Waves of Solar Activity

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Abstract: We develop a theory of the sunspot cycle predicated on the assumption that the observed bands of activity are packets of dynamo waves. An approximate equation is proposed to describe the dynamics of these packets, using standard ideas from bifurcation theory. We show that in a certain limit the system can be described in terms of a slowly-evolving solitary wave, and that periodic behavior, like that of the observed butterfly diagram, can easily be found. Generalizations of the theory are discussed.

1. Motivations

Elsewhere in this volume you can read Brandenburg's description of simulations of hydromagnetic convection, in which descending plumes pull down and stretch out magnetic field lines, defying (at least our) intuition about how magnetic buoyancy might drive the field upward and outward. This will have to be studied hard and understood, but it does seem to suggest that the bulk of the solar convection zone has the capability of performing some serious dynamo action.

On the other hand, participants in the conference also heard from Goode that there is a sharp change in solar angular velocity with radius that can be detected just under the convection zone. This tachycline, as such a feature may be called, may be expected to do a good job of stretching any field that dares to enter it into a strong toroidal field. From there magnetic buoyancy and penetrative convection can cause this enhanced field to intrude back into the zone where it may be swept up to the surface to emerge and form spots.

The interplay of two such mechanisms may be responsible for much of the complexity of the solar cycle and this promises to be a source of concern to stellar physicists for some time. But here, we are interested in studying the second of these two possible processes: subphotospheric dynamo action. Our plan is to study the dynamo waves in a thin fluid layer. We shall first write down the partial differential equations for an α–ω dynamo and recall how they are subject to the onset of

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overstability for a critical value of the dynamo number at a critical value of the horizontal wave number.

For dynamo numbers slightly in excess of the critical value, for a thin layer of fluid, such as that described by Goode, we expect that a packet of waves will be unstable. This packet will grow to finite amplitude until it causes back reactions on the fluid motions. The nature and the details of such feedbacks were much discussed in the conference that spawned this volume, but it is not clear that any general agreement was reached on what the principal mechanisms are. In this situation, we have decided to let the modern theory of nonlinear instability come to our aid.

To the student of nonlinear stability theory, a band of overstable modes such as we are describing is just a Hopf bifurcation in an extended system. In that case, bifurcation theory reveals what the form of the equation for the envelope of the packet must be, up to certain coefficients that depend on the details of the problem. We shall present this equation here and use it to study the propagation of nonlinear packets of dynamo waves in the manner of the butterfly diagram of solar physics. Of course, what we are describing here is but the beginning of a more extensive modelling process, but we find the results encouraging enough, not only to report them here, but to mention some possible extensions.

2. Equations

As we stated in the introduction, we are concerned with a thin layer of fluid under the convection zone. For this discussion we shall not worry about the sphericity and consider a plane-parallel osculating layer at mid-latitudes, in analogy to the f–plane of meteorology. In that layer we take x to be the North-South coordinate, y as the azimuthal coordinate and z as the vertical coordinate. We confine ourselves to axisymmetric situations.

The magnetic field is

$$\mathbf{B} = b\hat{\mathbf{y}} + \nabla\times(a\hat{\mathbf{y}}) \tag{2.1}$$

where $\hat{\mathbf{y}}$ is a unit vector in the y–direction; that is, $b\hat{\mathbf{y}}$ is the toroidal field and $a\hat{\mathbf{y}}$ is the vector potential for the poloidal field. The equations for $\mathbf{B}$ are these:

$$\partial_t a = \alpha b + \eta\nabla^2 a \tag{2.2}$$

$$\partial_t b = \Omega\partial_x a + \eta\nabla^2 b \tag{2.3}$$

where α measures the ability of helical turbulence to generate poloidal field from toroidal field, Ω is a measure of the solar differential rotation, and $\nabla^2 = \partial_x^2 + \partial_z^2$. We take the differential rotation in the x–direction for illustrative purposes so as to avoid detailed discussions here of what may involve vertical eigenfunctions. By taking this scientific license we are able to avoid technical details. These equations may either be taken as a mathematical statement of the familiar Babcock-Leighton-Parker ideas about solar activity or understood as a direct outcome of the simplest versions of mean field dynamo theory.

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Axisymmetry translates into the statement that the equations are invariant with respect to translations in y, so that we may look for solutions of the form $a(x,z,t) = A(x,t)\exp(i\ell z)$, $b(x,z,t) = B(x,t)\exp(i\ell z)$. The equations become

$$\dot{A} = \alpha B + \eta(A'' - \ell^2 A) \tag{2.4}$$

$$\dot{B} = \Omega A' + \eta(B'' - \ell^2 B) \tag{2.5}$$

where the prime denotes derivative with respect to x and the dot indicates time derivative.

To get an idea of the dynamics contained in these equations, we suppose, for the moment, that the coefficients, such as α, are constants. Then we can look for solutions of the form $A(x,t) = A_0 \exp(st + ikx,)$ and $B(x,t) = B_0 \exp(st + ikx)$, where A_0 and B_0 are constants. Then we get two equations for A_0 and B_0. The condition for solvability of these equations is the vanishing of the determinant of the coefficients. We obtain the result

$$s = -\eta\left(k^2 + \ell^2\right) \pm \sqrt{\alpha\Omega k/2}(1+i), \tag{2.6}$$

where we shall consider only cases where $\alpha\Omega > 0$. There is a Hopf bifurcation for wavenumber k when the dynamo number

$$D \equiv \sqrt{\frac{\alpha\Omega}{2\eta^2\ell^3}} \tag{2.7}$$

has the value $(1+K^2)/K^{\frac{1}{2}}$ where $K = k/\ell$. The minimum value of this expression over K is the critical dynamo number $D_c = 4\cdot 3^{-\frac{3}{4}}$ and it occurs when $K = K_c = 1/\sqrt{3}$. When $D \geq D_c$, we have dynamo action for a range of K in a neighborhood of K_c.

3. Wave packets

For K in the neighborhood of the critical value for the onset of instability, we can construct a wave packet that we propose as a model of the band of solar activity delineated in the butterfly diagram. To make this packet, it is convenient to simplify the expression for s for D close to D_c.

Let $D = D_c + \delta$ and $K = K_c + q$. For small δ and q the nondimensional growth rate $\sigma \equiv s/(\eta\ell^2) - 4/3$ is

$$\sigma = 3^{\frac{1}{4}}\delta - 2q^2 + i\left(3^{\frac{1}{4}}\delta + \frac{2}{\sqrt{3}}q - q^2\right). \tag{3.1}$$

Just as in the complete expression for s, the group and the phase velocities of the waves are towards the equator, which is at $x = 0$.

This expression for the growth rate was derived for constant coefficients in (2.4) and (2.5). We shall now assume that these coefficients may vary slowly in x and represent this effect by allowing such a dependence in δ. We can also simplify

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the expression for σ by factoring out a constant and reabsorbing it into the time. Then we can write down the linear equation for $\Psi(\xi,\tau)$, the amplitude function for a superposition of waves. This takes the form

$$\partial_\tau \Psi = 3^{\frac{1}{4}}(1+i)\delta\Psi + \frac{2}{\sqrt{3}}\partial_\xi \Psi + (2+i)\partial_\xi^2 \Psi. \tag{3.2}$$

Here we have introduced a new spatial coordinate ξ and time variable τ to emphasise the weak dependence of the wave envelope on position and time. Thus, if we replace Ψ by $\exp(i(K_c x + q\xi) + \sigma\tau)$, we recover the dispersion relation (3.1).

In the chosen frame, δ depends weakly on space and time, and the wave packet will deform as it propagates and grows in amplitude. Eventually nonlinear terms become significant, and we have to make allowance for this by *nonlinearizing* (3.2). The possible nonlinear processes arising through the action of the Lorentz force were much discussed at the Symposium. Nevertheless the generic forms of the nonlinear terms that can come into (3.2) are dictated by nonlinear stability theory; the leading one of these is $F(\xi,t)|\Psi|^2\Psi$ for some complex function F.

4. Envelopes of solar activity

Nonlinear waves arising from the modulation of overstability in thin layers are often very robust. Even though they may not arise as completely integrable systems, such waves often have sufficient stability to be considered as solitons. That is the sort of wave that we wish to liken to the waves of solar activity that propagate from mid-latitudes to the equator as the solar cycle unfolds. However the kind of theory for such envelope solitons that we have outlined in §III involves quite a few complications and the solution of the full amplitude evolution equation for Ψ will require numerical integrations. Here we are satisfied to see the qualitative content of such a theory by choosing the parameters of the problem to place the system in *nearly* integrable conditions.

There are two kinds of effects that will spoil the complete integrability of the activity envelope equation: (1) nonconstant coefficients of slow variability and (2) dissipative terms of small amplitude. Let us for the purposes of this discussion suppose that both kinds of smallness can be measured by the same parameter, ϵ. Then the equation formulated in the previous section can conveniently be expressed (in a frame moving with the leading order group velocity) as follows:

$$\partial_\tau \Psi - i\partial_\xi^2 \Psi - i|\Psi|^2\Psi = f(Z,[\Psi]) + \epsilon g(Z,[\Psi]), \tag{4.1}$$

where $Z = \epsilon(\xi - c_0\tau)$, and c_0 is the group velocity, so that Z gives the position of the wave packet. Thus $f(Z)$ represents effects of type (1) while the small amplitude, type (2) effects are summarized by g, and the square brackets indicate dependence on derivatives of Ψ as well as Ψ itself.

In this picture, for $\epsilon = 0$, (4.1) is the cubic Schrödinger equation, which has a solitary wave solution of the form

$$\Psi = \mathrm{Re}(\xi, \tau) e^{i\Theta(\xi,\tau)}, \tag{4.2}$$

where

$$\mathrm{Re} = \sqrt{2} R \operatorname{sech}[R(\xi - \xi_0)], \tag{4.3}$$

and

$$\Theta = U(\xi - \xi_0) + \int (U^2 + R^2) d\tau. \tag{4.4}$$

This Schrödinger soliton has two arbitrary parameters, U and R, with $\xi_0 = 2U\tau$.

The effect of the terms on the right side of (4.1) is to modify the behavior of this nonlinear wave and, when ϵ is small, we can expect to capture the effects of the modification by letting the constants vary slowly. So, to solve (4.1) for small ϵ, we let R and U vary slowly and redefine ξ_0 as $2\int U d\tau$. In short, instead of the simple soliton of the cubic Schrödinger equation with arbitrary values assigned to its basic parameters, we seek a similar object whose basic parameters are allowed to vary according to the dictates of the terms that destroy exact integrability. So we shall use (4.1) to find equations of motion for U, R and $X_0 = \epsilon\xi_0$. These equations control the proposed model for the evolution of the solar cycle. To get them in explicit form, we need to become more specific about f and g. We do that in the next section.

5. Equations of motion of the activity waves

To derive the dynamical equations for the parameters of the solitary waves that we associate to the solar cycle, we first need to give the forms of the terms that cause their variation. Terms of type (2), small dissipative terms, are familiar from studies of the complex Ginzburg-Landau equation for Hopf bifurcation in extended systems. They take the form

$$g = \mu(Z)\Psi + \partial_\xi^2 \Psi - \nu |\Psi|^2 \Psi. \tag{5.1}$$

The type (1) terms, formally of order unity though their average effect is of order ϵ, are

$$f = i\kappa(Z)\Psi + c(Z)\partial_\xi \Psi. \tag{5.2}$$

To find the slow evolution of R and U, we use the following integral relations, reminiscent of familiar results from dynamics:

$$\partial_\tau \int_{-\infty}^{\infty} |\Psi|^2 d\xi = 2\epsilon \int_{-\infty}^{\infty} [\mu(Z)|\Psi|^2 - \frac{dc}{dZ}|\Psi|^2 - |\partial_\xi \Psi|^2 - \nu|\Psi|^4] d\xi \tag{5.3}$$

$$\partial_\tau \int_{-\infty}^{\infty} \frac{1}{i}(\Psi^* \partial_\xi \Psi - \Psi \partial_\xi \Psi^*) d\xi = 2\epsilon \int_{-\infty}^{\infty} \frac{d\kappa}{dZ} |\Psi|^2 d\xi +$$

$$+\frac{2\epsilon}{i} \int_{-\infty}^{\infty} \left[(\mu(Z) - \nu|\Psi|^2)(\Psi^* \partial_\xi \Psi - \Psi \partial_\xi \Psi^*) + (\partial_\xi^2 \Psi^* \partial_\xi \Psi - \partial_\xi^2 \Psi \partial_\xi \Psi^*)\right] d\xi. \tag{5.4}$$

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The integrals on the left hand sides of equations (5.3) and (5.4) are easily evaluated in terms of R and U. In fact

$$\int_{-\infty}^{\infty} |\Psi|^2 d\xi = 4R \ , \ \int_{-\infty}^{\infty} \frac{1}{i}(\Psi^* \partial_\xi \Psi - \Psi \partial_\xi \Psi^*) d\xi = 8UR. \tag{5.5}$$

The terms on the right hand sides can be determined similarly, with the functions of Z considered constant while the integrals over the shorter ξ scale are being performed. Consequently we may now regard the spatial variable Z as giving the centre of the soliton, so that $Z \approx X_0 - \epsilon c_0 \tau$. We have thus reduced the problem of the slow evolution of the wave packet to the determination of the solutions to three simple nonlinear o.d.e.'s. The first two can be obtained from equations (5.4) and, after some obvious manipulations to separate the evolution of R and U, take the form

$$\partial_T R = 2R[\mu(Z) - \omega(Z) - U^2] - \frac{2}{3}(1 + 4\nu)R^3, \tag{5.6}$$

$$\partial_T U = U[2\omega(Z) - \frac{4}{3}R^2] + \lambda(Z), \tag{5.7}$$

while the third comes from the definitions of U and Z and can be written

$$\partial_T Z = 2U - c_0. \tag{5.8}$$

Here $T \equiv \epsilon\tau$ is the slow time on which the wave packet changes its form, and we have written $\omega(Z) \equiv dc/dZ$, $\lambda(Z) \equiv d\kappa/dZ$.

These three equations, then, give a simplified description of the evolution of a single wave packet in a rather special limit. Of course, we could and should elaborate the model by discussing the interaction of two or more packets, but such a program is far from trivial and we plan in conjunction with such a study to solve the full Ginzburg-Landau equation numerically in due time. For now we content ourselves with analysing (5.6)-(5.8) for simple forms of μ, ω, and λ so as to adumbrate the variety of possible behaviors. Before doing this, we may briefly discuss the physics of the system represented by the evolution equations.

Clearly the wave packet is being forced to the 'left' (Z decreasing) by the c_0 term. Were U to be exactly zero, the packet would continue on its merry way until R decreased to exponentially small values. Because of the λ term, however, U cannot be zero and, if circumstances are such that U is forced to be positive, then the packet is stopped in its tracks if U becomes large enough. This in turn is possible if ω is positive, so that U grows exponentially for small R. If μ is supposed to increase with Z, then we can find the following sequence of events: (i) dynamo action (represented by R) arises at some value of Z. Both Z and the amplitude of the packet then decrease, while U begins to grow. Eventually R falls to a small value and the (now almost invisible) disturbance makes its way back to larger Z, where the whole process repeats itself. Thus in this simple model the cyclical behavior of the dynamo is manifested, not in the oscillations of the dynamo waves themselves, but in the periodic motion of their envelope. These ideas are illustrated in the next section for particular forms of the variable coefficients in equations (5.6)-(5.8).

6. Butterfly diagrams and self-renovating envelopes

In this section we present some numerical solutions to equations (5.6)-(5.8). Since we have already made rather drastic simplifying assumptions about the magnitudes of the coefficients so as to treat the system as almost integrable, we do not worry too much about their correspondence with the forms given in earlier sections. In particular, we take ω to be a positive constant (and equal to 1 throughout as this can be accomplished by a simple rescaling). We also take λ to be constant and positive (and $c_0 > 0$ without loss of generality), since only when both ω and λ are positive can we achieve recurrent behavior (in other cases the soliton just moves towards the equator and decays away so that a more complicated model would be required). For purposes of illustration we use two different models for the growth rate $\mu(Z)$. In what follows we envisage that the 'equator' of our model is at some large negative value of Z, while the pole is represented by Z large and positive. For Model I, $\mu = qZ$ and for Model II, $\mu = q - Z^2$, where in each case q is a constant. The dynamics then depends on the four parameters q, ν, λ and c_0. We only investigate the case of positive q since this is the only one with a physical correspondence.

It proves convenient to use, instead of R, the variable $E \equiv R^2/3$. The equations have just one fixed point at

$$(E_0, U_0, Z_0) \equiv \left(\frac{1}{4}[2 + \frac{2\lambda}{c_0}], \frac{1}{2}c_0, \frac{1}{q}[U_0^2 + 1 + E_0(1 + 4\nu)]\right); \qquad (5.9)$$

provided $c_0^3 < (1 + 4\nu)\lambda$ this is stable for $0 < q < c_0^{-2}[\lambda + (1 + 4\nu)(c_0 + \lambda)((1 + 4\nu)\lambda - c_0^3)]$; for larger values of q (and for all positive q if the first inequality fails) it is unstable to an oscillatory instability. Depending on the initial conditions the solution may run away to infinity; otherwise it is attracted to a periodic orbit with the general characteristics described in the previous section. We have not been able to find any further bifurcations leading to bounded orbits with more complicated time dependence.

Figure 1 shows a typical orbit of the system, together with time traces of the three dependent variables. It is notable that (seemingly because c_0 is small), there are two distinct timescales; there is a slow progress towards negative Z during which the magnitude of R (equivalently, the magnetic energy) decreases; this timescale must then be identified with the 11 years of the solar cycle during which sunspot activity makes its way from mid-latitudes to the equator. There follows a rapid recovery phase during which the field strength is very small and the disturbance propagates rapidly back towards the pole. We can construct a butterfly diagram by plotting the contours of the soliton amplitude as a function of ξ and time, and this is shown for the same parameters in Figure 2.

For Model II there are two fixed points which appear at a saddle-node bifurcation at the origin as q passes through zero. That with $Z > 0$ is nonstable, while the other, with opposite sign has similar properties to the single fixed point in Model I. For moderate q there is again a stable limit cycle which for small c_0 spends most of its time close to a 'slow manifold'; now, however, this has parabolic shape in

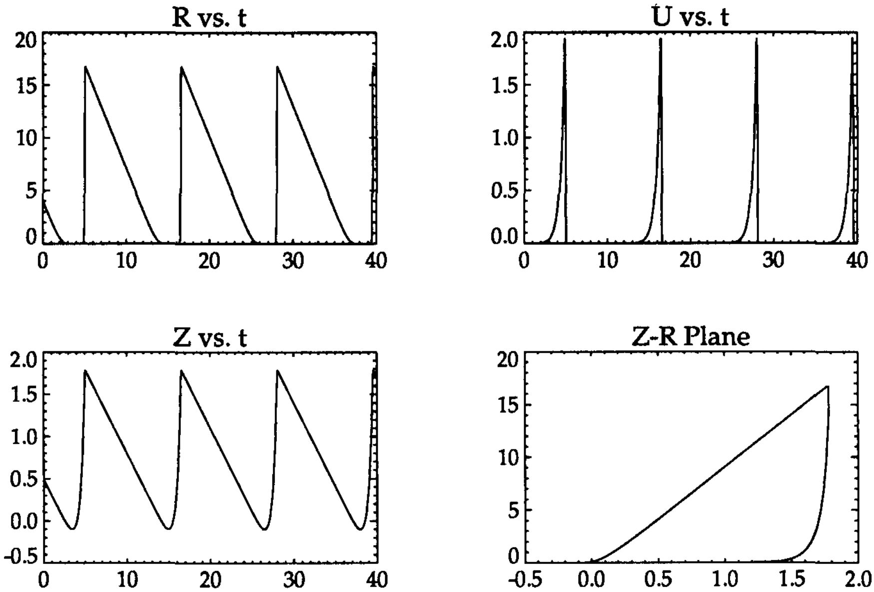

Fig. 1. A periodic solution for Model I with $q = 10$, $\lambda = .0093$, $\nu = 0$, $c_0 = 0.2$. Plots of each of the variables are shown as functions of time, together with a phase portrait of R against Z.

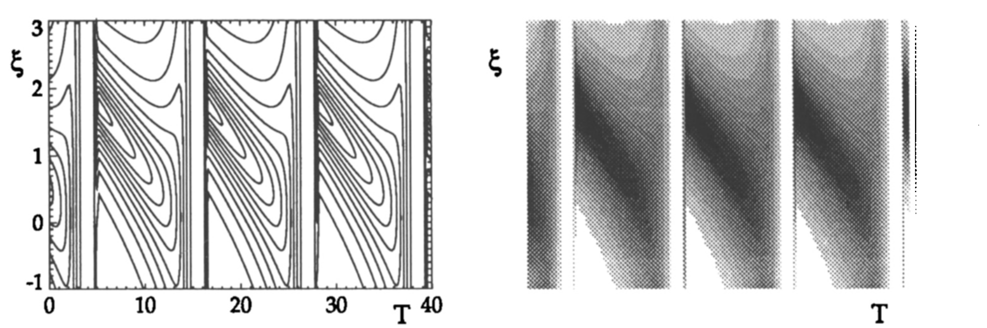

Fig. 2. As Figure 1, but showing a "butterfly diagram" constructed by plotting contours of the quantity Re as a function of ξ and T. A grayscale plot is also shown.

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(Z, R) space. Thus Model II gives a better picture of the magnitude of magnetic activity first waxing and then waning as the disturbance propagates towards the equator. Figures 3 and 4 show a typical case, in the same manner as for Figures 1 and 2.

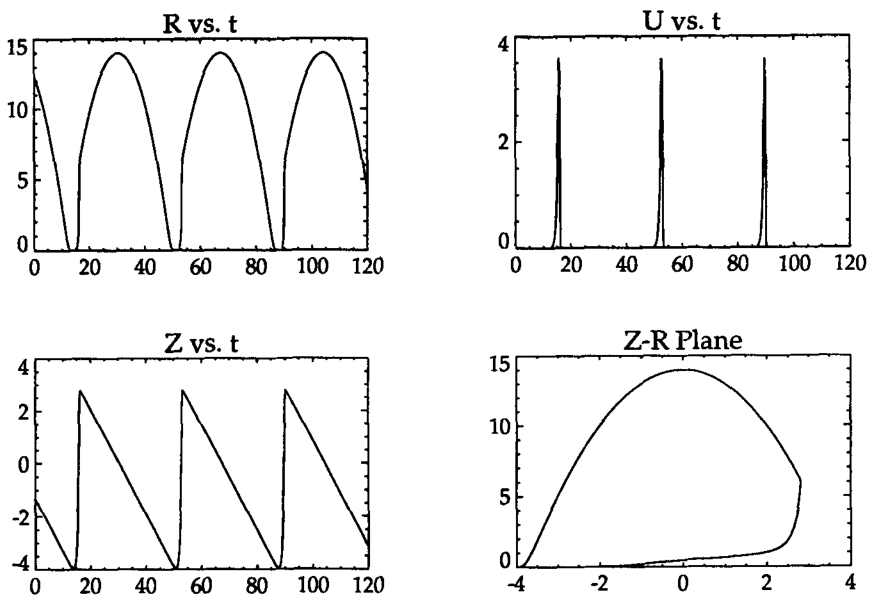

Fig. 3. As for Figure 1, but now for Model II with $q = 15,\ \lambda = .0064,\ \nu = 0,\ c_0 = 0.2$.

In both cases the resemblance to the observed butterfly diagram is striking, although our neglect of the (temporal) width of the wave packet means that the slight overlap (in time) between old waves at low latitude and new ones at high latitude cannot be modelled here.

7. Conclusions

The aim of this work is to make a mathematical model for the solar activity that is represented by the butterfly diagram. To do this, we have had to decide what the diagram actually represents. We are suggesting that the butterfly diagram is a spacetime diagram showing the cyclic appearance of waves of activity forming at ±37° and propagating toward the equator. With this image, there are several features that must be explained. When the waves arrive at the equator what happens to them? Why do they seem to disappear rather than move through or bounce back? Why do the waves in the two hemispheres remain so nearly

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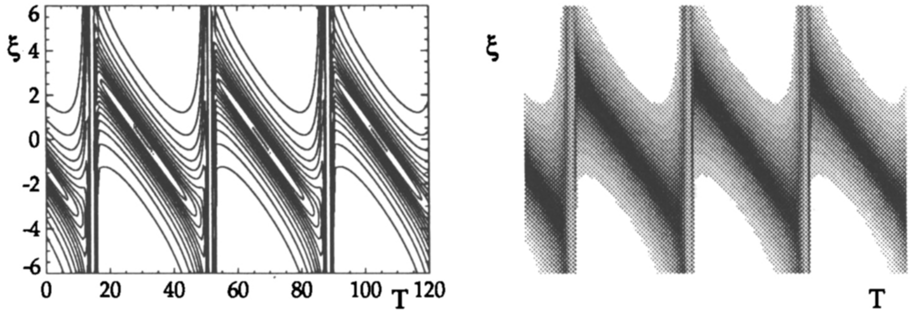

Fig. 4. Butterfly diagram for the calculation of Figure 3 (cf. Figure 2).

synchronous? Is it reasonable that the waves should have such a steady velocity as they move equatorward?

To be able to respond to such questions, we have chosen a particular kind of wave as our model. It is known that in thin-layer dynamo theory there are waves and that, in the standard $\alpha - \omega$ picture for example, these waves are overstable given suitable conditions. Our hypothesis is that the solar activity waves seen in the butterfly diagram are not simple dynamo waves but the envelopes of packets of overstable dynamo waves. The velocity of the observed waves is then not the phase velocity of any dynamo wave process, but the group velocity.

Once this picture is accepted it is not difficult to use qualitative mathematical arguments familiar in modern nonlinear stability theory to write an appropriate general model equation for the waves of solar activity. The main ingredient that we have added that is not usual in the more standard versions of such nonlinear instability theory is the possiblity of weak spatial dependence of the stability characteristics of the system. Given simple choices for such dependence, we have here examined, in a particularly tractable case, the evolution of just one wave packet and found that there do exist periodic solutions that are reminiscent of the observed butterfly diagrams. There are three (to us) pleasing outcomes of this modelling: (a) The velocity of the wave as it progresses toward the equator is reasonably constant; (b) Though the wave is reflected on reaching the equator, the reflected wave has such a small amplitude that we can understand its indetectability to date; and (c) The speed of motion toward the equator is much smaller than that of the return toward the higher latitudes, at least for our choice of parameters. Thus, in this model, a given wave can be thought to appear and move to the equator in eleven years but to make the return trip (to high latitudes) unnoticed in under a year, when it grows again in amplitude to begin a new half-cycle. But

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there is more to be understood and it remains to be seen whether extensions of the calculations done so far will work to provide the missing features.

Our present theoretical butterfly has only one wing. We need to study a pair of envelope waves, one in each hemisphere, to see if they lock into phase with each other to make a reasonably real butterfly. To do this, we can introduce the two waves in the near integrable case studied here and derive a coupled, sixth-order system for the properties of the two waves. Then we can see how well the two waves get in step with each other. With a sixth order system, we can of course expect at least a little chaos, which would bespeak some north-south asymmetry, but the main question at this stage is whether we can get a full butterfly.

We may also have to allow for two pairs of activity waves, since that is what is seen just at the end of the half-cycle. The possibilities for chaos and intermittency (including minima) in such cases are of course much enhanced, but so are the difficulties of the analysis. Indeed, at this stage we begin to confront harder questions, such as: why are there two pairs of waves active and not some other number? In addressing such questions, we are best advised to solve the evolution equation (4.1) numerically, without relying on the near-integrable approximation. This should yield more disordered behavior due to the interaction of several wave packets. In that case, we can contemplate the extension of the model to incorporate longitudinal dependence of the amplitudes. It is well known that two-dimensional wave patterns can be unstable to three-dimensional disturbances, and the resulting solutions could help us to understand the sector structure that is observed in solar activity. Such a program will of course require extensive computation, but we do hope (and expect) to be heard from again on these matters.

In the meantime, we must not neglect the physical modelling. We have based our derivations on the assumed presence of overstable, simple dynamo waves. These are most accessible in thin-layer dynamos. In line with current thinking, we have assumed that such a thin layer is subconvective. Given such a concrete vision, we ought to be able to derive explicitly the kind of nonlinear wave equation we are here postulating, thus adding some astrophysical input to our astromathematical discussion.

In conclusion, we acknowledge that the collaboration leading this work was made possible by the financial support of our solar research by the Air Force under grant AFOSR89-0012. The work has also been fostered by the N.S.F. under PHY87-04250. The preparation of the manuscript began during the Geophysical Fluid Dynamics School of 1990 at the Woods Hole Oceanographic Institution and was completed during a visit by EAS to the École Normale Supérieure de Lyon.

8. Bibliographical notes

In this informal sketch of our ideas on the mathematics of the solar cycle, we have not given any explicit references since we have assumed that our readers have in mind the basic ideas that we are using. We shall of course make a careful accounting of the provenance of the basic ideas in a more detailed treatment that we shall prepare in the relatively near future. What we shall simply do here is to give some references to basic topics that we have drawn on in this discussion, dynamo theory and nonlinear instability theory.

The main physical assumption of this work is that there is lurking somewhere inside the sun, but accessible to convective upwelling, a thin-layer dynamo. The theory of such dynamos has been elaborated in terms of the 'α-effect' or two-scale model by

Zel'dovich, Ya.B., Ruzmaikin, A.A. & Sokoloff, D.D.: 1983, *Magnetic Fields in Astrophysics*, Gordon & Breach.

Moffatt, H.K.: 1978, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press.

In particular, we relied on the property of these models of leading to instabilities in the form of propagating dynamo waves, according to the original ideas of

Parker, E. N.: 1980, *Cosmical Magnetic Fields: their Origin and Activity*, Oxford University Press.

Given the known result that such waves are overstable we studied the nonlinear propagation of a packet of them following the known procedures of nonlinear stability theory as described for instance in

Manneville, P.: 1990, *Dissipative Structures and Weak Turbulence*, Academic Press.

Then we applied standard nonlinear perturbation theory to the solitary envelope waves, a subject introduced in

Lamb, G.L, Jr.: 1980, *Elements of Soliton Theory*, J. Wiley and Sons.

That the solutions to the Ginzburg-Landau equations for which we have derived the governing dynamical system may indeed be prone to the development of more than one activity wave is a danger that is foreshadowed in

Bretherton, C.S. and Spiegel, E.A.: 1983, " Intermittency Through Modulational Instability," *Phys. Lett.* **A96**, 152