Canad. Math. Bull.Vol. **34** (4), 1991 pp. 480–484

SOME REMARKS ON ROTORS IN LINK THEORY

GYO TAEK JIN AND DALE ROLFSEN

ABSTRACT. We present examples showing that certain results on the invariance of link polynomials under generalized mutation are the best possible. They show, moreover, that this generalized mutation cannot be effected by a sequence of ordinary mutations. One of the examples also shows that the reduced Jones polynomial can be a more sensitive invariant than the Jones polynomial itself.

Mutation of knots and links, introduced by J. H. Conway [C], provides a means of creating examples of different links having the same polynomial invariants. Achieving the same effect, a more complicated procedure was described in [APR], involving *rotors* in a link. The idea is that one assumes that some part of a link diagram (the *rotor*) has n-fold rotational symmetry, $n \geq 3$, and then gives that part a dihedral flip, forming the *rotant* link. Unless the rotor itself has dihedral symmetry, the rotant is generally a different link than the original (but with the same number of components), and it was shown in [APR] that under certain assumptions, they have equal polynomial invariants. In particular, mutual rotants have the same Jones polynomial when $n \leq 5$, the same Homfly polynomial when $n \leq 4$, and the same Kauffman polynomial when $n = 3$. The three examples below show that these assumptions on n cannot be improved.

The Kauffman polynomial is defined in [K]. A basic reference for the others is [FYHLMO], but since there are several versions in the literature, we give the formulae defining the versions used here for the reader's convenience:

Jones polynomial: $t^{-1}V_+(t) - tV_-(t) = (t^{1/2} - t^{-1/2})V_0(t)$

Homfly polynomial: $lP_+(l,m) + l^{-1}P_-(l,m) + mP_0(l,m) = 0$

$V_{\text{unknot}} = P_{\text{unknot}} = 1.$

We also recall that the *reduced* Jones polynomial of the k-component link $L = L_1 \cup \cdots \cup L_k$ is the quotient of its polynomial by the product of those of the individual components.

$$\tilde{V}_L(t) = \frac{V_L(t)}{V_{L_1}(t)\cdots V_{L_k}(t)}$$

It is noted in [R1] that this rational function, and reduced versions of the other polynomials as well, are invariant under (nonambient) PL isotopy of the link L. They are also invariant under mutation.

Research supported in part by Canadian NSERC grant A8082. We would like to thank J. Hoste and J. Jenkins for developing software used for the electronic calculation of polynomial invariants, and K. Millett for verifying some of our calculations.

Received by the editors April 1, 1989; revised: September 14, 1990.

AMS subject classification: Primary: 57M25; secondary: 57M15.

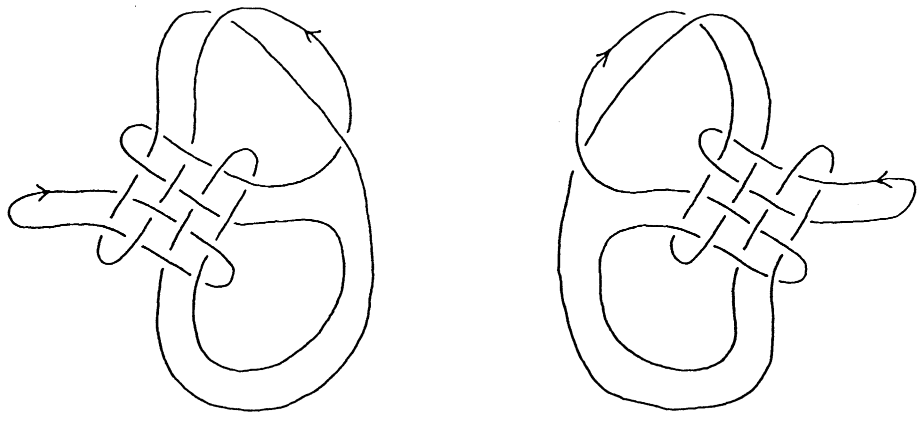


FIGURE 1

EXAMPLE 1. Figure 1 illustrates two 2-component links, L_4 and M_4. (The subscripts here refer to the fact that they are rotants of order 4, not to components as above). Instead of flipping the rotor, for visual clarity we instead flip the complementary tangle (the stator), an equivalent move. The equality of their Jones polynomials is expected:

$$\begin{aligned} V_{L_4}(t) = V_{M_4}(t) = \\ & - t^{-41/2} + 7t^{-39/2} - 26t^{-37/2} + 68t^{-35/2} - 139t^{-33/2} \\ & + 237t^{-31/2} - 348t^{-29/2} + 450t^{-27/2} - 518t^{-25/2} \\ & + 533t^{-23/2} - 494t^{-21/2} + 410t^{-19/2} - 302t^{-17/2} \\ & + 195t^{-15/2} - 109t^{-13/2} + 50t^{-11/2} - 19t^{-9/2} + 5t^{-7/2} - t^{-5/2}. \end{aligned}$$

Both links have one unknotted component and one knotted one. The knotted component of L_4 is of type 9_{20} in the table of [R2] and has Jones polynomial:

$$-t^{-9} + 3t^{-8} - 5t^{-7} + 6t^{-6} - 7t^{-5} + 7t^{-4} - 5t^{-3} + 4t^{-2} - 2t^{-1} + 1$$

The knotted component of M_4, on the other hand, is (the reverse of) 10_{72} with polynomial

$$t^{-10} - 4t^{-9} + 7t^{-8} - 10t^{-7} + 12t^{-6} - 12t^{-5} + 11t^{-4} - 8t^{-3} + 5t^{-2} - 2t^{-1} + 1$$

and we see that L_4 and M_4 have different reduced Jones polynomials. Similarly, these links have the same Alexander polynomials, but different reduced Alexander polynomials. It follows that they are not related by any sequence of ordinary mutation, which is also verified by the fact that their Kauffman polynomials differ:

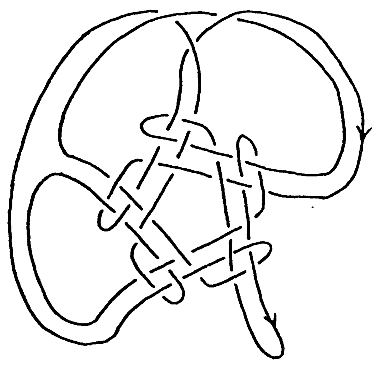
$$\begin{aligned}F_{L_4}(a,z) = {} & (-2a^9 - 4a^{11} - 4a^{13} - 3a^{15} - a^{17})z^{-1} \\ & + (2a^{10} + 2a^{12} + 4a^{14} + 5a^{16} + 2a^{18}) \\ & + (-2a^7 + 18a^9 + 37a^{11} + 36a^{13} + 24a^{15} + 4a^{17} - a^{19})z \\ & + O(z^2)\end{aligned}$$

$$\begin{aligned}F_{M_4}(a,z) = {} & (-2a^9 - 4a^{11} - 4a^{13} - 3a^{15} - a^{17})z^{-1} \\ & + (a^{10} - 2a^{12} - 2a^{14} + a^{16} + a^{18}) \\ & + (-2a^7 + 18a^9 + 37a^{11} + 37a^{13} + 27a^{15} + 7a^{17})z \\ & + O(z^2)\end{aligned}$$

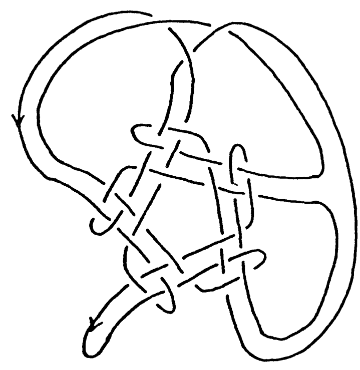
and

$$\begin{aligned}F_{L_4}(a,z) - F_{M_4}(a,z) = {} & (a^{10} + 4a^{12} + 6a^{14} + 4a^{16} + a^{18}) \\ & + (-a^{13} - 3a^{15} - 3a^{17} - a^{19})z + O(z^2)\end{aligned}$$

It may also be worth noting that, although the knots of L_4 and M_4 both have Murasugi signature 9, they are not cobordant. A cobordism between them would imply their Alexander polynomials agree after multiplying by factors of the form $f(t)f(t^{-1})$. But this contradicts the fact that their determinants are, respectively, 41 and 73, and there is no integral solution to the equation $73x^2 = 41y^2$.



L5



M5

FIGURE 2

EXAMPLE 2. The links L_5 and M_5 in Figure 2 are 5-rotants with different Homfly polynomials:

$$\begin{aligned}P_{L_5}(l,m) = {} & (6l^{-31}+22l^{-29}+24l^{-27}+5l^{-25}-3l^{-23})m^{-1}\\ & +(2l^{-35}+24l^{-33}+45l^{-31}-89l^{-29}-230l^{-27}-53l^{-25}+69l^{-23}-2l^{-21})m\\ & +O(m^3).\end{aligned}$$

$$\begin{aligned}P_{M_5}(l,m) = {} & (l^{-33}+11l^{-31}+32l^{-29}+34l^{-27}+10l^{-25}-2l^{-23})m^{-1}\\ & +(2l^{-35}+23l^{-33}+35l^{-31}-114l^{-29}-255l^{-27}-63l^{-25}+68l^{-23}-2l^{-21})m\\ & +O(m^3)\end{aligned}$$

Their difference is, more precisely,

$$\begin{aligned}& P_{L_5}(l,m)-P_{M_5}(l,m)\\ & \quad = (-l^{-33}-5l^{-31}-10l^{-29}-10l^{-27}-5l^{-25}-l^{-23})m^{-1}\\ & \qquad +(l^{-33}+10l^{-31}+25l^{-29}+25l^{-27}+10l^{-25}+l^{-23})m\\ & \qquad +(-6l^{-31}-22l^{-29}-22l^{-27}-6l^{-25})m^3\\ & \qquad +(l^{-31}+8l^{-29}+8l^{-27}+l^{-25})m^5+(-l^{-29}-l^{-27})m^7\end{aligned}$$

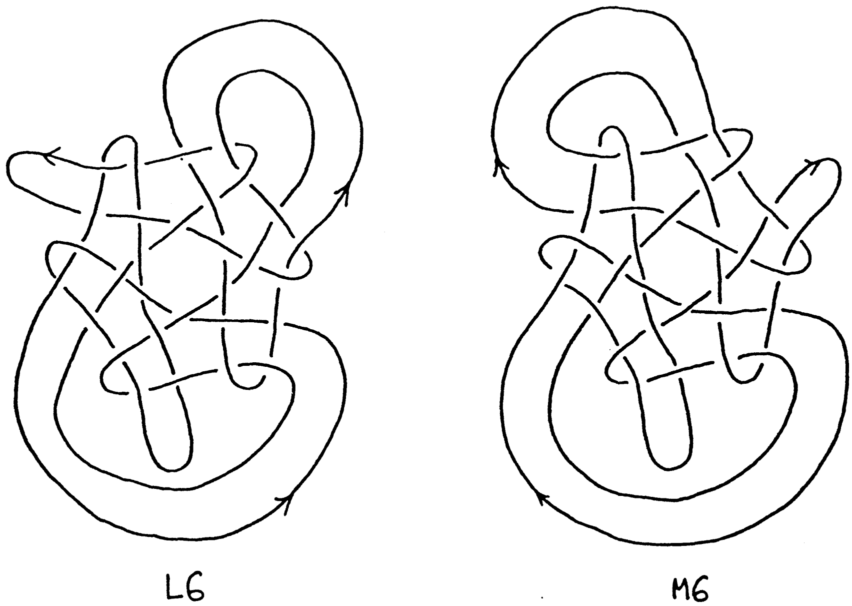


Figure 3

EXAMPLE 3. The links L_6 and M_6 of Figure 3 are 6-rotants with different Jones polynomials:

$$\begin{aligned}V_{L_6}(t) = t^{-1} &- 8 + 42t - 168t^2 + 552t^3 - 1555t^4 + 3846t^5 - 8481t^6 \\ &+ 16863t^7 - 30459t^8 + 50275t^9 - 76164t^{10} + 106279t^{11} \\ &- 136966t^{12} + 163352t^{13} - 180517t^{14} + 184917t^{15} \\ &- 175495t^{16} + 154062t^{17} - 124748t^{18} + 92778t^{19} \\ &- 63004t^{20} + 38756t^{21} - 21367t^{22} + 10408t^{23} - 4392t^{24} \\ &+ 1561t^{25} - 448t^{26} + 97t^{27} - 14t^{28} + t^{29}\end{aligned}$$

$$\begin{aligned}V_{M_6}(t) = t^{-1} &- 8 + 42t - 168t^2 + 552t^3 - 1555t^4 + 3845t^5 - 8478t^6 \\ &+ 16856t^7 - 30445t^8 + 50253t^9 - 76134t^{10} + 106247t^{11} \\ &- 136939t^{12} + 163337t^{13} - 180519t^{14} + 184934t^{15} \\ &- 175524t^{16} + 154095t^{17} - 124778t^{18} + 92800t^{19} \\ &- 63016t^{20} + 38761t^{21} - 21368t^{22} + 10408t^{23} - 4392t^{24} \\ &+ 1561t^{25} - 448t^{26} + 97t^{27} - 14t^{28} + t^{29}\end{aligned}$$

Therefore

$$\begin{aligned}&V_{L_6}(t) - V_{M_6}(t) \\ &= t^5 - 3t^6 + 7t^7 - 14t^8 + 22t^9 - 30t^{10} + 32t^{11} - 27t^{12} \\ &\quad + 15t^{13} + 2t^{14} - 17t^{15} + 29t^{16} - 33t^{17} + 30t^{18} - 22t^{19} \\ &\quad + 12t^{20} - 5t^{21} + t^{22}\end{aligned}$$

We close with the question of whether the Alexander and Conway polynomials must agree for n-rotants of *all* orders n. We have not discovered any examples to contradict this possibility.

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Korea Institute of Technology

Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Y4