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FUNCTION SPACES AND THE MOSCO TOPOLOGY

GERALD BEER AND ROBERT TAMAKI

Let X and Y be Banach spaces and let $C(X,Y)$ be the functions from X to Y continuous with respect to the weak topology on X and the strong topology on Y. By the Mosco topology τ_M on $C(X,Y)$ we mean the supremum of the Fell topologies determined by the weak and strong topologies on $X \times Y$, where functions are identified with their graphs. The function space is Hausdorff if and only if both X and Y are reflexive. Moreover, τ_M coincides with the stronger compact-open topology on $C(X,Y)$ provided X is reflexive and Y is finite dimensional. We also show convergence in either sense is properly weaker than continuous convergence, even for continuous linear functionals, whenever X is infinite dimensional. For real-valued weakly continuous functions, τ_M is the supremum of the Mosco epi-topology and the Mosco hypo-topology if and only if X is reflexive.

1. INTRODUCTION

Over the past twenty years, numerous articles have been written on Mosco convergence and the associated Mosco topology τ_M for closed convex sets and lower semicontinuous convex functions (as identified with their epigraphs) in the context of reflexive Banach spaces [**1, 12, 18, 3, 4, 6, 9, 25**]. Two recent articles by Beer and Pai [**7, 8**] show that these notions are also fruitful more generally, that is, for weakly closed sets and weakly lower semicontinuous functions. In this article, we consider Mosco convergence in the context of function spaces. Specifically, we look at the space of functions from X to Y continuous with respect to the weak topology on X and the strong topology on Y, as identified with their (weakly closed) graphs in $X \times Y$. How is Mosco convergence of graphs related to classical convergence notions in the theory of function spaces? The main result of this paper shows that τ_M coincides with the usually stronger compact-open topology provided X is reflexive and Y is finite dimensional.

2. NOTATION AND TERMINOLOGY

If X is a Banach space, we denote its origin by θ, and its closed unit ball by B, subscripting these by X when necessary. The continuous dual of X will be denoted by X^*. The sphere $\{x \in X : \|x - p\| = \varepsilon\}$ of radius ε around a point $p \in X$ will

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be denoted by $S(p,\varepsilon)$. The symbol $\mathrm{co}(A)$ represents the convex hull of a subset A of X. To avoid trivial counterexamples, we will always assume Banach spaces have more than one point. When necessary we will write (X,w) for X equipped with the weak topology. Repeatedly, we will make use of the fact that for Banach spaces X and Y, $(X,w)\times(Y,w)$ is $(X\times Y,w)$. The box norm on $X\times Y$ will be understood.

The *Mosco topology* τ_M, defined on the nonempty weakly closed subsets $WCL(X)$ of a Banach space X, has as a base all sets of the form

$$[U_1, U_2, \ldots, U_n; K] \equiv \{A \in WCL(X) \colon A \cap U_i \neq \emptyset \text{ for all } i \text{ and } A \cap K = \emptyset\}$$

where $\{U_1, U_2, \ldots, U_n\}$ is a finite set of norm open subsets of X and K is a weakly compact subset of X (this notation follows Klein and Thompson [**16**]). Observe that a subbase for the Mosco topology consists of all *hit sets* of the form $[U;\emptyset]$ with U norm open, and all *miss sets* of the form $[X;K]$ with K weakly compact. This topology was anticipated by the *Fell topology* [**13**, **14**, **19**], also called the *topology of closed convergence* [**16**]. In fact, the Mosco topology is simply the supremum of the Fell topologies determined by the strong and weak topologies on X. The rationale for its name is as follows: convergence of a sequence $\langle A_n\rangle$ in $WCL(X)$ to A in $WCL(X)$ with respect to τ_M agrees with *Mosco convergence of sequences*, as defined by Mosco in [**17**]: (i) for each $a\in A$, there exists $\langle a_n\rangle$ strongly convergent to a such that for $n\in Z^+$, we have $a_n\in A_n$, and (ii) whenever $n(1)<n(2)<n(3)<\cdots$, and $a_k\in A_{n(k)}$, then the weak convergence of $\langle a_k\rangle$ to $x\in X$ forces x to be in A (the reader is invited to modify the proof in the convex case supplied in Theorem 3.1 of [**3**]). Of course, the great interest in Mosco convergence stems from (but is not limited to) its *stability with respect to duality* in reflexive spaces as proved sequentially in [**18**] and topologically in [**4**].

As mentioned in the introduction, we will be interested in the space of functions from a Banach space X to a Banach space Y continuous with respect to the weak topology on X and the strong topology on Y. We pause to recall some basic notions from function spaces. An excellent reference for this subject is the recent monograph of McCoy and Ntantu [**20**].

For any topological spaces X and Y, we denote the continuous functions from X to Y by $C(X,Y)$ (the topologies on X and Y will either be explicitly stated or implied). As mentioned in the introduction, when X, Y are Banach spaces, the weak topology on X and the strong topology on Y will be understood. The *compact-open topology* on $C(X,Y)$ for any topological spaces X and Y, having as a subbasis the sets $(A,V)\equiv\{f\in C(X,Y) : f(A)\subset V\}$, where $A\subset X$ is compact and $V\subset Y$ is open, will be denoted by τ_C. When the target space Y is uniformisable, it is often advantageous to consider the compact-open topology τ_C from the point of view of uniformities. Specifically, if $\mathcal{U}$ is a compatible uniformity for Y, the induced compatible uniformity for the compact open topology has as a base all entourages of the form

$$\{(f,g) \in C(X, Y) \times C(X, Y): (f(x), g(x)) \in U \text{ for each } x \in K\}$$

where K runs over the compact subsets of X and U runs over the entourages in $\mathcal{U}$ [26, Theorem 43.7], or more generally, [20, Theorem 1.2.3]). When X is a reflexive Banach space with the weak topology, by the Banach–Alaoglu theorem, the sets $\{nB: n \in Z^+\}$ are cofinal in the (weakly) compact subsets of X. Thus, for any Banach space Y, all sets of the form

$$D[n] \equiv \{(f, g) \in C(X, Y) \times C(X, Y): \text{ for all } x \in nB, \|f(x) - g(x)\| < 1/n\},$$

where $n \in Z^+$ give a countable base of entourages for uniformity compatible with τ_C on $C(X, Y)$, when X is equipped with the weak topology and Y is equipped with the norm topology. In particular, τ_C is metrisable in this case [26], p.257.

The well-known concept of continuous convergence for sequences of functions ([11], p.268) can be equally well formulated for nets of functions (for a different but equivalent formulation, see ([20], p.40).

DEFINITION: The net of functions $\langle f_\lambda \rangle$ from a topological space X to a topological space Y is said to *converge continuously* to f if for all $x \in X$ and for each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that for all sufficiently large λ, $f_\lambda(U) \subset V$.

It is standard to call a topology τ on $C(X, Y)$ *splitting* (respectively *conjoining*) provided continuous convergence $\Rightarrow$ τ convergence (respectively τ convergence $\Rightarrow$ continuous convergence). Other characterisations of splitting/conjoining topologies are presented in [20]. That the compact-open topology is always splitting is an elementary fact ([20], Theorem 2.5.2), whose proof we include for completeness.

LEMMA 2.1. *If $\langle f_\lambda \rangle$ converges to f continuously, then it converges to f in the compact-open topology.*

PROOF: Let $f \in (A, V)$, a subbasic open set in the compact-open topology. For all $a \in A$, since V is a neighbourhood of $f(a)$, there is a neighbourhood U_a such that $f_\lambda(U_a) \subset V$ for sufficiently large λ. Extract a finite subcover $U_{a_1}, \ldots, U_{a_n}$ of A. Choose λ_0 large enough so that $\lambda \geqslant \lambda_0 \Rightarrow f_\lambda(U_{a_i}) \subset V$ for $i = 1, 2, 3, \ldots, n$. Thus for $\lambda \geqslant \lambda_0$, $f_\lambda(A) \subset f_\lambda(\cup U_{a_i}) \subset V$ as required. ∎

We mention in passing that the compact-open topology on $C(X, Y)$ is conjoining provided X is locally compact ([11], p.275, [20], p.32), and that $C(X, Y)$ can admit at most one topology that is both splitting and conjoining.

Consistent with our initial discussion of the Mosco topology, the Mosco topology on $C(X, Y)$ for the Banach spaces X and Y, denoted by τ_M, is that having as a base all sets of the form

$$[U_1, U_2, \dots, U_n; K] = \{f : f \cap U_i \neq \emptyset \text{ for all } i,\ f \cap K = \emptyset\}$$

for strongly open $U_i \subset X \times Y$ and weakly compact $K \subset X \times Y$. Notice that we identify functions with their graphs, which we will do freely in the sequel. This identification is in the spirit of [**21, 2, 22, 23, 15**].

3. Mosco convergence and the compact-open topology

We first wish to show that the Mosco topology on $C(X, Y)$ is Hausdorff precisely when we work in reflexive spaces. This is accomplished through a sequence of lemmas.

Lemma 3.1. *Let X be a completely regular Hausdorff space. Let K be a nonempty compact subset, $x_1, \dots, x_n$ distinct points in K^c. Let $\mu, \lambda_1, \dots, \lambda_n$ be real scalars. Then there exists $f \in C(X, R)$ with $f(K) = \mu$, and $f(x_i) = \lambda_i$ for each index i.*

Proof: Let βX be the Stone-Cech compactification of X ([**11**], p.243). $K \cup \{x_1, \dots, x_n\}$ is a compact (*a fortiori* closed) subset of βX; so, if $f_0 : K \cup \{x_1, \dots, x_n\} \to R$, where $f_0(K) = \mu$ and $f_0(x_i) = \lambda_i$, we can extend f_0 by Tietze ([**11**], p.149) to a continuous $f : \beta X \to R$. Now restrict f to X to obtain the desired function. ∎

Lemma 3.2. *Let X be a nonreflexive Banach space and Y any Banach space. Then $\langle C(X, Y), \tau_M \rangle$ is* not *Hausdorff.*

Proof: We begin by proving the theorem for the special case $Y = R$. Suppose X is not reflexive. It suffices to show that τ_M-open sets are dense. To this end, consider nonempty τ_M-open basic sets $[U_1, \dots, U_n; K_1]$ and $[U_{n+1}, \dots, U_m; K_2]$ (where the U_i's are norm open and K_1 and K_2 are weakly compact). Since X is not reflexive, the projection of the weakly compact K_i's onto X cannot contain any norm open subset [**12**], p.425. Therefore, the projection of the open U_i's, being norm open, must contain points outside of the projection $\pi_X(K_1 \cup K_2)$. Therefore, we can choose distinct $u_i \in \pi_X(U_i) - \pi_X(K_1 \cup K_2)$ and scalars α_i, so that $(u_i, \alpha_i) \in U_i$. Now, the complete regularity of the weak topology on X and Lemma 3.1 allow us to obtain a function f passing through all the (u_i, α_i)'s and, since $K_1 \cup K_2$ is norm bounded, above $K_1 \cup K_2$. This f is clearly seen to be in $[U_1, \dots, U_n; K_1] \cap [U_{n+1}, \dots, U_m; K_2]$.

Now for the general case, let $y \in Y - \{0\}$ and let L be the linear span of y. Since L is isometrically isomorphic to R, by what we just proved $\langle C(X, L), \tau_M \rangle$ is not Hausdorff. Therefore, since the Hausdorff property is hereditary, if we can embed $\langle C(X, L), \tau_M \rangle$ in $\langle C(X, Y), \tau_M \rangle$ we will be done. Since $C(X, L) \subset C(X, Y)$, we need only show that τ_M on $C(X, L)$ is precisely the Mosco topology on $C(X, Y)$ restricted to the subspace $C(X, L)$ (we'll temporarily call this restricted topology τ_r).

First, let $U_1, \ldots, U_n \subset X \times Y$ be norm open in $X \times Y$, and let $K \subset X \times Y$ be weakly compact. Since convex closed sets are weakly closed, $X \times L$ is weakly closed, so that $K \cap (X \times L)$ is weakly compact in $X \times L$ (here, we are implicitly using the fact that the weak topology on a subspace is the inherited weak topology from the parent space by the Hahn-Banach Theorem). Observe that

$$C(X, L) \cap [U_1, \ldots, U_n; K] = [U_1 \cap (X \times L), \ldots, U_n \cap (X \times L); K \cap (X \times L)]$$

where $[U_1, \ldots, U_n; K]$ is $\langle C(X, Y), \tau_M\rangle$-open and $[U_1 \cap (X \times L), \ldots, U_n \cap (X \times L); K \cap (X \times L)]$ is $\langle C(X, L), \tau_M\rangle$-open. Thus, $\tau_r \subset \tau_M$.

To see $\tau_M \subset \tau_r$, suppose $U_1', \ldots, U_n'$ are norm open in $X \times L$, and suppose K' is weakly compact in $X \times L$. Evidently, K' is weakly compact in $X \times Y$ and since there must be $U_1, \ldots, U_n$ norm open in $X \times Y$ with $U_i' = U_i \cap (X \times L)$, we have $[U_1', \ldots, U_n'; K'] = C(X, L) \cap [U_1, \ldots, U_n; K']$ as required. $\square$

We remark that the proof above shows that for *any* closed subspace L of Y, the Mosco topology on the subspace $C(X, L)$ of $C(X, Y)$ is precisely the restricted Mosco topology of the larger space. We will also need the following result, that appears as Lemma 4.1 of [**6**].

LEMMA 3.3. *Let Y be a nonreflexive Banach space. Let K be a weakly compact subset of Y and let $V_1, V_2, \ldots, V_n$ be nonempty norm open subsets of Y (not necessarily distinct). Then there exist distinct points $y_i \in V_i$ such that* $\operatorname{co}(\{y_1, y_2, \ldots, y_n\}) \cap K = \emptyset$.

In addition, we'll need this purely topological technical lemma:

LEMMA 3.4. *Let X be a completely regular Hausdorff space, and let Y be a path-connected space, and let $\{x_0, x_1, x_2, \ldots, x_n\}$ be a finite subset of X. Then any function $f\colon \{x_1, x_1, x_2, \ldots, x_n\} \to Y$ can be extended to a continuous function $\varphi\colon X \to Y$.*

PROOF: First, using the complete regularity of X, extend the map

$$g\colon \{x_0, x_1, x_2, \ldots, x_n\} \to [0, 1] \text{ defined by}$$
$$g(x_i) \equiv i/n$$

to $\psi\colon X \to [0, 1]$ (see Lemma 3.1). Now for each $i = 1, 2, \ldots, n$, since each $[(i-1)/n, i/n]$ is homeomorphic to $[0, 1]$, define a path $\pi_i\colon [(i-1)/n, i/n] \to Y$ from $f(x_{i-1})$ to $f(x_i)$, that is, we require $\pi_i((i-1)/n) = f(x_{i-1})$ and $\pi_i(i) = f(x_i)$. Let $\pi\colon [0, 1] \to Y$ be the "pasting" of these functions. Since

$$\{[(i-1)/n, i/n]\colon i = 1, \ldots, n\}$$

is a closed finite (*a fortiori* neighbourhood-finite) family, π is continuous ([**11**], p.83). Hence, $\varphi = \pi \circ \psi$ is the required extension. ∎

Lemma 3.5. *Let Y be a nonreflexive Banach space and X be any Banach space. Then $\langle C(X, Y), \tau_M\rangle$ is not Hausdorff.*

Proof: We show that nonempty τ_M-basic open sets $[V_1, \ldots, V_n; K_1]$ and $[V_{n+1}, \ldots, V_m; K_2]$ must intersect. Take the projections on the Y axis and observe that $\pi_Y(K_1 \cup K_2)$ is weakly compact, and that the $\pi_Y(V_i)$ are norm open. Applying Lemma 3.3 to Y, we find distinct $y_i \in \pi_Y(V_i)$ such that $\operatorname{co}(\{y_1, y_2, \ldots, y_m\}) \cap \pi_Y(K_1 \cup K_2) = \emptyset$. Now there are distinct $x_i \in X$ such that $(x_i, y_i) \in V_i$. Let f be a partial function mapping for each i the point x_i to the point y_i. Now using Lemma 3.4 and the fact that convex hulls are path-connected, extend f continuously to $f'\colon X \to \operatorname{co}(\{y_1, y_2, \ldots, y_m\})$. This extension f' is seen to be the required function lying the intersection. ∎

Theorem 3.6. *Let X, Y be Banach spaces. Then the following are equivalent:*

(a) *τ_M is Hausdorff on $C(X, Y)$;*

(b) *X, Y are both reflexive.*

Proof: (a) $\Rightarrow$ (b): This is immediate from Lemmas 3.2 and 3.5 above.

(b) $\Rightarrow$ (a): Let $f, g \in C(X, Y)$ and suppose $f(a) \neq g(a)$. Since the graph of g is weakly closed, consider $A \equiv (a, f(a)) + \varepsilon B$ with $\varepsilon > 0$ small enough such that $A \cap g = \emptyset$. Then since closed balls are weakly compact in $X \times Y$, $f \in [\operatorname{Int} A; \emptyset]$, $g \in [X \times Y; A]$ yields the desired separation. ∎

We now shall show that when X is reflexive and Y is finite dimensional, the Mosco topology is precisely the compact-open topology with respect to the weak topology on X. To this end we need

Lemma 3.7. *Let X be a reflexive Banach space, and let $f\colon X \to R^n$ be weakly continuous. For $m \in Z^+$, the "tube" T_m around $f \mid mB_X$ given by*

$$T_m \equiv \cup\{\{x\} \times S(f(x), 1/m)\colon x \in mB_X\}$$

is weakly compact.

Proof: By the Eberlein-Smulyan result ([**12**], p.430), it is only necessary to show that T_m is weakly *sequentially* compact. Thus, let $\langle (x_n, y_n)\rangle$ be a sequence in T_m, that is, $\|f(x_n) - y_n\| = 1/m$ for all $n \in Z^+$. By reflexivity of X, mB_X is weakly compact, so we can assume without loss of generality (by passing to a subsequence) that $\langle x_n\rangle \to x$ weakly. Since f is weakly continuous, $\langle f(x_n)\rangle \to f(x)$. Now since $\|y_n - f(x_n)\| = 1/m$, and $\langle f(x_n)\rangle$ is bounded, $\langle y_n\rangle$ is also bounded in R^n. So by refining again, we can assume without loss of generality that $\langle y_n\rangle \to y$ for some y.

Thus, by the continuity of the distance function, $\|y - f(x)\| = 1/m$ whence $(x, y) \in T_m$ as required. ∎

THEOREM 3.8. *Let X be a reflexive Banach space. Then the Mosco topology on $C(X, R^n)$ equals the compact-open topology with respect to the weak topology on X.*

PROOF: $\tau_C \subset \tau_M$. We work with the countable base for the usual uniformity for τ_C on $C(X, R^n)$ whose typical entourage is of the form:

$$D[n] = \{(f, g) \in C(X, R^n) \times C(X, R^n)\colon \text{ for all } x \in nB_X,\ \|f(x) - g(x)\| < 1/n\}.$$

Fix $f \in C(X, R^n)$ and fix n and let $T = \cup\{\{x\} \times S(f(x), 1/n)\colon x \in nB_X\}$ be the weakly compact tube as defined in Lemma 3.7. Choose a strong neighbourhood V of the origin θ contained in nB_X such that

$$f(V) \subset f(\theta) + (1/2n)(\text{Int } B_{R^n}).$$

Let $W = V \times (f(\theta) + (1/2n))(\text{Int } B_{R^n})$. We claim that $f \in [W;T] \subset D[n](f)$; to wit, let $g \in [W;T]$. To prove that $g \in D[n](f)$, we need to show that $\|f(x) - g(x)\| < 1/n$ on nB_X. Since g hits W, there is some $x_0 \in V$ such that $g(x_0) \in f(\theta) + (1/2n)(\text{Int } B_{R^n})$. Now $\|f(x_0) - f(\theta)\| < 1/2n$ since $x_0 \in V$. Combining this with $\|g(x_0) - f(\theta)\| < 1/2n$ and the triangle inequality, we obtain $\|f(x_0) - g(x_0)\| < 1/n$. To finally see that $\|f(x) - g(x)\| < 1/n$ on nB_X observe that $\|f(\cdot) - g(\cdot)\|$ is a weakly continuous function on nB_X (using that the difference of continuous functions is continuous and that the norm function is continuous). Since g does not meet T, $\|f(x) - g(x)\|$ cannot equal $1/n$. By the connectedness of nB_X, that $\|f(x) - g(x)\| < 1/n$ at one point (namely x_0) must guarantee $\|f(x) - g(x)\| < 1/n$ throughout nB_X. This is precisely what we need to show for g to be in $D[n](f)$.

$\tau_M \subset \tau_C$: Fix the τ_M-basic open set $[U_1, \ldots, U_m; K]$ and $f \in [U_1, \ldots, U_m; K]$. We seek a τ_C-open neighbourhood of f contained in $[U_1, \ldots, U_m; K]$. Now since f is a weakly continuous map, the graph of f is weakly closed in $X \times R^n$. Since f does not intersect the weakly compact set K, there exists $\varepsilon > 0$ such that $(K + \varepsilon B) \cap f = \emptyset$. Now for every $i = 1, 2, \ldots, m$, choose a point $(x_i, y_i) \in f \cap U_i$ and $\delta > 0$ such that $(x_i, y_i) + \delta B \subset U_i$. Choosing an n large enough so that $\pi_X(K) \cup \{x_i\}_i \subset nB_X$ *and* $1/n < \min\{\delta, \varepsilon\}$, we have

$$f \in D[n](f) \subset [U_1, \ldots, U_m; K]$$

as required. ∎

Notice that the inclusion $\tau_M \subset \tau_C$ does not require reflexivity. Also notice that when X is reflexive, the Mosco topology τ_M on $C(X, R^n)$ is *metrisable* because there

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is a countable basis for the uniformity of $\langle C(X, R^n), \tau_C\rangle$. The same statement for the closed convex subsets of X requires that X be reflexive *and separable* [6, 3].

That $\tau_C = \tau_M$ may fail on $C(X, Y)$ with X reflexive and Y infinite dimensional is confirmed by the following example:

EXAMPLE. There is a sequence of functions $\langle f_n\rangle$ in $C(R, l_2)$ τ_M-convergent to the identically zero function f that is not strongly pointwise convergent (hence not τ_C-convergent). Let $\langle e_n\rangle$ be the usual orthonormal base for the sequence space, and define $f_n : R \to l_2$ by

$$f_n(x)\begin{cases} = 0 & \text{if } x > 1/n \text{ or } x < -1/n, \\ = (1 - nx)e_n & \text{if } 0 < x \leqslant 1/n, \\ = (1 + nx)e_n & \text{if } -1/n \leqslant x \leqslant 0. \end{cases}$$

Let $f \in C(R, l_2)$ be the identically zero function $f(x) \equiv 0$. We claim $\langle f_n\rangle \to f$ in τ_M but $\langle f_n\rangle$ does not converge to the zero function pointwise. This last claim is clear since $f_n(0) = e_n$ for all n. To see that $\langle f_n\rangle \to f$ in τ_M, let $[U_1, \ldots, U_n; K]$ be a τ_M-basic open neighbourhood of f. Since each strongly open U_i hits f (the "R-axis"), let $(\alpha_i, \theta) \in U_i$ where $\alpha_i \neq 0$ and choose N such that for all i, $1/N < |\alpha_i|$. Then for all $n > N$, $f_n \cap U_i \neq \emptyset$. Now consider the weakly compact K which misses f. We claim that eventually $\langle f_n\rangle$ misses K; otherwise, (by passing to a subsequence) we can find an n_0 such that if $n > n_0$, $f_n \cap K \neq \emptyset$. For each $n > n_0$ select a point $(\lambda_n, \mu_n e_n) \in f_n \cap K$. Since each $\lambda_n \in]-1/n, 1/n[$, we have $\langle \lambda_n\rangle \to 0$. Since $\{\mu_n : n \in Z^+\}$ is a bounded set of scalars (in particular, never exceeding unity) and $\langle e_n\rangle \to \theta$ weakly, $\langle \mu_n e_n\rangle$ converges weakly to $\theta \in l_2$. The weak compactness of K would force $(0, \theta) \in K$, a contradiction.

Let X be a normed linear space. We may of course view elements of X^* as sitting in $C(X, R)$. Norm convergence of linear functionals means nothing more than uniform convergence on bounded sets, and thus is stronger than τ_C-convergence. But with reflexivity, we have as a corollary to Theorem 3.8 the following informal observation made in sequential form in [5].

COROLLARY 3.9. *Suppose $\langle y_\alpha\rangle$ is a net in X^* with X reflexive. The following are equivalent:*

(1) $\lim_\alpha \|y_\alpha - y\| = 0$;
(2) $\langle y_\alpha\rangle \to y$ *in* τ_C;
(3) $\langle y_\alpha\rangle \to y$ *in* τ_M.

PROOF: This corollary is immediate from what we know so far. Condition (1) is equivalent to uniform convergence on bounded subsets, while (2) is equivalent to uniform convergence on weakly compact subsets. Thus (1) and (2) are equivalent by reflexivity. That (2) holds if and only if (3) holds is the content of Theorem 3.8. ∎

As noted in Section 2, by local compactness, τ_C is conjoining on $C(X, Y)$ for finite dimensional X and any Banach space Y. The next example shows that τ_C (and hence the weaker τ_M) on $C(X, R)$ for infinite dimensional X is never conjoining.

EXAMPLE. Let X be an infinite dimensional Banach space. There exists a net $\langle y_\lambda \rangle$ in X^* that converges to the origin $\theta^* \in X^*$ in norm (thus in τ_C and in τ_M), yet $\langle y_\lambda \rangle$ does *not* converge weakly continuously to θ^*. We first construct a net $\langle x_\lambda \rangle$ such that $\langle \|x_\lambda\| \rangle \to \infty$, yet $\langle x_\lambda \rangle \to \theta$ weakly. Consider the directed set

$$D = \{(\{f_1, \dots, f_n\}, k) : f_i \in X^*, \quad f_i \neq \theta^*, \quad k \in Z^+\}$$

where $(\{f_1, \dots, f_n\}, k) \leqslant (\{g_1, \dots, g_m\}, l) \equiv \{f_i\} \subset \{g_i\}$ and $k \leqslant l$. Map each $(\{f_1, \dots, f_n\}, k)$ to any vector in $\cap \ker f_i$ having norm k. This is possible since each kernel has codimension 1, so $\cap \ker f_i$ cannot be trivial. Thus, while this net clearly goes to ∞ in norm, it nevertheless converges weakly to θ, to wit: for any basic open neighbourhood of θ determined by functionals $f_1, \dots, f_n$, then x_{λ_0} is in the neighbourhood where $\lambda_0 = (\{f_1, \dots, f_n\}, 1)$. Furthermore, if $\lambda \geqslant \lambda_0$, then x_λ is also in that same neighbourhood.

Now, for each λ, using the Hahn-Banach theorem, we can find a $y_\lambda \in X^*$ such that $y_\lambda(x_\lambda) = 1$ and $\|y_\lambda\| = 1/\|x_\lambda\|$. Since $\|y_\lambda - \theta^*\| = \|y_\lambda\| = 1/\|x_\lambda\|$, $\langle y_\lambda \rangle$ converges to θ^* in norm, yet we claim $\langle y_\lambda \rangle$ does *not* converge weakly continuously to θ^*. If it did, for the point θ and the neighbourhood $(-1/2, 1/2)$ of 0, there would be a weakly open neighbourhood $V = V(\theta)$ and a $\lambda_0 \in D$ such that $\lambda \geqslant \lambda_0 \Rightarrow y_\lambda(V) \subset (-1/2, 1/2)$. Since $\langle x_\lambda \rangle \to \theta$ weakly, $\langle x_\lambda \rangle$ is eventually in V. Therefore, there is some large λ_1 such that $y_{\lambda_1}(V) \subset (-1/2, 1/2)$ and $x_{\lambda_1} \in V$, a contradiction, since $y_{\lambda_1}(x_{\lambda_1}) = 1$.

The last example shows more, namely that τ_C is not conjoining for $C(X, Y)$ whenever X is infinite dimensional and Y is any nontrivial Banach space, for Y contains a one dimensional subspace. But there are attractive weaker versions of continuous convergence which are equivalent to τ_C-convergence in $C(X, Y)$ with X reflexive:

THEOREM 3.10. *Let X be a reflexive Banach space and let Y be a Banach space, with f, $\langle f_\lambda \rangle_{\lambda \in L}$ in $C(X, Y)$. Then the following are equivalent:*

(1) *$\langle f_\lambda \rangle_{\lambda \in L}$ converges to f in the compact-open topology;*
(2) *for all $x \in X$, $\varepsilon > 0$, $\rho > 0$, there exists a weak neighbourhood $W = W(x, \varepsilon, \rho)$ of x and there exists $\lambda_0 \in L$ such that $\lambda \geqslant \lambda_0 \Rightarrow f_\lambda(W \cap \rho B_X) \subset f(x) + \varepsilon B_Y$;*
(3) *for each cofinal function $\varphi: M \to L$, where M is a directed set, if $\langle x_{\varphi(\mu)} \rangle$ is eventually bounded and $\langle x_{\varphi(\mu)} \rangle \to x$ weakly, then $\langle f_{\varphi(\mu)}(x_{\varphi(\mu)}) \rangle \to f(x)$ strongly.*

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PROOF: (1) $\Rightarrow$ (2). Suppose $f = \tau_C - \lim f_\lambda$ and suppose that $x \in X$, $\varepsilon > 0$, $\rho > 0$ are given. We find a weakly open neighbourhood of x, say $W = W(x)$, such that $f(W) \subset f(x) + (\varepsilon/2)B_Y$. Consider $W \cap \rho B_X$. Since the weak closure of a bounded set is bounded, the weak closure of $W \cap \rho B_X$ is weakly compact. Then, using the notation for subbasic open sets of the compact-open topology, $f \in (A, V) \equiv (w - \text{cl } W \cap \rho B_X, f(x) + \varepsilon(\text{Int } B_Y))$ and $\langle f_\lambda \rangle$ is eventually in (A, V). Therefore there exists a λ_0 such that $\lambda \geqslant \lambda_0$ implies $f_\lambda \in (A, V)$ so that $f_\lambda(W \cap \rho B_X) \subset f(x) + \varepsilon B_Y$.

(2) $\Rightarrow$ (3). Let $\varphi\colon M \to L$ be cofinal, $\langle x_{\varphi(\mu)} \rangle$ be eventually bounded, and let $x = w - \lim x_{\varphi(\mu)}$. We need to show that $\langle f_{\varphi(\mu)}(x_{\varphi(\mu)}) \rangle \to f(x)$ strongly. To this end let $\varepsilon > 0$ and let ρ be large enough so that $x_{\varphi(\mu)} \in \rho B_X$ eventually. By (2) there is a weak neighbourhood W of x and $\mu_0 \in M$ such that when $\mu \geqslant \mu_0$ we have

$$f_{\varphi(\mu)}(W \cap \rho B_X) \subset f(x) + \varepsilon B_Y.$$

Since $\langle x_{\varphi(\mu)} \rangle \to x$ weakly, choose μ_1 large enough so that $\mu \geqslant \mu_1$ implies $x_{\varphi(\mu)} \in W$ and let μ_2 be large enough so $\mu \geqslant \mu_2$ implies $x_{\varphi(\mu)} \in \rho B_X$. Let $\mu^* \in M$ majorise each of μ_0, μ_1 and μ_2. Then for $\mu \geqslant \mu^*$:

1. $f_{\varphi(\mu)}(W \cap \rho U) \subset f(x) + \varepsilon B_Y$;
2. $x_{\varphi(\mu)} \in W$;
3. $x_{\varphi(\mu)} \in \rho B_X$.

We conclude $f_{\varphi(\mu)}(x_{\varphi(\mu)}) \in f(x) + \varepsilon B_Y$ for $\mu \geqslant \mu^*$, as required.

(3) $\Rightarrow$ (1). Suppose (1) fails; then for some weakly compact subset A of X and a norm open subset V of Y, we have $f \in (A, V)$ but $f_\lambda \notin (A, V)$ frequently. Thus, there is a cofinal subset L' of L and for each $l \in L'$ a point $x_l \in A$ such that $f(x_l) \in V^c$. By weak compactness, $\langle x_l \rangle$ has a subnet convergent weakly to some $x \in A$. This means precisely that there is a directed set M, and a cofinal map $\alpha\colon M \to L'$ so that the aforementioned subnet converging weakly to x is $\langle x_{\alpha(\mu)} \rangle$. Now write $\varphi = i \circ \alpha$ where $i : L' \to L$ is the inclusion map. Evidently, $\langle x_{\alpha(\mu)} \rangle$ is bounded because A is. The strong convergence of $\langle f_{\varphi(\mu)}(x_{\varphi(\mu)}) \rangle$ to $f(x)$ is impossible, because this net lies in the norm closed set V^c and $f(x) \in V$. Thus, condition (3) must fail. $\square$

4. Epigraphical and hypographical convergence

Our next theorem relates the Mosco epi- and hypo- topologies for real valued continuous functions to the Mosco topology for graphs. The idea is to identify $f: X \to R$ with its ***epigraph*** epi f (respectively its ***hypograph*** hypo f), that is, the points lying on or above [respectively below] the graph of f, rather than with the graph of f itself. This point of view is standard in one-sided (for example, convex) analysis [**1**, **10**, **24**, **4**].

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DEFINITION: Let X be a Banach space. The *epi-Mosco topology* τ_{epi} on $C(X, R)$ is generated by the miss sets $\{f\colon \operatorname{epi} f \cap K = \emptyset\}$ for weakly compact $K \subset X \times R$, and the hit sets $\{f\colon \operatorname{epi} f \cap V \neq \emptyset\}$ for strongly open $V \subset X \times R$. The *hypo-Mosco topology* τ_{hypo} is defined similarly, replacing $\operatorname{epi} f$ by $\operatorname{hypo} f$.

We will use the notation $[U_1, \ldots, U_n; K]_e$ for our basic open sets in the epi-Mosco topology (for U_i strongly open and K weakly compact) and $[U_1, \ldots, U_n; K]_h$ for our basic open sets in the hypo-Mosco topology.

THEOREM 4.1. *Let X be a reflexive Banach space. Then*

$$\tau_M = \tau_{\mathrm{epi}} \vee \tau_{\mathrm{hypo}}.$$

on the function space $C(X, R)$.

PROOF: Observe that $\tau_{\mathrm{epi}} \vee \tau_{hypo}$ is generated by all sets of the form $[U_1, \ldots, U_s; K_1]e \cap [V_1, \ldots, V_t; K_2]_h$.

$\tau_{\mathrm{M}} \subset \tau_{\mathrm{epi}} \vee \tau_{\mathrm{hypo}}$: Sets of the form $[V; K]$ where $V = U \times (\alpha, \beta)$ with U norm open and convex in X and K weakly compact in $X \times R$ determine a subbase for the Mosco topology τ_M on $C(X, R)$. Let $f \in [V; K]$. Then we claim

$$f \in [V; K \cap \operatorname{hypo} f]_e \cap [V; K \cap \operatorname{epi} f]_h \subset [V; K]$$

as required. If $g \in [V; K \cap \operatorname{hypo} f]_e \cap [V; K \cap \operatorname{epi} f]_h$, there exists an $x_1 \in U$ with $g(x_1) < \beta$ and $x_2 \in U$ with $g(x_2) > \alpha$. By the intermediate value property for continuous functions, there is a $\lambda \in [0, 1]$ such that $g(\lambda x_1 + [1 - \lambda]x_2) \in (\alpha, \beta)$. This guarantees $g \in [V; K]$.

$\tau_{\mathrm{epi}} \vee \tau_{\mathrm{hypo}} \subset \tau_{\mathrm{M}}$: Suppose $f \in [U_1, \ldots, U_s; K_1]_e \cap [V_1, \ldots, V_t; K_2]_h$. We may assume without loss of generality that the U_i are of the form $W_i \times (-\infty, \alpha)$ where W_i is open in X and the V_j are of the form $G_j \times (\beta, \infty)$ for G_j open in X (see [4, Lemma 2.1]). We seek a τ_M-neighbourhood W of f contained in the set above. To this end, choose

$$(x_i, \alpha_i) \in f \cap U_i \text{ and } (z_j, \beta_j) \in f \cap V_j.$$

for each $i = 1, 2, \ldots, s$; $j = 1, 2, \ldots, t$. Choose $\varepsilon > 0$ small enough so $(x_i, \alpha_i) + \varepsilon B \subset U_i$ and $(z_j, \beta_j) + \varepsilon B \subset V_j$ for all i and j.

Consider the weakly compact set $K \equiv \pi_X(K_1 \cup K_2) \cup \{x_i\} \cup \{z_j\}$. By Theorem 3.8, $C(X, R)$ with the Mosco topology τ_M is homeomorphic to $C(X, R)$ with the compact-open topology τ_C. The required neighbourhood of f is

$$W \equiv \{g\colon |f(x) - g(x)| < \delta \text{ for all } x \in K\}$$

where $\delta = \min\{\varepsilon, \delta_1, \delta_2\}$ and δ_i is the distance from the weakly closed set f to K_i. These δ_i are positive because disjoint weakly closed and weakly compact sets are a positive distance apart. ∎

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THEOREM 4.2. (Converse) *Let X be a Banach space. If $\tau_M = \tau_{\text{epi}} \vee \tau_{\text{hypo}}$ on the function space $C(X, R)$, then X is reflexive.*

PROOF: Suppose on the contrary that X is not reflexive. We will show that τ_M does not contain τ_{epi}.

We again use the fact that in nonreflexive spaces, every weakly compact set has empty norm interior. We will show that the τ_{epi}-open set

$$\Omega \equiv \{f : \text{epi} f \cap K = \emptyset\}$$

where $K \subset X \times R$ is weakly compact is *not* τ_M-open. To this end, we show that $[V_1, V_2, \ldots, V_n; K_1] - \Omega$ is nonempty for every nonempty τ_M-basic open set $[V_1, V_2, \ldots, V_n; K_1]$.

Since $K_1 \cup K$ is weakly compact and π_X is weakly continuous, $\pi_X(K_1 \cup K)$ is also weakly compact and hence has empty norm interior. In particular, none of the sets $\pi_X(V_i)$ are contained in $\pi_X(K_1 \cup K)$; so, select $v_i \in \pi_X(V_i) - \pi_X(K_1 \cup K)$ and an $\alpha_i \in R$ such that $(v_i, \alpha_i) \in V_i$. Let $\beta_0 = \inf\{\beta : (x, \beta) \in K \cup K_1\} = \inf \pi_R(K \cup K_1)$. Using the complete regularity of X with the weak topology and Lemma 3.1, define $g \in C(X, R)$ so that $g(v_i) = \alpha_i$, and $g(\pi_X(K \cup K_1)) = \beta_0 - 1$. Thus $g \in [V_1, V_2, \ldots, V_n; K_1\} - \Omega$, as required. ∎

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Department of Mathematics
California State University,Los Angeles
Los Angeles, CA 90032
United States of America