x, v)$, where $M \models \wedge_{1\leq i\leq n}I^*(x_i)$ & $x = b^n ax_1 \ldots x_n$. From 7.1(e) we have that

$$M \models wBb^n \vee w = b^n \vee w = b^n a \vee \exists z_1(z_1Bx_1 \;\&\; w = b^n az_1) \vee$$
$$\vee \vee_{1\leq i<n} w = b^n ax_1 \ldots x_i \vee \vee_{1\leq i\leq n}\exists z_i(z_iBx_i \;\&\; w = b^n ax_1 \ldots x_{i-1}z_i).$$

We then follow the pattern of the second part of the proof of 2.7, formalizing in each case the argument that $M \models u \leq v$. We omit the details.

(e) Assume that $M \models \wedge_{1\leq i\leq n}I^*(x_i)$ & $I^*(y)$. Assume further that $M \models ay\subseteq_p b^n ax_1 \ldots x_n$. We show that for each j, $0 \leq j < n-1$,

$$(\$)\quad M \models ay\subseteq_p b^{n-j}ax_1 \ldots x_{n-j} \to ay\subseteq_p x_{n-j} \vee y = x_{n-j} \vee ay\subseteq_p b^{n-(j+1)}ax_1 \ldots x_{n-(j+1)}.$$

Assume that $M \models ay\subseteq_p b^{n-j}ax_1 \ldots x_{n-j}$. Then

$$M \models ay = b^{n-j}ax_1 \ldots x_{n-j} \vee ayB(b^{n-j}ax_1 \ldots x_{n-j}) \vee ayE(b^{n-j}ax_1 \ldots x_{n-j}) \vee$$
$$\vee \exists y_1, y_2 b^{n-j}ax_1 \ldots x_{n-j} = y_1(ay)y_2.$$

Now, $M \models ay = b^{n-j}ax_1 \ldots x_{n-j} \vee ayB(b^{n-j}ax_1 \ldots x_{n-j})$ is ruled out immediately by (QT4).

Suppose that (1) $M \models ayE(b^{n-j}ax_1 \ldots x_{n-j})$. Then, from $M \models x_{n-j}E(b^{n-j}ax_1 \ldots x_{n-j})$ we have, by 7.1(b), $M \models ayEx_{n-j} \vee ay = x_{n-j} \vee x_{n-j}Eay$. If $M \models ayEx_{n-j} \vee ay = x_{n-j}$, then $M \models ay\subseteq_p x_{n-j}$ and we are done. So we may assume that $M \models x_{n-j}Eay$.

Then $M \models \exists y_1 ay = y_1 x_{n-j}$, whence $M \models y_1 = a \vee aBy_1$. If $M \models y_1 = a$, then $M \models ay = y_1 x_{n-j} = ax_{n-j}$; hence $M \models y = x_{n-j}$, as needed.

If $M \models aBy_1$, then $M \models \exists y_2 y_1 = ay_2$. That is, $M \models ay = y_1 x_{n-j} = ay_2 x_{n-j}$, whence $M \models y = y_2 x_{n-j}$. Then $M \models y_2 By$. Now, from hypothesis $M \models ayE(b^{n-j} ax_1 \dots x_{n-j})$ we have

$$M \models \exists z_1 b^{n-j} ax_1 \dots x_{n-(j+1)} x_{n-j} = z_1 ay = z_1 ay_2 x_{n-j},$$

whence, by 5.1(f), $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} = z_1 ay_2$. Then $M \models z_1 = b \vee bBz_1$, so we have $M \models bb^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = bay_2 \vee \exists z_2 bb^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = (bz_2)ay_2$, whence $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = ay_2 \vee b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = z_2 ay_2$. But $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = ay_2$ is ruled out. Hence $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = z_2 ay_2$, and so $M \models y_2 E(b^{n-(j+1)} ax_1 \dots x_{n-(j+1)})$. But by (d) from $M \models \wedge_{1 \le i \le n-(j+1)} I^*(x_i)$ we have $M \models I^*(b^{n-(j+1)} ax_1 \dots x_{n-(j+1)})$. This, however, contradicts $M \models I^*(y)$ & $y_2 By$ by 5.1(d). Hence subcase $M \models aBy_1$ is ruled out.

Suppose that (2) $M \models \exists y_1, y_2 b^{n-j} ax_1 \dots x_{n-j} = y_1(ay)y_2$. Then $M \models y_2 E(b^{n-j} ax_1 \dots x_{n-j})$, whence by 7.1(b), $M \models x_{n-j} Ey_2 \vee x_{n-j} = y_2 \vee y_2 Ex_{n-j}$. We distinguish the subcases:

(2a) $$M \models x_{n-j} Ey_2.$$

Then $M \models \exists y_4 y_2 = y_4 x_{n-j}$, that is, $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} x_{n-j} = y_1 ay(y_4 x_{n-j})$. By 5.1(f), we get $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} = y_1 ayy_4$. Then $M \models y_1 = b \vee bBy_1$; hence

$M \models bb^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = bayy_4 \vee \exists y_3 bb^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = by_3 ayy_4$, and further, $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = ayy_4 \vee b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = y_3 ayy_4$. But $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = ayy_4$ is ruled out. Hence $M \models b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = y_3(ay)y_4$, and so we obtain $M \models ay \subseteq_p b^{n-(j+1)} ax_1 \dots x_{n-(j+1)}$, as needed.

(2b) $$M \models x_{n-j} = y_2.$$

Then $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} x_{n-j} = y_1 ayx_{n-j}$, and by 5.1(f), $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} = y_1 ay$. But then $M \models y_1 = b \vee bBy_1$. We proceed as in (2a) to show that $M \models \exists y_3 b^{n-(j+1)} ax_1 \dots x_{n-(j+1)} = y_3 ay$, whence $M \models ay \subseteq_p b^{n-(j+1)} ax_1 \dots x_{n-(j+1)}$, as needed.

(2c) $$M \models y_2 Ex_{n-j}.$$

Then $M \models \exists y_3 x_{n-j} = y_3 y_2$, so $M \models y_3 Bx_{n-j}$. Also, $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} (y_3 y_2) = y_1 ayy_2$, and by 5.1(f), $M \models b^{n-j} ax_1 \dots x_{n-(j+1)} y_3 = y_1 ay$.

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So $M \models y_3E(b^{n-j}ax_1 \dots x_{n-(j+1)}y_3)$ & $yE(b^{n-j}ax_1 \dots x_{n-(j+1)}y_3)$, whence by 7.1(b), $M \models y_3Ey \vee y_3 = y \vee yEy_3$.

(2ci) $$M \models y_3Ey \vee y_3 = y.$$

Then from $M \models I^*(y)$ and $M \models I^*(x_{n-j})$ & y_3Bx_{n-j} we obtain a contradiction, either by 5.1(d) or 5.1(c). Thus (2ci) is ruled out.

(2cii) $$M \models yEy_3.$$

Then $M \models \exists y_5 y_3 = y_5y$, so $M \models b^{n-j}ax_1 \dots x_{n-(j+1)}(y_5y) = y_1ay$. By 5.1(f), $M \models b^{n-j}ax_1 \dots x_{n-(j+1)}y_5 = y_1a$. Then $M \models y_5 = a \vee aEy_5$. From $M \models y_3 = y_5y$, we then have $M \models y_3 = ay \vee ayEy_3$, whence from $M \models y_3Bx_{n-j}$, we obtain $M \models ay \subseteq_p x_{n-j}$, as needed.

This completes the proof of ($).

After n–1 applications of ($) we have from hypothesis $M \models ay \subseteq_p b^n ax_1 \dots x_n$ that

$$M \models \vee_{0 \leq i \leq n-2}(y = x_{n-i} \vee ay \subseteq_p x_{n-i}) \vee ay \subseteq_p bax_1.$$

If $M \models ay \subseteq_p bax_1$, we have exactly as in the proof of 5.3(a) that

$$M \models y = x_1 \vee ay \subseteq_p x_1.$$

Hence $M \models \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i)$.

But then $M \models ay \subseteq_p b^n ax_1 \dots x_n \to \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i)$.

Conversely, suppose that $M \models \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i)$. If $M \models ay \subseteq_p x_j$ where $1 \leq j \leq n$, we immediately have that $M \models ay \subseteq_p x_j \subseteq_p b^n ax_1 \dots x_n$. Assume now that $M \models \vee_{1 \leq i \leq n} y = x_i$. If $M \models y = x_1$, then $M \models ay = ax_1 \subseteq_p b^n ax_1 \dots x_n$. If $j = k + 1 \leq n$, we have from hypothesis $M \models I^*(x_k)$, that $M \models x_k = a \vee aEx_k$. But then if $M \models y = x_j$, then $M \models ay = ax_j \subseteq_p b^n ax_1 \dots x_k x_{j \dots} x_n$, as required. Hence we also have

$$M \models \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i) \to ay \subseteq_p b^n ax_1 \dots x_n.$$

But then $M \models ay \subseteq_p b^n ax_1 \dots x_n \leftrightarrow \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i)$, whence $M \models y = b^n ax_1 \dots x_n \vee ay \subseteq_p b^n ax_1 \dots x_n \leftrightarrow y = b^n ax_1 \dots x_n \vee \vee_{1 \leq i \leq n}(y = x_i \vee ay \subseteq_p x_i)$, as required. ⊣

We have dealt with (T5) – (T8) in 5.3(c) and (d). Hence from 7.2 we have:

THEOREM 7.3. T^* *is interpretable in* QT^+.

§8. Some theories of dyadic trees. The theory T introduced by Kristiansen and Murwanashyaka in [4] is formulated in the vocabulary $\mathcal{L}_T = \{0, \tau, \sqsubseteq\}$, and has as its axioms:

$$\forall x, y \neg\tau(x, y) = 0 \tag{T1}$$

$$\forall x, y, z, w\, [\tau(x, y) = \tau(z, w) \to x = z \,\&\, y = w] \tag{T2}$$

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$$\forall x\, [x \sqsubseteq 0 \leftrightarrow x = 0] \tag{T3}$$

$$\forall x, y, z\, [x \sqsubseteq \tau(y, z) \leftrightarrow x = \tau(y, z) \vee x \sqsubseteq y \vee x \sqsubseteq z] \tag{T4}$$

Let $I(x)$ be an $\mathcal{L}_T$-formula with x as sole free variable. We say that $I(x)$ is a (dyadic) *tree form* if $T \vdash I(0)$ and $T \vdash \forall x(I(x) \,\&\, I(y) \rightarrow I(\tau(x, y)))$.

Note that if J_1 and J_2 are tree forms, then so is $J_1 \,\&\, J_2$.

Let $I_Z(x) \equiv 0 \sqsubseteq x$, $I_{REF\sqsubseteq}(x) \equiv x \sqsubseteq x \;\&\; \forall x_1, x_2(x = \tau(x_1, x_2) \rightarrow x_1 \sqsubseteq x_1 \,\&\, x_2 \sqsubseteq x_2)$, and $I_{TRANS\sqsubseteq}(x) \equiv \forall y, z(y \sqsubseteq z \,\&\, z \sqsubseteq x \rightarrow y \sqsubseteq x)$.

Throughout this section we let M be any model of T. Then we have:

8.1 $I_Z(x)$, $I_{REF\sqsubseteq}(x)$, and $I_{TRANS\sqsubseteq}(x)$ are tree forms.

PROOF. That $I_Z(x)$ is a tree form easily follows from (T3) and (T4). For $I_{REF\sqsubseteq}(x)$, note that $T \vdash I_{REF\sqsubseteq}(0)$ follows immediately from (T3) and (T1). Assume $M \models I_{REF\sqsubseteq}(u) \,\&\, I_{REF\sqsubseteq}(v)$ and consider $x = \tau(u, v)$. We have that $M \models \tau(u, v) \sqsubseteq \tau(u, v)$ by (T4). On the other hand, assume $M \models \tau(u, v) = \tau(x_1, x_2)$. Then $M \models u = x_1 \,\&\, v = x_2$ by (T2), whence from $M \models I_{REF\sqsubseteq}(u) \,\&\, I_{REF\sqsubseteq}(v)$ we obtain $M \models x_1 \sqsubseteq x_1 \,\&\, x_2 \sqsubseteq x_2$. Hence we have

$$M \models \forall x_1, x_2(\tau(u, v) = \tau(x_1, x_2) \rightarrow x_1 \sqsubseteq x_1 \,\&\, x_2 \sqsubseteq x_2),$$

that is, $M \models I_{REF\sqsubseteq}(\tau(u, v))$. We now deal with $I_{TRANS\sqsubseteq}(x)$. Let $x = 0$ and assume $M \models y \sqsubseteq z \,\&\, z \sqsubseteq 0$. Then $M \models z = 0$ by (T3), whence $M \models y \sqsubseteq 0$, and we obtain $M \models I_{TRANS\sqsubseteq}(0)$. Assume now that $M \models I_{TRANS\sqsubseteq}(u) \,\&\, I_{TRANS\sqsubseteq}(v)$, and suppose $M \models y \sqsubseteq z \,\&\, z \sqsubseteq \tau(u, v)$. By (T4) we have that $M \models z = \tau(u, v) \vee z \sqsubseteq u \vee z \sqsubseteq v$. If $M \models z = \tau(u, v)$, then $M \models y \sqsubseteq \tau(u, v)$. If $M \models z \sqsubseteq u$, then $M \models y \sqsubseteq u$ follows from $M \models I_{TRANS\sqsubseteq}(u)$. From (T4) we then obtain that $M \models y \sqsubseteq \tau(u, v)$. Analogously if $M \models z \sqsubseteq v$. Therefore, $M \models \forall y, z(y \sqsubseteq z \,\&\, z \sqsubseteq \tau(u, v) \rightarrow y \sqsubseteq \tau(u, v))$, that is, $M \models I_{TRANS\sqsubseteq}(\tau(u, v))$. ⊣

If $I(x)$ and $J(x)$ are $\mathcal{L}_T$-formulae with x as sole free variable, we write $J \subseteq I$ if $T \vdash \forall x\, (J(x) \rightarrow I(x))$. And we write "$\forall x \in J(\dots)$" for "$\forall x\, (J(x) \rightarrow \dots)$" and "$\forall x \sqsubseteq t(\dots)$" for "$\forall x\, (x \sqsubseteq t \rightarrow \dots)$" for an $\mathcal{L}_T$-term t.

8.2. (a) For any tree form I there is a tree form $J \subseteq I$ such that

$$T \vdash J(x) \,\&\, y \sqsubseteq x \rightarrow J(y).$$

(b) For any tree form I there is a tree form $J \subseteq I$ such that

$$T \vdash \forall x \in J\, \forall y(\tau(x, y) \neq x \,\&\, \tau(y, x) \neq x).$$

(c) For any tree form I there is a tree form $J \subseteq I$ such that

$$T \vdash \forall y \in J\, (\forall z(z \sqsubseteq y \rightarrow z = y) \vee \exists y_1, y_2 y = \tau(y_1, y_2)).$$

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(d) For any tree form I there is a tree form $J \subseteq I$ such that

$$T \vdash \forall x \in J\, \forall y\, (x \sqsubseteq y\ \&\ y \sqsubseteq x \rightarrow x = y).$$

(e) For any tree form I there is a tree form $J \subseteq I$ such that

$$T \vdash \forall x \in J\, \forall u, v\, (x \sqsubseteq u \vee x \sqsubseteq v \rightarrow x \neq \tau(u, v)).$$

(f) For any tree form I there is a tree form $J \subseteq I$ such that

$$\begin{aligned} T \vdash \forall x \in J\, (x = 0 \vee \exists y, z\, (y \sqsubseteq x\ \&\ z \sqsubseteq x\ \&\ y \neq x\ \&\ z \neq x\ \& \\ \&\ (y \sqsubseteq z \vee \forall w(y \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = y)) \\ \&\ (z \sqsubseteq y \vee \forall w(z \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = z)))). \end{aligned}$$

PROOF. (a) Let $J(x) \equiv I_{\sqsubseteq}(x) \equiv I(x)\ \&\ \forall z \sqsubseteq x\, I(z)$.

Assume $M \models z \sqsubseteq 0$. Then $M \models z = 0$ by (T3) and $M \models I(z)$ holds since $I(x)$ is a tree form. Hence $M \models I_{\sqsubseteq}(0)$. Assume now that $M \models I_{\sqsubseteq}(u)\ \&\ I_{\sqsubseteq}(v)$ and suppose that $M \models z \sqsubseteq \tau(u, v)$. By (T4) we have that $M \models z = \tau(u, v) \vee z \sqsubseteq u \vee z \sqsubseteq v$. If $M \models z \sqsubseteq u \vee z \sqsubseteq v$ we have that $M \models I(z)$ follows from hypothesis $M \models I_{\sqsubseteq}(u)\ \&\ I_{\sqsubseteq}(v)$. If $M \models z = \tau(u, v)$ then $M \models I(z)$ again from hypothesis $M \models I_{\sqsubseteq}(u)\ \&\ I_{\sqsubseteq}(v)$ and the fact that $I(x)$ is a tree form. Hence $M \models I_{\sqsubseteq}(\tau(u, v))$.

(b) Let $J(x) \equiv I(x)\ \&\ I_{REF\sqsubseteq}(x)\ \&\ \forall z \sqsubseteq x \forall y(\tau(z, y) \neq z\ \&\ \tau(y, z) \neq z)$. Assume $M \models z \sqsubseteq 0$. Then $M \models z = 0$ by (T3), and $M \models J(0)$ follows from (T1) and $M \models I_{REF\sqsubseteq}(0)$. Now, assume $M \models J(u)\ \&\ J(v)$, and let $M \models z \sqsubseteq \tau(u, v)$. Then $M \models z = \tau(u, v) \vee z \sqsubseteq u \vee z \sqsubseteq v$ by (T4). Suppose (i) $M \models z = \tau(u, v)$, and assume, for a reductio, that $M \models \tau(z, y) = z$, that is, $M \models \tau(\tau(u, v), y) = \tau(u, v)$. Then by (T2) we have $M \models \tau(u, v) = u$, contradicting the hypothesis $M \models J(u)$. Similarly if $M \models \tau(y, z) = z$. Hence (i) is ruled out. Suppose (ii) $M \models z \sqsubseteq u \vee z \sqsubseteq v$. Then from hypothesis $M \models J(u)\ \&\ J(v)$ we have $M \models \forall y(\tau(z, y) \neq z\ \&\ \tau(y, z) \neq z)$. Therefore, $M \models \forall z \sqsubseteq \tau(u, v)(\forall y(\tau(z, y) \neq z\ \&\ \tau(y, z) \neq z))$. Given that $I_{REF\sqsubseteq}$ is a tree form, we thus have $M \models J(\tau(u, v))$. Hence $J(x)$ is also a tree form.

(c) Let $J(x) \equiv I(x)\ \&\ (\forall y(y \sqsubseteq x \rightarrow y = x) \vee \exists x_1, x_2 x = \tau(x_1, x_2))$. That $J(x)$ is a tree form follows from (T3).

(d) Let $J(x) \equiv I_{REF\sqsubseteq}(x)\ \&\ I_{TRANS\sqsubseteq}(x)\ \&\ I_{8.2(b)}(x)\ \&\ \forall y\, (x \sqsubseteq y\ \&\ y \sqsubseteq x \rightarrow x = y)$. Assume $M \models 0 \sqsubseteq y\ \&\ y \sqsubseteq 0$. Then $M \models y = 0$, by (T3). Hence for $x = 0$, we have $M \models x \sqsubseteq y\ \&\ y \sqsubseteq x \rightarrow x = y$. Thus $M \models J(0)$. Assume now that $M \models J(u)\ \&\ J(v)$, and let $x = \tau(u, v)$. Suppose $M \models \tau(u, v) \sqsubseteq y\ \&\ y \sqsubseteq \tau(u, v)$. By (T4) we have that $M \models y = \tau(u, v) \vee y \sqsubseteq u \vee y \sqsubseteq v$. If $M \models y = \tau(u, v)$, then $M \models x = y$. Suppose, on the other hand, that $M \models y \sqsubseteq u$. From $M \models J(u)$ we have

$$M \models I_{REF\sqsubseteq}(u)\ \&\ I_{TRANS\sqsubseteq}(u)\ \&\ I_{8.2(b)}(u).$$

By (T4) we have that $M \models u \sqsubseteq \tau(u, v)$. And from hypothesis $M \models \tau(u, v) \sqsubseteq y$ and $M \models y \sqsubseteq u$ we obtain $M \models \tau(u, v) \sqsubseteq u$. Then $M \models u =$

$\tau(u, v)$ follows from hypothesis $M \models J(u)$. But this is a contradiction since also $M \models I_{8.2(b)}(u)$. Hence $M \models y \sqsubseteq u$ is ruled out. A completely analogous argument rules out $M \models y \sqsubseteq v$. Therefore, $M \models \forall y\ (\tau(u, v) \sqsubseteq y\ \&\ y \sqsubseteq \tau(u, v) \rightarrow \tau(u, v) = y)$, and so $M \models J(\tau(u, v))$.

(e) Let $J(x) \equiv I_{8.2(d)}(x)\ \&\ \forall u, v\ (x \sqsubseteq u \vee x \sqsubseteq v \rightarrow x \neq \tau(u, v))$. We have $M \models J(0)$ by (T1). Assume $M \models J(y)\ \&\ J(z)$ and consider $x = \tau(y, z)$. Suppose, for a reductio, that $M \models \tau(y, z) \sqsubseteq u\ \&\ \tau(y, z) = \tau(u, v)$. Then $M \models y = u\ \&\ z = v$ by (T2); hence $M \models J(u)$, whence $M \models I_{\mathrm{REF}\sqsubseteq}(u)$. By (T4) then $M \models u \sqsubseteq \tau(u, v)$. On the other hand, also $M \models \tau(u, v) \sqsubseteq u$. Hence, again from $M \models J(u)$, we obtain $M \models u = \tau(u, v)$. But this contradicts $M \models J(u)$. Thus, $M \models \tau(y, z) \sqsubseteq u \rightarrow \tau(y, z) \neq \tau(u, v)$. Exactly analogously we derive $M \models \tau(y, z) \sqsubseteq v \rightarrow \tau(y, z) \neq \tau(u, v)$. Therefore,

$$M \models \forall u, v\ (\tau(y, z) \sqsubseteq u \vee \tau(y, z) \sqsubseteq v \rightarrow \tau(y, z) \neq \tau(u, v)),$$

which suffices to show that J is a tree form.

(f) Let

$$\begin{aligned} J(x) \equiv\ & I_{8.2(d)}(x)\ \&\ (x = 0 \vee \exists y, z\ (y \sqsubseteq x\ \&\ z \sqsubseteq x\ \&\ y \neq x\ \&\ z \neq x\ \&\\ & \&\ (y \sqsubseteq z \vee \forall w(y \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = y))\ \&\\ & (z \sqsubseteq y \vee \forall w(z \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = z)))). \end{aligned}$$

We clearly have $M \models J(0)$. Assume $M \models J(u)\ \&\ J(v)$ and consider $x = \tau(u, v)$. Then from $M \models I_{\mathrm{REF}\sqsubseteq}(u)\ \&\ I_{\mathrm{REF}\sqsubseteq}(v)$ we have $M \models u \sqsubseteq \tau(u, v)\ \&\ v \sqsubseteq \tau(u, v)$, and from $M \models I_{8.2(b)}(u)\ \&\ I_{8.2(b)}(v)$, that $M \models u \neq \tau(u, v)\ \&\ v \neq \tau(u, v)$. Suppose that $M \models \neg u \sqsubseteq v$, and assume $M \models u \sqsubseteq w \sqsubseteq x$. Then $M \models w \sqsubseteq \tau(u, v)$, and by (T4) we have $M \models w = \tau(u, v) \vee w \sqsubseteq u \vee w \sqsubseteq v$. If $M \models w = \tau(u, v) = x$, we are done. So assume $M \models w \sqsubseteq u$. Then from hypothesis $M \models u \sqsubseteq w$ we have $M \models w = u$ since $M \models I_{8.2(d)}(u)$. Suppose, on the other hand, that $M \models w \sqsubseteq v$. Then from hypothesis $M \models u \sqsubseteq w$ we have $M \models u \sqsubseteq v$ since $M \models I_{\mathrm{TRANS}\sqsubseteq}(v)$. But this contradicts hypothesis $M \models \neg u \sqsubseteq v$. Therefore

$$M \models u \sqsubseteq v \vee \forall w\ (u \sqsubseteq w \sqsubseteq \tau(u, v) \rightarrow w = \tau(u, v) \vee w = u).$$

Exactly analogously we establish that

$$M \models v \sqsubseteq u \vee \forall w\ (v \sqsubseteq w \sqsubseteq \tau(u, v) \rightarrow w = \tau(u, v) \vee w = v).$$

Since $I_{8.2(d)}(x)$ is a tree form this suffices to show that $M \models J(\tau(u, v))$. ⊣

We now consider several extensions of T postulating additional "natural" properties of dyadic trees. Let:

$$\forall x, y\ (\tau(x, y) \neq x\ \&\ \tau(y, x) \neq x). \tag{T9}$$

$$\forall x\ (\forall y(y \sqsubseteq x \rightarrow y = x) \vee \exists y, z\ x = \tau(y, z)). \tag{T10}$$

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$$\forall x, u, v\ (x \sqsubseteq u \ \&\ x \sqsubseteq v \rightarrow x \neq \tau(u, v)). \tag{T11}$$

$$\begin{aligned} &\forall x\ (x = 0 \vee \exists y, z\ (y \sqsubseteq x \ \&\ z \sqsubseteq x \ \&\ y \neq x \ \&\ z \neq x\ \& \\ &\ \&\ (y \sqsubseteq z \vee \forall w (y \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = y))\ \& \\ &\ \&\ (z \sqsubseteq y \vee \forall w (z \sqsubseteq w \sqsubseteq x \rightarrow w = x \vee w = z)))). \end{aligned} \tag{T12}$$

Let D_0 stand for the theory T. ("D" is for "dyadic.") Further, let:

$$D_1 =: D_0 + (T5), (T8).$$

$$D_2 =: D_1 + (T6).$$

$$D_3 =: D_2 + (T7)(= T_2).$$

$$D_4 =: D_3 + (T9).$$

$$D_5 =: D_4 + (T10).$$

$$D_6 =: D_0 + (T5), (T7), (T11).$$

$$D_7 =: D_0 + (T5), (T7), (T8), (T9), (T12).$$

Then we have:

THEOREM 8.3. $D_0 \equiv_I D_1 \equiv_I D_2 \equiv_I D_3 \equiv_I D_4 \equiv_I D_5 \equiv_I D_6 \equiv_I D_7$.

In particular, $T \equiv_I T_2$.

PROOF. Note that, with the exception of D_5 and D_7, all of the theories considered are universal. We introduce a series of different translations A^{φ} of $\mathcal{L}_T$-formulae into $\mathcal{L}_T$-formulae where $\varphi(x)$ is a given $\mathcal{L}_T$-formula with x as sole free variable. In each case, the translation relativizes all free and bound variables in A to $\varphi(x)$, otherwise leaving the formula unchanged. In the resulting interpretations, we generally let 0 and τ be interpreted by themselves, and we let $\varphi(x)$ define the domain of the interpretation. The chosen formula will in each case be a tree form, ensuring that the domain of the interpretation contains 0 and is closed under τ. In most cases, the validation of the axioms in T of the φ-translations of the axioms of the interpreted theory is immediate given the choice of φ.

(a) $D_1 \leq_I D_0$.
Let $\varphi(x) \equiv I_{REF\sqsubseteq}(x)\ \&\ I_{TRANS\sqsubseteq}(x)$. It suffices to note that $T \vdash (T5)^{\varphi}$ and $T \vdash (T8)^{\varphi}$.

(b) $D_2 \leq_I D_0$.
Let $\varphi(x) \equiv I_{REF\sqsubseteq}(x)\ \&\ I_{TRANS\sqsubseteq}(x)\ \&\ I_Z(x)$. Then also $T \vdash (T6)^{\varphi}$.

(c) $D_3 \leq_I D_0$.
Let $\varphi(x) \equiv I_{REF\sqsubseteq}(x)\ \&\ I_{TRANS\sqsubseteq}(x)\ \&\ I_Z(x)\ \&\ I_{8.2(b)}(x)\ \&\ \forall y\ (x \sqsubseteq y\ \&\ y \sqsubseteq x \rightarrow x = y)$. Then also $T \vdash (T7)^{\varphi}$.

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(d) $D_4 \leq_I D_0$.
Let $\varphi(x)$ be as in (c). Then also $T \vdash (T9)^{\varphi}$.
(e) $D_5 \leq_I D_0$.
Let $\varphi(x)$ abbreviate

$$I_{REF\sqsubseteq}(x) \,\&\, I_{TRANS\sqsubseteq}(x) \,\&\, I_Z(x) \,\&\, I_{8.2(b)}(x) \,\&\, \forall y\,(x \sqsubseteq y \,\&\, y \sqsubseteq x \to x = y) \,\&\\ \&\, (\forall y(y \sqsubseteq x \to y = x) \vee \exists y, z\; x = \tau(y, z)).$$

Suppose $M \models \varphi(x)$. Then $M \models \forall y(y \sqsubseteq x \to y = x) \vee \exists z_1, z_2 x = \tau(z_1, z_2)\big)$.
If $M \models \forall y(y \sqsubseteq x \to y = x)$, then $M \models \forall y(\varphi(y) \to (y \sqsubseteq x \to y = x))$.
Suppose $M \models \neg\forall y(y \sqsubseteq x \to y = x)$. From $M \models I_{REF\sqsubseteq}(x)$ we have
$M \models \forall x_1, x_2(x = \tau(x_1, x_2) \to x_1 \sqsubseteq x_1 \,\&\, x_2 \sqsubseteq x_2)$. Hence
$M \models z_1 \sqsubseteq z_1 \,\&\, z_2 \sqsubseteq z_2$. But then, by (T4), $M \models z_1 \sqsubseteq \tau(z_1, z_2) = x \,\&\, z_2 \sqsubseteq \tau(z_1, z_2) = x$.
Since by 8.2(a) we may assume that $\varphi(x)$ is downward closed with respect to $\sqsubseteq$, it follows that $M \models \varphi(z_1) \,\&\, \varphi(z_2)$. Therefore,

$$M \models \forall y(\varphi(y) \to (y \sqsubseteq x \to y = x)) \vee \exists z_1, z_2(\varphi(z_1) \,\&\, \varphi(z_2) \,\&\, x = \tau(z_1, z_2)),$$

that is, $M \models (T10)^{\varphi}$.

(f) $D_6 \leq_I D_0$.

Let $\varphi(x) \equiv I_{8.2(e)}$.

(g) $D_7 \leq_I D_0$.

Let $\varphi(x) \equiv I_{8.2(f)}$. By 8.2(a) we may assume that $\varphi(x)$ is downward closed under $\sqsubseteq$. Then it is easily seen that we also have $T \vdash (T12)^{\varphi}$.
From (a)–(g) we have that $D_j \leq_I D_0 \leq_I D_k$, where $1 \leq j, k \leq 7$. This completes the proof of the theorem. ⊣

§9. Right or left?. Some authors define binary trees more broadly so as to allow differentiating between left-hand and right-hand single branchings: a binary tree either consists of a null tree, e, or of a single vertex, a, or has as its "left child" a binary tree T_1 and as its "right child" a binary tree T_2, in which case the tree is of the form $\tau(T_1, T_2)$. This gives us, e.g., two 2-vertex and five 3-vertex binary trees:

We axiomatize the corresponding theory T_e in the language $\mathcal{L}_{T,e} = \{e, a, \tau, \sqsubseteq\}$ with two individual constants e and a, a single binary function symbol τ, and a 2-place relational symbol $\sqsubseteq$:

$$\forall x, y \neg \tau(x, y) = e. \tag{T0$_e$}$$

$$\tau(e, e) = a. \tag{T1$_e$}$$

$$\forall x, y, z, w\, [\tau(x, y) = \tau(z, w) \rightarrow x = z \,\&\, y = w]. \tag{T2}$$

$$\forall x\, (x \sqsubseteq e \leftrightarrow x = e). \tag{T3$_e$}$$

$$\forall x, y, z\, [x \sqsubseteq \tau(y, z) \leftrightarrow x = \tau(y, z) \vee x \sqsubseteq y \vee x \sqsubseteq z]. \tag{T4}$$

Thinking of the null tree as an invisible branching to a "phantom" (or anti-node) 0 as opposed to a "real" node 1, the trees

$$\tau(e, a), \tau(a, e), \tau(\tau(a, e), e), \tau(\tau(e, a), e), \tau(e, \tau(a, e)), \tau(e, \tau(e, a)), \tau(a, a)$$

listed above may be described using the obvious parenthetical notation, omitting the outermost parentheses, as

$$01, 10, (10)0, (01)0, 0(10), 0(01), 11,$$

respectively. We are in effect identifying these trees with 2-colored dyadic trees:

the red branches indicating the invisible branchings to a phantom node. Taking a bold step into abstraction and thinking of a single real node as the result of conjoining two anti-nodes into a single tree (or, alternatively, of collapsing two invisible branchings into a single point/vertex), we obtain the parenthetical notations

$$0(00), (00)0, ((00)0)0, (0(00))0, 0((00)0), 0(0(00)), (00)(00), \ldots.$$

Each of these, on the other hand, represents an ordinary (monochromatic) dyadic tree:

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Thus we obtain the following direct interpretation $^{\varepsilon}$ of T_e in Kristiansen and Murwanashyaka's T. Let

$$e^{\varepsilon} =: 0, \quad a^{\varepsilon} =: \tau_2(0,0), [\tau(x,y)]^{\varepsilon} =: \tau_2(x,y) \text{ and } [x \sqsubseteq y]^{\varepsilon} \equiv: x \sqsubseteq y.$$

The ε-translations of the axioms of T_e are immediately derivable in T. Hence $T_e \leq_I T$. On the other hand, T_e obviously extends T in the expanded vocabulary $\mathcal{L}_{T,e}$ modulo relabeling 0 in T as e. So we have

THEOREM 9.1. $T_e \equiv_I T$. ⊣

We analogously obtain theories $T_{\leq n,e}$, for $n \geq 2$, and $T^*{}_e$ allowing for left-hand and right-hand single branchings in $\leq$ n-ary trees, by expanding the vocabularies of $T_{\leq n}$ and T^* with an additional constant e, relabeling 0 as a, and by adding axioms $(T0_e)$, $(T1_e)$, and $(T3_e)$, written with τ_2 in place of τ, while omitting $(T1_2)$ and replacing (T6) with $(T6_e)$: $e \sqsubseteq x$. Then a slight variant of the above argument shows that

$$T_{\leq n,e} \leq_I T_{\leq n} \text{ and } T^*{}_e \leq_I T^*.$$

On the other hand, by Theorems 4.2, 8.3, and 9.1 we also have

$$T_{\leq n} \leq_I T_2 \leq_I T \leq_I T_e \leq_I T_{\leq n,e}.$$

Thus we also obtain

THEOREM 9.2. *For each* $n \geq 2$, $T_{\leq n,e} \equiv_I T_{\leq n}$.

Taking into account Theorem 7.3 and [3], we also have

$$T^* \leq_I QT^+ \leq_I T \leq_I T_{\leq 2,e} \leq_I T^*{}_e$$

since $T^*{}_e$ extends $T_{\leq 2,e}$. Thus we have:

THEOREM 9.3. $T^* \equiv_I T^*{}_e$.

§10. The big picture. In [1] we used the concatenation theory QT^+ as a linchpin to establish mutual interpretability of several well-known weak theories of numbers, sets, and strings. In [3], we added T to that list, in part relying on Kristiansen and Murwanashyaka's interpretation of Robinson arithmetic Q in T. We are now in the position to expand the list to include the theories of trees considered in this paper.

From Theorem 7.3 and [3], we have that $T^* \leq_I QT^+ \leq_I T_2$. Since T_2 is a subtheory of T^*, we then obtain from Theorems 6.3, 8.3, 9.1, and 9.2:

THEOREM 10.1. *For each* $n \geq 2$,

$$T_{\leq n,e} \equiv_I T_{\leq n} \equiv_I T_{\leq 2} \equiv_I T \equiv_I T_e \equiv_I T_{\mathbf{2}} \equiv_I T_n \equiv_I T^* \equiv_I T^*{}_e \equiv_I QT^+ \equiv_I Q \equiv_I AST + EXT.$$

Here AST + EXT is Adjunctive Set Theory with Extensionality described in [1]. See [1, 3] for more theories that belong in this chain. To summarize, even though on the surface these theories, in their intended interpretations, refer to objects of as diverse kinds as numbers, sets, strings, and finite trees of different arities, and to corresponding relations and operations associated with those objects, it turns out that each one of these theories contains expressive and deductive resources sufficient to allow it to formally simulate reasoning in any one of the other theories.

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