A THEORY OF THE CONSOLIDATION OF SNOW

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Abstract. A consolidation theory is developed for an age-hardened snow under uniaxial stress in the porosity range of 35 to 55 per cent by considering one mechanism, viz. viscous flow of interparticle bonds. For a uniaxial stress σ, the differential equation for porosity n in terms of time t is shown to be

$$\frac{1}{(1-n)}\frac{dn}{dt} = -\frac{\nu\sigma n}{\eta(1-an)}$$

where a and ν are structural parameters and η is the coefficient of viscosity of ice. Comparison of this equation and the integrated form with existing data predicts consistent and reasonable values for a. The predicted values of η/ν range from 10^{-2} to 10^{2} times the published values for η, which may indicate that ν, and hence the consolidation rate, is greatly affected by the diagenetic history of the snow and the conditions of experimentation.

Résumé. *Une théorie de la consolidation de la neige.* Une théorie de la consolidation de la neige durcie avec l'âge, soumise à une contrainte non-axiale et dont la porosité s'échelonne de 35 à 55%, a été développée en considérant un mécanisme unique, la fluidité du ciment interparticulaire. Pour une contrainte uniaxiale σ, l'équation différentielle de la porosité n en fonction du temps t sera:

$$\frac{1}{(1-n)}\frac{dn}{dt} = -\frac{\nu\sigma n}{\eta(1-an)}$$

dans laquelle a et ν sont des paramètres relatifs à la structure, et η est le coefficient de viscosité de la glace. Si l'on compare cette équation et la forme intégrée avec les données connues, on peut prévoir des valeurs de logiques et raisonnables de a. Les valeurs établies de η/ν s'échelonnent de 10^{-2} à 10^{2} fois les valeurs publiées de η, ce qui voudrait dire que ν, et par là le taux de consolidation, est grandement affecté par l'évolution diagénétique de la neige et les conditions de l'expérience.

Zusammenfassung. *Eine Theorie der Verfestigung von Schnee.* Entwickelt wird eine Verfestigungstheorie für altersgehärteten Schnee unter einachsigem Druck im Porositätsbereich von 35 bis 55% unter Berücksichtigung nur eines mechanischen Vorganges, nämlich des viskosen Flusses von Bindemitteln der Teilchen. Als Differentialgleichung für die Porosität n als Funktion der Zeit t bei einachsigem Druck σ ergibt sich:

$$\frac{1}{(1-n)}\frac{dn}{dt} = -\frac{\nu\sigma n}{\eta(1-an)}$$

wobei a und ν strukturelle Parameter sind und η der Koeffizient der Viskosität von Eis ist. Ein Vergleich dieser Gleichung und ihrer integrierten Form mit bekannten Werten ergibt folgerichtige und annehmbare Werte für a. Die berechneten Werte von η/ν weichen vom 10^{-2}- bis 10^{2}-fachen von den bekannten Werten für η ab, woraus man schliessen kann, dass ν und damit auch das Tempo der Verfestigung weitgehend von der diagenetischen Vorgeschichte des Schnees und von den Versuchsbedingungen abhängt.

Introduction

Previous work on the deformation and consolidation of snow has concentrated on the macroscopic rheological behavior and has resulted in the introduction of macroscopic rheological parameters that are snow-type, temperature and density dependent.

Extensive experimental work has been done to investigate small deformations under low-stress conditions. Landauer (1955[b]) and Ramseier and Pavlak (1964) empirically expressed the time-dependent component of the deformation, for a given constant stress, as a simple power law in terms of time. Other researchers (including Ramseier and Pavlak) compared experimental results with the equations of certain rheological models: Bucher (1948) discussed the use of the Maxwell model; and de Quervain (1946), Yosida and others (1956) and Ramseier and Pavlak (1964) employed the Maxwell-Voigt model. It was found that the power law and the rheological equations agreed quite well with experimental data; however, these investigators showed that the macroscopic elasticity and viscosity coefficients as well as the experimental parameters in the power law are definite functions of at least temperature and density. Application of these equations to a consolidation process in which there is a significant density change necessitates knowing the functions relating the viscosity coefficient or the parameters in the power law to density and temperature.

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The following investigators have advanced simple, yet very different, empirical equations for the macroscopic coefficient of viscosity of snow as a function of density: Landauer (1955[b]), Kojima (1956, 1957, 1958), Bader (1963), and Mellor and Hendrickson (1965).

Examining recent densification theories one finds that Costes (1963) circumvented the problem of an explicit viscosity–density–temperature relationship by developing a rather complicated empirical equation, involving several parameters, for the rate of densification. Bader (1963) used the empirical viscosity–density equation of Kojima, and modified it by arbitrarily introducing a function which satisfied certain boundary conditions. Kojima (1964) showed that Bader's modified equation was approximately of the same form as his original equation for a large density range, and proceeded to use the original in the analysis of his densification studies in the Antarctic.

In order to gain further insight into the process of snow consolidation and to allow formulation of a theoretical equation describing the density-dependent viscous behavior of snow, it is suggested that the theory should originate on the microscopic level by considering the actual phenomena occurring in the grains and grain bonds.

In this way the properties and behavior of ice are immediately incorporated into the theory. By considering the snow mass to be composed of a finite number of ice particles joined together by a finite number of ice bonds one can introduce parameters related to grain and pore-space geometry. Then introducing the macroscopic variable porosity, which is a measure of the mass of ice in an elemental volume, the density dependence is incorporated and a transition to the phenomenalistic level of interest is achieved.

Consolidation Theory

The theory of consolidation is developed for a laterally confined cylindrical snow mass subjected to an axial load. The lateral stresses which develop are considered negligible.* Shear components of this axial force in the grain bonds produce a viscous flow of the ice composing the bonds, and pore space is decreased by the relative movement of the individual snow particles as they slide along intergranular boundaries into a closer configuration. Consolidation is considered to proceed with little change in the shape and size of the snow grains until the closest possible arrangement of the particles is reached. This closest configuration corresponds to a bulk porosity somewhere in the range of 36 to 43 per cent (Benson, 1962) and further consolidation can only continue by mutual intrusion of the particles, which results in a marked transition in the rate of consolidation.

It is assumed that the given snow mass is statistically homogeneous, and contains a large number of particles joined together by k bonds. The total volume of the voids, V_v, in the mass is imagined to be partitioned into k equal volume elements with a common dimension, r, i.e. the elemental volume is represented by

$$\frac{V_v}{k} = \alpha r^3 \tag{1}$$

where the proportionality constant α may be considered to be a volume shape factor.

Let $\delta\epsilon_i$ denote the small shear strains which occur in the bonds when V_v is decreased by a small amount δV_v. It is assumed that there exist k geometrical parameters λ_i which enable one to approximate the change in the geometrically complicated pore volume as

$$-\delta V_v = \sum_{i=1}^{k} \alpha r^3 \lambda_i \delta\epsilon_i, \tag{2}$$

then from equations (1) and (2) the unit change in pore volume is

$$\frac{-\delta V_v}{V_v} = \frac{1}{k} \sum_{i=1}^{k} \lambda_i \delta\epsilon_i. \tag{3}$$

* Landauer (1958) showed that lateral stresses are small for a confined specimen under uniaxial strain.

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It has been found experimentally that the rheological behavior of ice is simple Newtonian at low stresses, i.e. the relationship between shear stress and strain-rate is linear. Jellinek and Brill (1956) report this linearity to exist for the stress range of $3\cdot4\times10^5$ to $2\cdot3\times10^6$ dyn./cm.^{2} and Butkovich and Landauer (1960) found the linear relationship in the stress range of approximately 10^4 to 10^5 dyn./cm.^{2}. In this paper the forces of consolidation are considered small enough that the grain bond in the snow mass will exhibit this linear Newtonian behavior.

Let τ_i represent the shearing stresses acting in the grain bonds which produce the strains $\delta\epsilon_i$ in time δt. Then

$$\delta\epsilon_i = \frac{\tau_i \delta t}{\eta}$$

where η is the coefficient of viscosity for ice. Equation (3) becomes,

$$\frac{-\delta V_v}{V_v} = \frac{\delta t}{\eta k} \sum_{i=1}^{k} \lambda_i \tau_i. \tag{4}$$

For the confined snow mass let P be the axial force and define a surface S, in general perpendicular to P, which intersects only grain bonds in the planes of τ_i. Defining the number of bonds intersected by S to be l, the area of each bond cross-section to be A_i, and the angle between the normal to A_i and the direction of P to be θ_i, and assuming that the vertical force on A_i is $PA_i \cos\theta_i / \sum_{i=1}^{k} A_i \cos\theta_i$, then

$$\tau_i = \frac{P \cos\theta_i \sin\theta_i}{\sum_{i=1}^{l} A_i \cos\theta_i}.$$

It is now assumed that the "two-dimensional porosity" of the surface S projected on a plane perpendicular to P is identical with the effective porosity, n_f, of a potential failure surface as is defined by Ballard and McGaw (1965). If A is the right cross-sectional area of the mass then

$$n_f = \frac{A - \sum_{i=1}^{l} A_i \cos\theta_i}{A}$$

and

$$\tau_i = \frac{\sigma \sin\theta_i \cos\theta_i}{1 - n_f} \tag{5}$$

where σ is the axial external stress P/A. Substituting equation (5) in (4)

$$\frac{-\delta V_v}{V_v \delta t} = \frac{\sigma}{\eta(1-n_f)k} \sum_{i=1}^{k} \lambda_i \sin\theta_i \cos\theta_i .$$

Allowing δt to approach zero

$$\frac{-dV_v}{V_v dt} = \frac{\sigma}{\eta(1-n_f)k} \sum_{i=1}^{k} \lambda_i \sin\theta_i \cos\theta_i . \tag{6}$$

If V represents the bulk volume of the snow mass with bulk porosity n, then the insignificant compressibility of ice allows the substitution of dV for dV_v, nV for V_v and $dn/(1-n)$ for dV/V. Substituting these relationships in equation (6)

$$\frac{(1-n_f)}{n(1-n)} \frac{dn}{dt} = \frac{-\sigma}{\eta k} \sum_{i=1}^{k} \lambda_i \sin\theta_i \cos\theta_i. \tag{7}$$

The dependence of n_f and $\frac{1}{k} \sum_{i=1}^{k} \lambda_i \sin\theta_i \cos\theta_i$ on n will now be discussed. According to Ballard and McGaw (1965) n_f for an age-hardened snow can be represented by an where a

is the reciprocal of the limiting porosity n_1. In the summation k increases as consolidation progresses and the quantities λ_i and θ_i will in general change as n decreases; however, if the separate distributions of the values of λ_i and θ_i are invariant with porosity for k infinite, then when k is large $\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sin\theta_i \cos\theta_i$ will not change appreciably with porosity.

There is very little evidence in the literature to substantiate the invariance of the distributions of θ_i and λ_i. Certainly a porosity-dependent distribution of θ_i would require a preferred orientation of the bonds in a naturally consolidating snow mass, a condition which to date has not been observed. The exact physical significance of the geometrical parameters λ_i is not yet clearly visualized; however, if the geometry of pore space remains similar for some range of n as n decreases, then it seems reasonable to assume that the distribution of the values of λ_i is independent of n for this range of n. Replacing $\sum_{i=1}^{k}\lambda_i \sin\theta_i \cos\theta_i/k$ by a constant ν and n_f by an in equation (7) produces the equation

$$\frac{(1-an)}{n(1-n)}\frac{dn}{dt} = -\frac{\nu\sigma}{\eta} \tag{8}$$

which is now restricted to age-hardened snow.

Representing the height of the snow mass by z, then since

$$\frac{1}{(1-n)}\frac{dn}{dt} = \frac{1}{z}\frac{dz}{dt} \tag{9}$$

equation (8) may be rewritten as

$$\frac{1}{z}\frac{dz}{dt} = \frac{-\nu}{\eta}\left(\frac{n}{1-an}\right)\sigma \tag{10}$$

to give the expression for the rate of consolidation.

The Coefficient of Viscosity

The term $\frac{\eta}{\nu}\left(\frac{1-an}{n}\right)$ in equation (10) is the macroscopic coefficient of compressive viscosity for the conditions specified in the development: the confining stresses are negligible; the macroscopic stress condition is uniaxial; and the porosity is such that consolidation can occur through particle rearrangement, but is less than $1/a$. Denoting this macroscopic coefficient of compressive viscosity as η_c, then

$$\eta_c = \frac{\eta}{\nu}\left(\frac{1-an}{n}\right). \tag{11}$$

The dependence of η_c on temperature, snow-type, and porosity is immediately apparent: temperature dependence is inherent in the temperature-dependent coefficient of viscosity for ice, η, and variation with different snow-types is introduced by the parameters ν and a.

Equation (11) was compared with the viscosity data of Mellor and Hendrickson (1965) from confined creep tests at "Byrd" station, Antarctica. The porosity range of these data, $0{\cdot}35 < n < 0{\cdot}55$, is within the range of applicability. Making the substitution

$$X = \frac{1}{n} - a$$

reduces equation (11) to a linearized equation through the origin,

$$\eta_c = \frac{\eta}{\nu}X. \tag{12}$$

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A series of regression analyses of η_c on X for $1\cdot70 \leqq a \leqq 1\cdot80$ in increments of $0\cdot01$ showed a maximum correlation coefficient of $0\cdot940$ for an a of $1\cdot78$. The reciprocal of this value of a predicts a limiting porosity, n_1, of $0\cdot561$, which agrees very well with the values found by Ballard and McGaw. The corresponding value of the slope, η/ν, was $1\cdot23\times10^{15}$ poise. Figure 1 shows a comparison of the theoretical equation developed here with the empirical equation of Mellor and Hendrickson for their "Byrd" station data.

Haefeli (1939) showed that the effect of lateral confinement is small for unit deformations up to several per cent; hence equation (11) should be applicable to unconfined consolidation for small deformations. From viscosity experiments (unconfined case) Landauer (1955[a], p. 24–26) found that the compressive viscosity of snow varied as exp $(-4\cdot4e)$, where e is the void ratio. From equation (11)

$$\eta_c \propto \frac{1-e(a-1)}{e} = f(e)\,. \tag{13}$$

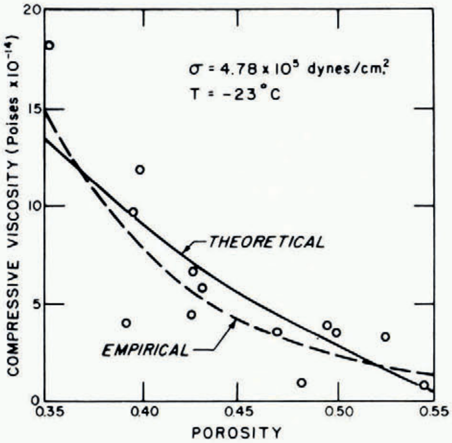

Fig. 1. Comparison of theoretical compressive viscosity $\eta_c = \frac{\eta}{\nu}\left(\frac{1-an}{n}\right)$, *for* $a = 1\cdot78$, *with empirical relationship,* $\eta_c = 2\cdot27\times10^{11}\left(\frac{1-n}{n}\right)^{3\cdot03}$ *for "Byrd" station data (Mellor and Hendrickson, 1965)*

The function $f(e)$ is well represented by an equation of the form exp $(-4\cdot4e)$ for $a = 1\cdot82$ (or $n_1 = 0\cdot566$) in the porosity range of 38 to 53 per cent (Fig. 2).

Equation (11) is also compared in Figure 3 with the experimentally determined viscosity curve of Ramseier and Pavlak (1964) and the empirical relationship of Kojima (1964), $\eta_c \propto \exp[20(1-n)]$ which was derived from depth–density data. In general $d\eta_c/dn$ for the theoretical curve is inconsistent with the slopes of these other curves.

Porosity—Time Relationship

Equation (8) can be integrated for a constant value of σ to give

$$\ln\left[\left(\frac{1-n_0}{1-n}\right)^{a-1}\left(\frac{n_0}{n}\right)\right] = \frac{\nu\sigma t}{\eta} \tag{14}$$

where n_0 is the value of n at $t = 0$. Figure 4 is a graph of equation (14) in units of $\nu\sigma/\eta$ for

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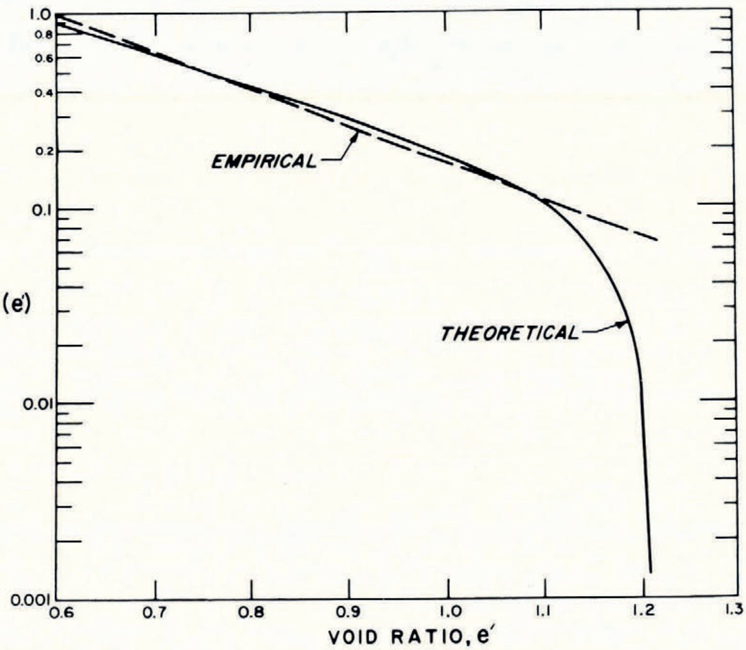

Fig. 2. Comparison of theoretical compressive viscosity $\eta_c \propto f(e) = \frac{1-e(a-1)}{e}$, *for* $a = 1 \cdot 82$, *with empirical relationship,* $\eta_c \propto \exp(-4 \cdot 4\, e)$ *of Landauer (1955[a])*

$a = 1 \cdot 8$ and $n_0 = 0 \cdot 555$. To investigate the validity of equation (14) porosity-time data may be analyzed statistically by using the linearized form

$$Y = MX + B \tag{15}$$

where

$$Y = \frac{1}{t} \ln \left(\frac{1-n}{1-n_0}\right), \; X = \frac{1}{t} \ln \left(\frac{n_0}{n}\right), \; B = \frac{-\nu\sigma}{\eta(a-1)}, \; M = \frac{1}{a-1}.$$

Data from a U.S. Cold Regions Research and Engineering Laboratory Research Report in preparation by Ballard and Feldt were analyzed according to equation (15) by regressing Y on X. The results are summarized in Table I and a typical example is shown in Figure 5. Ballard and Feldt reported data for a large series of tests, but only the data for the lower axial loads appeared to be explained by equation (14).

Table I. Summation of Regression Analysis of Porosity—Time Data according to Equation (15)

T °C.	σ dyne/cm.$^2 \times 10^{-4}$	M	B sec$^{-1} \times 10^8$	*Correlation coefficient*	$\frac{1}{a} = n_1$	η/ν poise $\times 10^{-12}$
−3·3	0·98	1·35	1·62	0·9989	0·575	0·82
	1·96	1·26	3·79	0·9997	0·557	0·65
−6·7	0·98	1·31	0·82	0·9986	0·567	1·56
	1·96	1·27	2·10	0·9990	0·559	1·19
	4·90	1·23	6·47	0·9998	0·551	0·93
−10	0·98	1·50	2·89	0·9994	0·601	0·51
	1·96	1·23	0·13	0·9993	0·551	18·1
	4·90	1·24	0·78	0·9999	0·554	7·79

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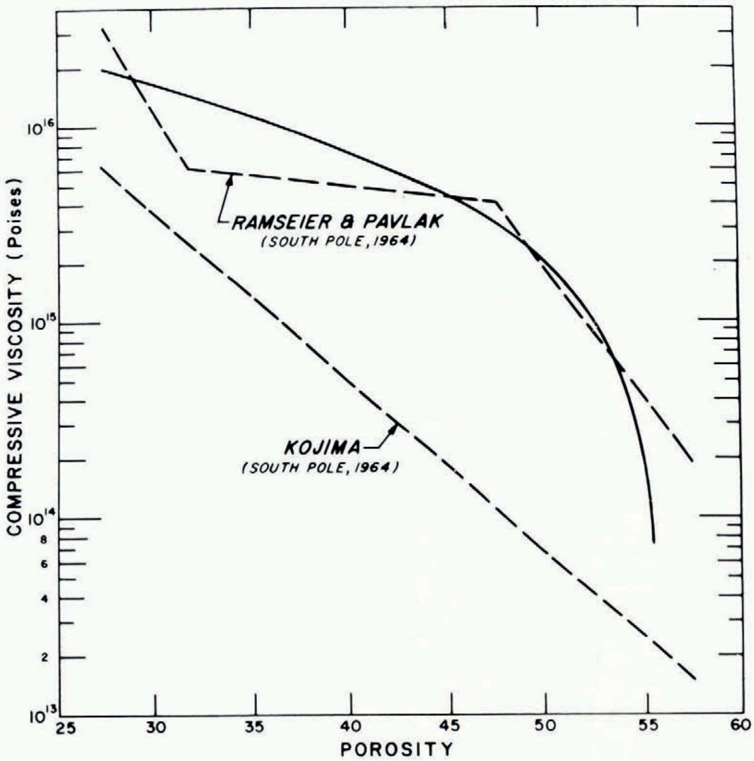

Fig. 3. Comparison of the theoretical compressive viscosity $\eta_c = \frac{\eta}{\nu}\left(\frac{1-an}{n}\right)$, *for* $a = 1 \cdot 8$, *with the experimentally determined viscosity curve of Ramseier and Pavlak (1964) and the empirical relationship of Kojima (1964)* $\eta_c \propto \exp[20(1-n)]$

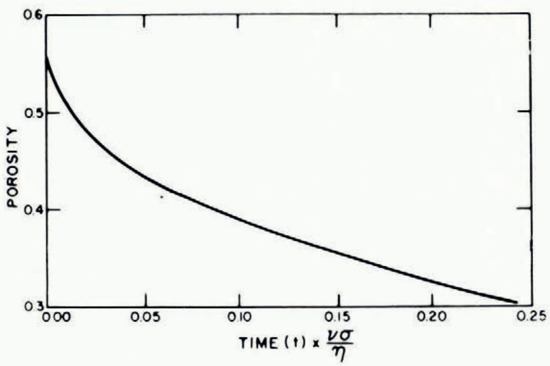

Fig. 4. Theoretical porosity—time curve $\ln\left[\left(\frac{1-n_0}{1-n}\right)^{a-1}\left(\frac{n_0}{n}\right)\right] = \frac{\nu\sigma}{\eta}\, t$ *for* $a = 1 \cdot 80$ *and* $n_0 = 0 \cdot 555$

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The consistently high correlation coefficients and the narrow range of values of the limiting porosity with a mean of 0·564 support the validity of equation (14). Values of η/ν are plotted against the reciprocal of absolute temperature in Figure 6. An Arrhenius-type relationship for an activation energy of 16 kcal./mole appears to fit the data very well. This agrees with Jellinek and Brill (1956), who found that the variation of the coefficient of viscosity of polycrystalline ice with temperature predicted an activation energy for creep of 16·1 kcal./mole. It therefore appears that ν is indeed constant for the particular type of snow used by Ballard and Feldt.

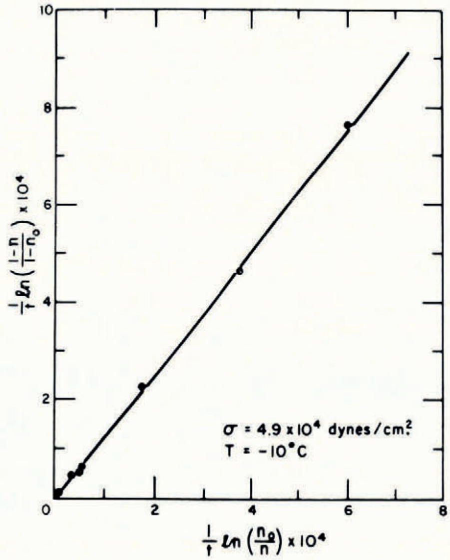

Fig. 5. Linearized time—porosity equation compared with data from a U.S. Cold Regions Research and Engineering Laboratory Report in preparation by Ballard and Feldt

Unit Deformation

Representing the height of the snow mass at time $t = 0$ and t by z_0 and $z_0 - \Delta z$ respectively, and making the substitutions

$$\frac{1-n_0}{1-n} = \frac{z}{z_0}$$

and

$$\frac{n_0}{n} = \frac{n_0(1-\Delta z/z_0)}{n_0-\Delta z/z_0},$$

then equation (14) may be written in terms of the unit deformation $\Delta z/z_0$ as

$$\ln\left[\frac{n_0(1-\Delta z/z_0)^a}{n_0-\Delta z/z_0}\right] = \frac{\nu\sigma}{\eta}t. \qquad (16)$$

Equation 16 was checked against the creep curves given by Landauer (1955[b]). Choosing $a = 1·80$ (a value which is consistent with previous predictions) and using $n_0 = 0·542$ (Landauer's initial porosity), a logarithmic plot was made in Figure 7 of the left side of equation (16), designated as $F(\Delta z/z_0)$, against $\Delta z/z_0$. Landauer's creep curves (Landauer,

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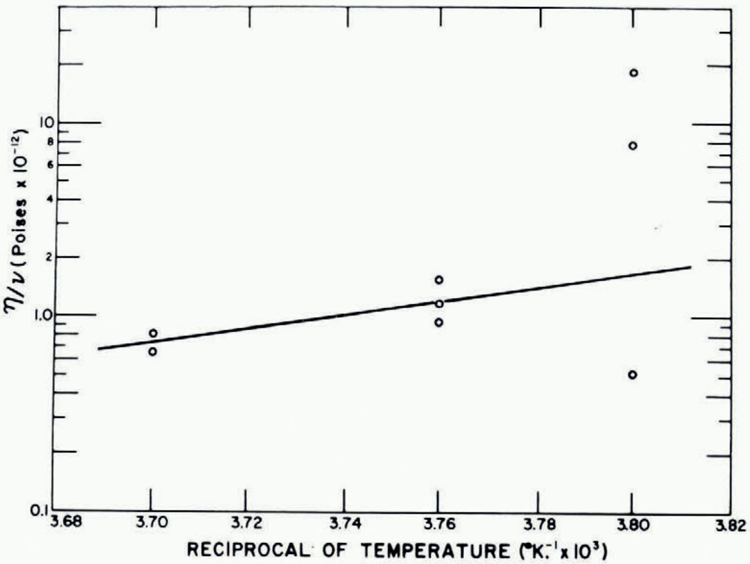

Fig. 6. Arrhenius-type relationship for an activation energy of 16 kcal./mole compared with data of Ballard and Feldt

1955[b], fig. 6, p. 7) can be represented by an equation of the form $t = b(\Delta z/z_0)^{1 \cdot 25}$ where b varies with σ. This empirical relationship is a very good approximation to equation (16) in the range $0 \cdot 001 < \Delta z/z_0 < 0 \cdot 1$ as can be seen in Figure 7 where the straight line is the

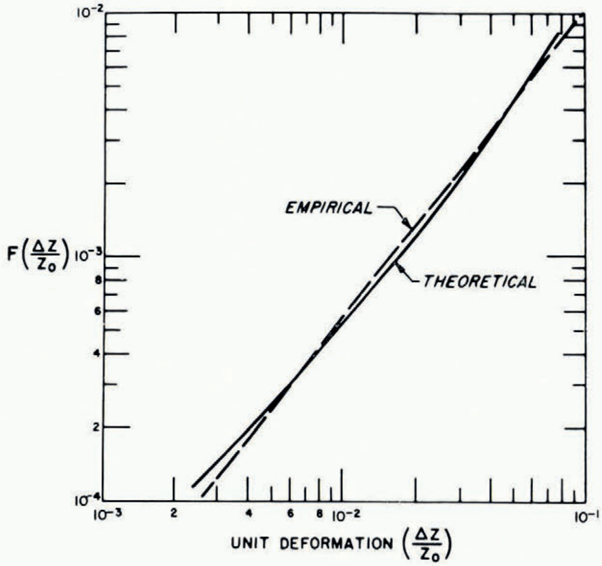

Fig. 7. Comparison of theoretical unit deformation equation $F(\Delta z/z_0) = \ln\left[\frac{n_0(1-\Delta z/z_0)^a}{n_0 - \Delta z/z_0}\right] = \frac{\nu\sigma}{\eta}t$, *for* $a = 1 \cdot 80$ *and* $n_0 = 0 \cdot 542$, *with power function approximation* $F(\Delta z/z_0) = 0 \cdot 179\,(\Delta z/z_0)^{1 \cdot 25}$

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power function $0 \cdot 179(\Delta z/z_0)^{1 \cdot 25}$. Replacing $F(\Delta z/z_0)$ by this approximation one has

$$t = 0 \cdot 179 \frac{\eta}{\nu\sigma} \left(\frac{\Delta z}{z_0}\right)^{1 \cdot 25} \tag{17}$$

which requires

$$0 \cdot 179 \frac{\eta}{\nu\sigma} = b.$$

Values of b from Landauer's curves and the calculated values of η/ν are shown in Table II.

TABLE II. VALUES OF η/ν PREDICTED FROM LANDAUER'S CREEP CURVES

σ dyne/cm.^{2}	b sec.	η/ν poise
$8 \cdot 0 \times 10^4$	$2 \cdot 73 \times 10^7$	$1 \cdot 25 \times 10^{13}$
$4 \cdot 31 \times 10^4$	$7 \cdot 05 \times 10^7$	$1 \cdot 73 \times 10^{13}$
$2 \cdot 41 \times 10^4$	$11 \cdot 5 \times 10^7$	$1 \cdot 58 \times 10^{13}$

Equation 16 was also compared with the data from Ramseier and Pavlak (1964, fig. 1, p. 326) by using the linearized form of the equation,

$$Y = aX + d, \tag{18}$$

where

$$Y = \frac{1}{t} \ln \left(1 - \frac{\Delta z}{n_0 z_0}\right),$$

$$X = \frac{1}{t} \ln \left(1 - \frac{\Delta z}{z_0}\right),$$

$$d = \frac{-\nu\sigma}{\eta}.$$

A regression of Y on X produced the results shown in Table III.

TABLE III. SUMMATION OF REGRESSION ANALYSIS ON CREEP DATA OF RAMSEIER AND PAVLAK ACCORDING TO EQUATION (18)

Location	T °C.	σ dyne/cm.$^2 \times 10^{-5}$	$1/a = n_1$	d sec.$^{-1} \times 10^{11}$	*Correlation coefficient*	η/ν poise $\times 10^{-16}$
South Pole	$-48 \cdot 0$	$0 \cdot 847$	$0 \cdot 49$	-423	$1 \cdot 0000$	200
"Camp Century"	$-22 \cdot 5$	$1 \cdot 78$	$0 \cdot 50$	$-1 \cdot 19$	$0 \cdot 9999$	$1 \cdot 5$
"Byrd" station	$-25 \cdot 0$	$2 \cdot 10$	$0 \cdot 50$	$-1 \cdot 05$	$1 \cdot 0000$	$2 \cdot 0$

Although widely separated areas are represented the values of n_1 are considerably lower than previously predicted.

DENSIFICATION OF NATURAL SNOW COVER

Finally, application of the result of this theory to densification of a natural snow cover should indicate whether the imposed conditions used in the development of the theory are applicable to *in situ* consolidation.

Assuming that the snow is accumulating at a constant rate A, then a snow layer which fell at time $t = 0$ will be subject to a vertical stress $\sigma = At$ at time t, and for the given snow layer equation (8) becomes

$$\frac{(1-an)}{n(1-n)}\frac{dn}{dt} = \frac{-\nu At}{\eta}. \tag{19}$$

Integrating gives

$$t = \left\{\frac{2\eta}{A\nu}\ln\left[\left(\frac{1-n_0}{1-n}\right)^{a-1}\frac{n_0}{n}\right]\right\}^{\frac{1}{2}} \tag{20}$$

where n_0 is now the surface snow porosity.

Denoting the depth as h, the density of the snow as γ and the density of ice as γ_i then from Sorge's Law (Bader, 1954)

$$\frac{A}{\gamma^2}\frac{d\gamma}{dh} = \frac{1}{\gamma}\frac{d\gamma}{dt}$$

or in terms of n

$$\frac{A}{\gamma_i(1-n)^2}\frac{dn}{dh} = \frac{1}{(1-n)}\frac{dn}{dt}. \tag{21}$$

Substituting from equations (19) and (20) in (21) and integrating from the surface to depth h produces the depth–porosity curve for a constant temperature,

$$h = \frac{-1}{\gamma_i}\left(\frac{A\eta}{2\nu}\right)^{\frac{1}{2}}\int_{n_0}^{n}\frac{(1-an)dn}{n(1-n)^2\left\{\ln\left[\left(\frac{1-n_0}{1-n}\right)^{a-1}\frac{n_0}{n}\right]\right\}^{\frac{1}{2}}}. \tag{22}$$

A depth–porosity curve from Kojima (1964, p. 186, fig. 14, BH58) was selected as representative for snow with a low surface porosity for comparison with equation (22). Estimating $n_0 = 0{\cdot}525$ and choosing $a = 1{\cdot}80$, the integral in equation (22) was evaluated numerically by changing the lower limit to $0{\cdot}5249$ to avoid the point $n = n_0$ where the integrand becomes infinite. Then a choice of

$$\frac{1}{\gamma_i}\sqrt{\frac{\eta A}{2\nu}} = 430$$

produced the depth–porosity curve shown in Figure 8. The curve expresses the data quite well down to a depth of about 450 cm., where the porosity is just within the range indicated

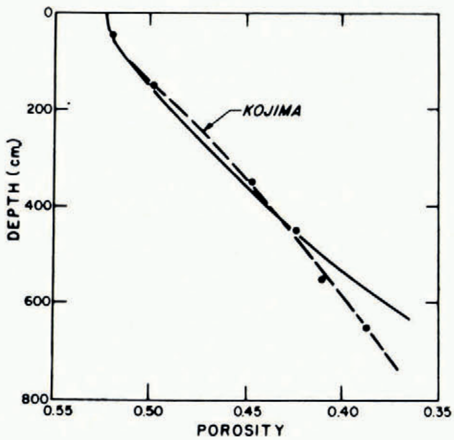

Fig. 8. Depth–porosity curve predicted in this paper compared with data and depth–porosity curve of Kojima (1964)

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by Benson (1962), for close packing. A value of $\eta/\nu = 4\cdot 25 \times 10^{13}$ poise was calculated from Kojima's estimated accumulation rate.

Discussion of Parameters

Two parameters, a and ν, were introduced in the development of the consolidation theory. The reciprocal of the first parameter, $1/a$, was assumed to be equivalent to the limiting porosity, n_1, as defined by Ballard and McGaw. Comparison of the theory with consolidation data from several different sources has predicted values of $1/a$ that agree very closely with the values of n_1, predicted by Ballard and McGaw from strength data.

The parameter, ν, defined by

$$\nu = \frac{1}{K} \sum_{i=1}^{k} \lambda_i \sin \theta_i \cos \theta_i \tag{23}$$

is not directly related to any quantity that has been previously defined in snow mechanics. It was introduced to relate the combined individual infinitesimal displacements in the structure of the snow mass to the total change in pore volume. Its value cannot be predicted precisely from consolidation data, but rather the quotient η/ν. A knowledge of the value of η for a particular temperature allows an estimation of ν. Jellinek and Brill found that η for polycrystalline ice could be represented by

$$\eta = 7\cdot 5 \exp \frac{16\cdot 1}{RT} \text{ poise} \tag{24}$$

where R is the ideal gas constant in kcal./mole °K. and T is the absolute temperature. Using equation (24) and the values of η/ν predicted in this paper, the values of ν shown in Table IV were calculated. It is seen that the order of magnitude of ν ranges from 10^{-2} to 10^2. This inconsistency in the values of ν as predicted from the data of different investigators may indicate that the value of ν is greatly affected by the diagenetic history of the snow and the specific conditions of the experiment. The magnitude of ν (equation (23)) is equal to the mean value of the term $\lambda_i \sin \theta_i \cos \theta_i$. From equation (2) the magnitude of λ_i should be of the order of 1; and if θ_i can assume all possible values between 0 and $\pi/2$, then the average value of $\sin \theta_i \cos \theta_i$ is $1/\pi$. Therefore the magnitude of ν should be of the order of 1. From Table IV it is seen that only the data of Mellor and Hendrickson predict a value of ν of this magnitude. It may be noted, however, that their experiment was the only one which actually

Table IV. Estimated Values of the Parameter ν

η/ν poise	T °C.	η poise	ν	*References*
$8\cdot 19 \times 10^{11}$	−3	$7\cdot 2 \times 10^{13}$	88	Ballard and Feldt
$6\cdot 51 \times 10^{11}$	−3	$7\cdot 2 \times 10^{13}$	111	U.S. Army C.R.R.E.L. Research
$1\cdot 56 \times 10^{12}$	−7	$1\cdot 1 \times 10^{14}$	71	Report in preparation
$1\cdot 19 \times 10^{12}$	−7	$1\cdot 1 \times 10^{14}$	93	
$9\cdot 31 \times 10^{11}$	−7	$1\cdot 1 \times 10^{14}$	118	
$5\cdot 1 \times 10^{11}$	−10	$1\cdot 6 \times 10^{14}$	314	
$1\cdot 81 \times 10^{13}$	−10	$1\cdot 6 \times 10^{14}$	8·85	
$7\cdot 79 \times 10^{12}$	−10	$1\cdot 6 \times 10^{14}$	20·5	
$4\cdot 25 \times 10^{13}$	−26	$1\cdot 25 \times 10^{15}$	29·4	Kojima (1964)
$1\cdot 25 \times 10^{13}$	−10	$1\cdot 6 \times 10^{14}$	12·8	Landauer
$1\cdot 73 \times 10^{13}$	−10		9·3	(1955[b])
$1\cdot 58 \times 10^{13}$	−10		10·1	
$1\cdot 23 \times 10^{15}$	−23	$8\cdot 5 \times 10^{14}$	0·69	Mellor and Hendrickson (1965)
2×10^{16}	−23	$8\cdot 5 \times 10^{14}$	0·0424	Ramseier and Pavlak
$1\cdot 5 \times 10^{16}$	−25	$1\cdot 2 \times 10^{15}$	0·08	(1964)
2×10^{18}	−48	$3\cdot 5 \times 10^{16}$	0·0175	

satisfied the conditions of the theory in that the samples were fully age-hardened snow and were confined during the tests.

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