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A NEIGHBOURHOOD CONDITION FOR GRAPHS TO BE FRACTIONAL (k, m)-DELETED GRAPHS*

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Abstract. Let G be a connected graph of order n, and let $k \geq 2$ and $m \geq 0$ be two integers. In this paper, we show that G is a fractional (k, m)-deleted graph if $\delta(G) \geq k + m + \frac{(m+1)^2-1}{4k}$, $n \geq 9k - 1 - 4\sqrt{2(k-1)^2+2} + 2(2k+1)m$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of non-adjacent vertices x, y of G. This result is an extension of the previous result of Zhou [**11**].

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1. Introduction. The graphs considered here will be finite undirected simple graphs. We refer the readers to [**1**] for the terminologies not defined here. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, we denote the degree of x in G by $d_G(x)$. For $X \subseteq V(G)$, we define $d_G(X) = \sum_{x \in X} d_G(x)$. We write $N_G(x)$ for the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For $X \subseteq V(G)$, we use $G[X]$ and $G - S$ to denote the subgraph of G induced by X and $V(G) - X$, respectively. Let X and Y be two disjoint vertex subsets of G, we denote the number of edges from X to Y by $e_G(X, Y)$. Instead of $e_G(\{x\}, Y)$, we just write $e_G(x, Y)$. We use $\delta(G)$ for the minimum degree of G.

Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor, if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [**6**]. In this paper we introduce first the definition of a fractional (k, m)-deleted graph, that is, a graph G is called a fractional (k, m)-deleted graph, if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph, if $m = 1$.

Iida and Nishimura gave a neighbourhood condition for a graph to have a k-factor [**3**]. Zhou obtained some sufficient conditions for graphs to have factors [**8**–**10**]. Correa and Matamala showed a necessary and sufficient condition for graphs to have factors [**2**]. Yu and the co-authors gave a degree condition for graphs to have fractional k-factors [**7**]. Liu and Zhang showed a toughness condition for graphs to have fractional k-factors [**5**].

The following results on k-factors and fractional k-factors are known.

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THEOREM 1. (*Iida and Nishimura* [**3**]). *Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, kn is even and the minimum degree is at least k. If G satisfies*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices $x, y \in V(G)$, then G has a k-factor.

THEOREM 2. (*Zhou and Liu* [11]). *Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

In this paper, we obtain a neighbourhood condition for a graph to be a fractional (k, m)-deleted graph. The result will be given in the following section.

2. Main theorems and proofs. Now, we give our main theorem which is an extension of Theorem 2.

THEOREM 3. *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2 - 1}{4k}$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

From Theorem 3, we get immediately Theorem 2 if $m = 0$. If $m = 1$ in Theorem 3, we get the following corollary.

COROLLARY 1. *Let $k \geq 2$ be an integer. Let G be a connected graph of order n with $n \geq 13k + 1 - 4\sqrt{2(k-1)^2 + 2}$, $\delta(G) \geq k + 2$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional k-deleted graph.

In order to prove Theorem 3, we depend on the following lemmas.

FACT 2.1. [3] *Let k be an integer such that $k \geq 1$. Then*

$$9k - 1 - 4\sqrt{2(k-1)^2 + 2} \begin{cases} > 3k + 5, & \text{for } k \geq 4 \\ > 3k + 4, & \text{for } k = 3 \\ = 3k + 3, & \text{for } k = 2 \\ > 2, & \text{for } k = 1 \end{cases}.$$

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LEMMA 2.1. (*Liu and Zhang* [4]). *Let G be a graph, then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k - 1\}$.

LEMMA 2.2. *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k, m)-deleted graph if and only if for any subset S of $V(G)$,*

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T),$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) - d_H(x) + e_H(x, S) \leq k - 1\}$.

Proof. Let $G' = G - E(H)$. Then G is a fractional (k, m)-deleted graph if and only if G' has a fractional k-factor. According to Lemma 2.1, this is true if and only if for any subset S of $V(G)$,

$$\delta_{G'}(S, T') = k|S| + d_{G'-S}(T') - k|T'| \geq 0,$$

where $T' = \{x : x \in V(G) \setminus S, d_{G'-S}(x) \leq k - 1\}$.

It is easy to see that $d_{G'-S}(x) = d_{G-S}(x) - d_H(x) + e_H(x, S)$ for any $x \in T'$. By the definitions of T' and T, we have $T' = T$. Hence, we obtain $\delta_{G'}(S, T') = \delta_G(S, T) - \sum_{x \in T} d_H(x) + e_H(S, T)$. Thus, $\delta_{G'}(S, T') \geq 0$ if and only if $\delta_G(S, T) \geq \sum_{x \in T} d_H(x) - e_H(S, T)$. It follows that G is a fractional (k, m)-deleted graph, if and only if $\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T)$. □

Proof of Theorem 3. According to Theorem 2, the theorem is trivial for $m = 0$. In the following, we consider $m \geq 1$.

Suppose that G satisfies the conditions of Theorem 3, but is not a fractional (k, m)-deleted graph. From Lemma 2.2 there exists a subset S of $V(G)$ such that

$$k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \leq -1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) - d_H(x) + e_H(x, S) \leq k - 1\}$ and H is any subgraph of G with m edges. □

At first, we prove the following claims.

Claim 1. $|S| \geq 1$.

Proof. If $S = \emptyset$, then according to equation (1), $d_H(x) \leq m$ and $\delta(G) \geq k + m + \frac{(m+1)^2 - 1}{4k}$, we get

$$-1 \geq \sum_{x \in T} (d_G(x) - d_H(x) - k) \geq \sum_{x \in T} (\delta(G) - m - k) \geq \sum_{x \in T} \frac{(m+1)^2 - 1}{4k} \geq 0,$$

this is a contradiction. □

Claim 2. $|T| \geq k + 1$.

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Proof. If $|T| \le k$, then by equation (1), Claim 1, $d_H(x) \le m$ and $\delta(G) \ge k + m + \frac{(m+1)^2-1}{4k}$, we obtain

$$\begin{aligned} -1 &\ge k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ &\ge |T||S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ &= \sum_{x\in T}(|S| + d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ &\ge \sum_{x\in T}(d_G(x) - d_H(x) + e_H(x,S) - k) \\ &\ge \sum_{x\in T}(\delta(G) - m - k) \\ &\ge \sum_{x\in T}\frac{(m+1)^2-1}{4k} \\ &\ge 0, \end{aligned}$$

a contradiction. □

From Claim 2, $T \ne \emptyset$. Let

$$h_1 = \min\{d_{G-S}(x) - d_H(x) + e_H(x,S) | x \in T\},$$

and choose $x_1 \in T$ with $d_{G-S}(x_1) - d_H(x_1) + e_H(x_1,S) = h_1$ and $d_H(x_1) - e_H(x_1,S)$ is minimum. Further, if $T \setminus N_T[x_1] \ne \emptyset$, we define

$$h_2 = \min\{d_{G-S}(x) - d_H(x) + e_H(x,S) | x \in T \setminus N_T[x_1]\},$$

and choose $x_2 \in T \setminus N_T[x_1]$ with $d_{G-S}(x_2) - d_H(x_2) + e_H(x_2,S) = h_2$ and $d_H(x_2) - e_H(x_2,S)$ is minimum. Then we obtain $0 \le h_1 \le h_2 \le k-1$ by the definition of T.

In view of the choice of x_1, x_2, we have $x_1x_2 \notin E(G)$. Thus, by the condition of Theorem 3, the following inequalities hold:

$$\begin{aligned} \frac{n+k-2}{2} &\le |N_G(x_1) \cup N_G(x_2)| \\ &\le d_{G-S}(x_1) + d_{G-S}(x_2) + |S| \\ &= |S| + h_1 + d_H(x_1) - e_H(x_1,S) + h_2 + d_H(x_2) - e_H(x_2,S), \end{aligned}$$

which implies

$$|S| \ge \frac{n+k-2}{2} - (h_1 + h_2 + d_H(x_1) + d_H(x_2) - e_H(x_1,S) - e_H(x_2,S)). \qquad (2)$$

Now in order to prove the theorem, we shall deduce some contradictions according to the following two cases.

Case 1: $T = N_T[x_1]$.

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Clearly, the following inequalities hold by $d_H(x_1) \le m$:

$$|T| = |N_T[x_1]| \le d_{G-S}(x_1) + 1 = h_1 + d_H(x_1) - e_H(x_1, S) + 1 \le h_1 + m + 1. \quad (3)$$

In view of $\delta(G) \le d_G(x_1) \le |S| + d_{G-S}(x_1) = |S| + h_1 + d_H(x_1) - e_H(x_1, S)$ and $d_H(x_1) \le m$, then we have

$$|S| \ge \delta(G) - h_1 - d_H(x_1) + e_H(x_1, S) \ge \delta(G) - h_1 - m. \quad (4)$$

By equations (1), (3), (4) and $0 \le h_1 \le k - 1$, we get

$$\begin{aligned}
-1 &\ge k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \\
&\ge k|S| + (h_1 - k)|T| \\
&\ge k(\delta(G) - h_1 - m) + (h_1 - k)(h_1 + m + 1) \\
&\ge k\left(k + m + \frac{(m+1)^2 - 1}{4k} - h_1 - m\right) + (h_1 - k)(h_1 + m + 1) \\
&= h_1^2 - (2k - m - 1)h_1 + k^2 - (m+1)k + \frac{(m+1)^2 - 1}{4} \\
&= \left(h_1 - k + \frac{m+1}{2}\right)^2 - \frac{1}{4} \\
&\ge -\frac{1}{4} > -1.
\end{aligned}$$

This is a contradiction.

Case 2. $T \setminus N_T[x_1] \ne \varnothing$.

From $|E(H)| = m$ and $x_1x_2 \notin E(G)$, we get

$$d_H(x_1) + d_H(x_2) \le m. \quad (5)$$

Subcase 2.1. $h_2 = 0$.

Clearly, $h_1 = 0$. By (1), (2) and $|S| + |T| \le n$, we obtain

$$\begin{aligned}
-1 &\ge k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \\
&\ge k|S| - k|T| \ge k|S| - k(n - |S|) = 2k|S| - kn \\
&\ge 2k\left(\frac{n+k-2}{2} - (d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S))\right) - kn \\
&= k^2 - 2k - 2k(d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S)),
\end{aligned}$$

that is,

$$d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S) \ge \frac{k^2 - 2k + 1}{2k} > 0.$$

According to the integrity of $d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S)$, we have

$$d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S) \ge 1.$$

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In view of the choice of x_1 and x_2, one of (a) and (b) holds for any $u \in T \setminus (\{x_1, x_2\} \cup N_H(\{x_1, x_2\}))$:

(a) $d_{G-S}(u) - d_H(u) + e_H(u, S) \geq 1$, or
(b) $d_{G-S}(u) - d_H(u) + e_H(u, S) = 0$ and $d_H(u) - e_H(u, S) \geq 1$.

Since $\{x_1, x_2\} \cap V(H) \neq \varnothing$ and any vertex $v \in T \setminus (\{x_1, x_2\} \cup V(H))$ satisfies (a), we have

$$\sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S)) \geq |T| - 2 - 2m + 1 = |T| - 2m - 1. \tag{6}$$

Using equations (1), (2), (5), (6), $|S| + |T| \leq n$, $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$ and Fact 2.1, we obtain

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \\
&\geq k|S| + |T| - 2m - 1 - k|T| \\
&= k|S| - (k-1)|T| - 2m - 1 \\
&\geq k|S| - (k-1)(n - |S|) - 2m - 1 \\
&= (2k-1)|S| - (k-1)n - 2m - 1 \\
&\geq (2k-1)\left(\frac{n+k-2}{2} - (d_H(x_1) + d_H(x_2) - e_H(x_1, S) - e_H(x_2, S))\right) \\
&\quad -(k-1)n - 2m - 1 \\
&\geq (2k-1)\left(\frac{n+k-2}{2} - m\right) - (k-1)n - 2m - 1 \\
&= \frac{n}{2} + \frac{(2k-1)(k-2)}{2} - (2k+1)m - 1 \\
&\geq \frac{n}{2} - (2k+1)m - 1 \\
&\geq \frac{9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m}{2} - (2k+1)m - 1 \\
&= \frac{9k - 1 - 4\sqrt{2(k-1)^2 + 2}}{2} - 1 \\
&> 0,
\end{aligned}$$

this is a contradiction.

Subcase 2.2. $1 \leq h_2 \leq k - 1$.

According to $d_H(x_1) \leq m$, we get $|N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + d_H(x_1) - e_H(x_1, S) + 1 \leq h_1 + m + 1$. Complying this with equations (1), (2), (5), $m \geq 1$, $0 \leq h_1 \leq h_2 \leq k - 1$, $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$ and $|S| + |T| \leq n$, we have

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \\
&\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - k|T| \\
&= k|S| + (h_1 - h_2)|N_T[x_1]| + (h_2 - k)|T| \\
&\geq k|S| + (h_1 - h_2)(h_1 + m + 1) + (h_2 - k)(n - |S|) \\
&= (2k - h_2)|S| + (h_1 - h_2)(h_1 + m + 1) - (k - h_2)n \\
&\geq (2k - h_2)\left(\frac{n + k - 2}{2} - h_1 - h_2 - m\right) + (h_1 - h_2)(h_1 + m + 1) - (k - h_2)n \\
&= h_2^2 + \frac{n - 5k}{2}h_2 + h_1^2 + (m + 1 - 2k)h_1 + k(k - 2) - 2km \\
&\geq h_2^2 + \frac{n - 5k}{2}h_2 + h_1^2 + (2 - 2k)h_1 + k(k - 2) - 2km \\
&\geq h_2^2 + \frac{n - 5k}{2}h_2 + h_2^2 + (2 - 2k)h_2 + k(k - 2) - 2km \\
&= 2h_2^2 + \frac{n - 9k + 4}{2}h_2 + k(k - 2) - 2km \\
&\geq 2h_2^2 - 2\sqrt{2(k - 1)^2 + 2}h_2 + (2k + 1)mh_2 + \frac{3}{2}h_2 + k(k - 2) - 2km \\
&\geq 2h_2^2 - 2\sqrt{2(k - 1)^2 + 2}h_2 + (2k + 1)m + \frac{3}{2}h_2 + k(k - 2) - 2km \\
&\geq 2h_2^2 - 2\sqrt{2(k - 1)^2 + 2}h_2 + \frac{3}{2}h_2 + k(k - 2) + 1 \\
&= \frac{1}{2}\left(2h_2 - \sqrt{2(k - 1)^2 + 2}\right)^2 + \frac{3}{2}h_2 - 1 \\
&\geq \frac{3}{2}h_2 - 1 \geq \frac{1}{2} \\
&> 0.
\end{aligned}$$

It is a contradiction.
This completes the proof of Theorem 3.

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