

## ON HERMITE-FEJÉR INTERPOLATION WITH EQUIDISTANT NODES

G. B. BAKER, T. M. MILLS and P. VÉRTESI

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### Abstract

This paper deals with Hermite-Fejér interpolation of functions defined on a semi-infinite interval but the nodes are equally spaced. It is shown that, under certain conditions, the interpolation process has poor approximation properties.

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### 1. Introduction

Suppose that  $I$  is an interval on the real line:  $I$  may be open, closed or semi-open, finite or infinite. Let

$$M = (x_{k,n} : k = 1, 2, \dots, n; n = 1, 2, 3, \dots)$$

be an infinite triangular array such that

$$x_{k,n} \in I \quad (k = 1, 2, \dots, n; n = 1, 2, \dots)$$

and, for  $n = 1, 2, \dots$ ,

$$x_{1,n} < x_{2,n} < \dots < x_{n,n}.$$

For  $f: I \rightarrow (-\infty, \infty)$  we define the Hermite-Fejér interpolation polynomial  $H_{2n-1}(f, x)$  to be the unique polynomial of degree  $2n-1$  or less which satisfies the  $2n$  conditions

$$H_{2n-1}(f, x_{k,n}) = f(x_{k,n}) \quad (k = 1, 2, \dots, n),$$

$$H'_{2n-1}(f, x_{k,n}) = 0 \quad (k = 1, 2, \dots, n).$$

These polynomials came into prominence in 1916 when L. Fejér [2] considered the special case where  $I = [-1, 1]$  and  $x_{k,n} = \cos((2k-1)\pi/(2n))$  ( $k = 1, 2, \dots, n; n = 1, 2, \dots$ ). In this case we describe  $M$  as being the matrix of Chebyshev nodes because, for each  $n$ , the points  $x_{k,n}$  ( $k = 1, 2, \dots, n$ ) are the zeros of the Chebyshev polynomial  $T_n(x) = \cos(n \arccos x)$ . For these nodes, Fejér proved that, if  $f \in C([-1, 1])$ , then

$$\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\| = 0$$

where  $\|\cdot\|$  denotes the uniform norm on the space  $C([-1, 1])$ . Hence, Fejér proved Weierstrass' approximation theorem by using interpolation polynomials.

Since 1916, a vast literature has grown up around these polynomials: see the bibliographies in Gonska and Knoop [3] and Mills [6].

In 1958, D. L. Berman [1] investigated these polynomials where  $I = [-1, 1]$  and the nodes were equidistant: that is

$$x_{k,n} = -1 + 2(k-1)/(n-1) \quad (k = 1, 2, \dots, n).$$

This choice of nodes is interesting because it is simple, and is the natural choice of the practical person. However, Berman shows that, in this case, the interpolation polynomials have poor approximation properties. In fact, he proves the rather remarkable fact that, if  $g(t) \equiv t$ , then the sequence  $\{H_{2n-1}(g, x) : n = 1, 2, 3, \dots\}$  diverges at all points in  $[-1, 1] \setminus \{-1, 0, 1\}$ . As we know that  $x_{1,n} = -1$  and  $x_{n,n} = 1$  for all  $n$ , the polynomial  $H_{2n-1}(g, x)$  and the function  $g(x)$  agree at these two points. We note here that some interesting convergence results concerning this choice of nodes have been stated by P. O. Runck [7].

In 1932, Szegő ([8]; see also Szegő [9, Theorem 14.7]) considered the Hermite-Fejér interpolation polynomials when the nodes of interpolation were the zeros of the Laguerre polynomial  $L_n^\alpha(x)$ . He proved that if  $f \in C([0, \infty))$  and  $f(x) = O(x^m)$  as  $x \rightarrow \infty$  for some fixed positive number  $m$ , then, as  $n \rightarrow \infty$ ,  $H_{2n-1}(f, x) \rightarrow f(x)$  for any fixed  $x \in (0, \infty)$  and the convergence is uniform in any fixed interval  $[a, b]$  where  $0 < a < b < \infty$ .

Since Szegő proved a convergence theorem which is similar to that of Fejér but set in the interval  $[0, \infty)$ , it is natural to ask whether there is a divergence theorem similar to that of Berman, but set in the interval  $[0, \infty)$ . In this paper, we will prove two such results, one in each of the following two sections. These results do not follow from applying a linear transformation to Berman's result. In the final section we will present some examples which illustrate the applicability of these results. Throughout this paper,  $c$  or  $c_1, c_2, \dots$  denote absolute, positive constants. However, repeated use of

any of these symbols in different expressions does not imply equality of the constants.

### 2. A divergence theorem

Before presenting our first divergence theorem, we prove the following technical lemma.

LEMMA 1. *Given sequences  $\{h_n : n = 1, 2, 3, \dots\}$ ,  $\{j_n : n = 1, 2, 3, \dots\}$  and  $\{\theta_n : n = 1, 2, 3, \dots\}$  such that*

$$(2.1) \quad h_n > 0 \quad (n = 1, 2, 3, \dots),$$

$$(2.2) \quad \lim_{n \rightarrow \infty} h_n = 0, \quad \lim_{n \rightarrow \infty} nh_n = \infty,$$

$$(2.3) \quad \lim_{n \rightarrow \infty} j_n h_n = c > 0,$$

$$(2.4) \quad 0 < c_1 < \theta_n < c_2 < 1 \quad (n = 1, 2, 3, \dots),$$

let

$$(2.5) \quad K_n = A_n/B_n$$

where

$$A_n = 2^{2n} h(j - 1 + \theta)^{2j-3+2\theta} (n - j + 2 - 2\theta)^{2n-2j+3-2\theta},$$

$$B_n = (n - 2j + 2 - 2\theta)n^{n-1}(n + 3)^{n+2},$$

$$h = h_n, \quad j = j_n, \quad \text{and} \quad \theta = \theta_n.$$

Then  $\lim_{n \rightarrow \infty} K_n = \infty$ .

PROOF. From (2.1)–(2.3) it follows that

$$(2.6) \quad \lim_{n \rightarrow \infty} j = \infty, \quad \lim_{n \rightarrow \infty} (j/n) = 0.$$

Using (2.6) and

$$(2.7) \quad \lim_{t \rightarrow \infty} (1 + t^{-1})^t = e$$

we obtain, for  $n$  sufficiently large,

$$(2.8) \quad (j - 1 + \theta)^{2j-3+2\theta} > c j^{2j-3+2\theta},$$

$$(2.9) \quad (n - j + 2 - 2\theta)^{2n-2j+3+2\theta} > c(n - j)^{2n-2j+3-2\theta}.$$

Thus, using (2.8), (2.9), (2.3), and (2.7) several times, we find that for values of  $n$  sufficiently large,

$$\begin{aligned}
 K_n &> \frac{c2^{2n}h_j^{2j-3+2\theta}(n-j)^{2n-2j+3-2\theta}}{(n-2j+2-2\theta)n^{n-1}(n+3)^{n+2}} \\
 &= \frac{c2^{2n}}{(n-2j+2-2\theta)} \left(\frac{h}{j}\right) \left(\frac{n-j}{n}\right)^{n-1} \left(\frac{n-j}{n+3}\right)^{n+2} \left(\frac{j}{n-j}\right)^{2j+2\theta-2} \\
 &> \frac{c_12^{2n}h^2c_2^j}{(n-2j+2-2\theta)} \left(\frac{j}{n-j}\right)^{2j+2\theta-2} \\
 &> (c_12^n) \left(\frac{2^{n/4}}{n-2j+2-2\theta}\right) (2^{n/4}h^2)(2^{n/4}c_2^j) \left(2^{n/4}\left(\frac{j}{n-j}\right)^{2j+2\theta-2}\right).
 \end{aligned}$$

Now each of these five factors increases without bound as  $n$  increases. Hence  $\lim_{n \rightarrow \infty} K_n = \infty$  and the lemma is proved.  $\square$

We now state the main result of this section.

**THEOREM 1.** *Suppose that, for  $n = 1, 2, 3, \dots$*

$$(2.10) \quad x_k = x_{k,n} = kh_n \quad (k = 1, 2, \dots, n)$$

where

$$(2.11) \quad h_n > 0 \quad (n = 1, 2, 3, \dots),$$

$$(2.12) \quad \lim_{n \rightarrow \infty} h_n = 0,$$

$$(2.13) \quad \lim_{n \rightarrow \infty} nh_n = \infty.$$

Let  $x$  be a positive number such that

$$(2.14) \quad \{\{x/h_n\}: n = 1, 2, 3, \dots\} \text{ has a limit point in } (0, 1)$$

and let  $g(t) \equiv t$ .

Then the sequence  $\{H_{2n-1}(g, x): n = 1, 2, 3, \dots\}$  is unbounded.

**REMARK.** In condition (2.14),  $\{x/h_n\} = x/h_n - [x/h_n]$  denotes the fractional part of  $x/h_n$ .

**PROOF.** If we define  $J_{2n-1}(g, x)$  to be the unique polynomial of degree  $2n - 1$  or less such that

$$J_{2n-1}(g, x_k) = g(x_k); \quad J'_{2n-1}(g, x_k) = g'(x_k) \quad (k = 1, 2, \dots, n),$$

then obviously  $J_{2n-1}(g, x) \equiv g(x)$ . However, by the Hermite interpolation

formula (see, for example, Szegő [9, (14.19)], we also have

$$\begin{aligned} J_{2n-1}(g, x) &= H_{2n-1}(g, x) + \sum_{k=1}^n g'(x_k)(x - x_k)l_k(x)^2 \\ &= H_{2n-1}(g, x) + \sum_{k=1}^n (x - x_k)l_k(x)^2 \end{aligned}$$

where

$$(2.15) \quad l_k(x) = \omega(x)/((x - x_k)\omega'(x_k)) \quad (k = 1, 2, \dots, n),$$

$$(2.16) \quad \omega(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Hence

$$g(x) = H_{2n-1}(g, x) + \sum_{k=1}^n (x - x_k)l_k(x)^2.$$

Thus, the sequence  $\{H_{2n-1}(g, x): n = 1, 2, 3, \dots\}$  converges (or diverges) if and only if the sequence  $\{E_n(x): n = 1, 2, 3, \dots\}$  converges (or diverges) where

$$(2.17) \quad E_n(x) = \sum_{k=1}^n (x - x_k)l_k(x)^2.$$

So we focus our attention on  $E_n(x)$ .

Let  $x \in (0, \infty)$  satisfy condition (2.14). Define  $j \geq 1$  to be the index such that  $x_{j-1} \leq x < x_j$  (taking  $x_0 = 0$ ) and write

$$(2.18) \quad x = (j - 1)h + \theta h \quad (0 \leq \theta < 1)$$

where  $h = h_n$ . Like  $h$ , the variables  $j, \theta$  are functions of  $n$  but we will not indicate this unless it is necessary.

Using (2.18), we see that the assumption (2.14) is equivalent to assuming that the sequence  $\{\theta_n: n = 1, 2, 3, \dots\}$  has a limit point in the interval  $(0, 1)$ . So there are two constants  $c_1, c_2$  such that  $0 < c_1 < c_2 < 1$  and an infinite sequence of integers  $n_1 < n_2 < n_3 < \dots$  such that

$$(2.19) \quad 0 < c_1 < \theta_n < c_2 < 1 \quad (n = n_1, n_2, \dots).$$

We shall restrict  $n$  to this sequence and thus makes use of (2.19) henceforth.

We now have from (2.10), (2.15)–(2.18) that

$$(2.20) \quad \omega(x) = \frac{(-1)^{n-j+1} h^n \Gamma(j - 1 + \theta) \Gamma(n - j + 2 - \theta)}{\Gamma(\theta) \Gamma(1 - \theta)}$$

and

$$(2.21) \quad \omega'(x_k) = (-1)^{n-k} h^{n-1} \Gamma(k) \Gamma(n - k + 1) \quad (k = 1, 2, \dots, n)$$

where  $\Gamma$  is the gamma function.

From (2.15)–(2.17), (2.20)–(2.21) we obtain

$$(2.22) \quad E_n(x) = G \sum_{k=1}^n U_k$$

where

$$(2.23) \quad G = G(n, x) = \frac{h\Gamma(j-1+\theta)^2\Gamma(n-j+2-\theta)^2}{\Gamma(\theta)^2\Gamma(1-\theta)^2},$$

$$(2.24) \quad U_k = 1/\{(j-k-1+\theta)\Gamma(k)^2\Gamma(n-k+1)^2\}, \quad k = 1, 2, \dots, n.$$

Note that

$$(2.25) \quad U_k > 0 \quad (k = 1, 2, \dots, j-1),$$

$$(2.26) \quad U_k < 0 \quad (k = j, j+1, \dots, n).$$

From (2.11), (2.12), (2.18) we obtain

$$(2.27) \quad \lim_{n \rightarrow \infty} hj = \lim_{n \rightarrow \infty} h_n j_n = x,$$

$$(2.28) \quad \lim_{n \rightarrow \infty} j = \infty,$$

$$(2.29) \quad \lim_{n \rightarrow \infty} (j/n) = 0.$$

So, for large values of  $n$ ,  $j$  is much smaller than  $n$ . Now using (2.22) for  $n$  sufficiently large,

$$(2.30) \quad E_n(x) = G \sum_{k=1}^n U_k = G \sum_{k=1}^{j-1} (U_k + U_{n-k+1}) + \left( G \sum_{k=j}^{n-j} U_k \right) \\ = S_1 + S_2, \quad \text{say}$$

We estimate  $S_1$  and  $S_2$  separately. Using (2.23), (2.24), (2.29) we find that, for  $n$  sufficiently large and  $1 \leq k \leq j-1$ ,

$$(GU_k + GU_{n-k+1})\Gamma(\theta)^2\Gamma(1-\theta)^2 \\ = \frac{h\Gamma(j-1+\theta)^2\Gamma(n-j+2-\theta)^2}{\Gamma(k)^2\Gamma(n-k+1)^2} \left\{ \frac{n-2j+3}{(j-1-k+\theta)(n-j-k+2-\theta)} \right\} \\ > 0.$$

Furthermore, it is not difficult to show that, when  $n$  is sufficiently large, for  $1 \leq k \leq (j-1)$  we have

$$GU_k + GU_{n-k+1} \leq GU_{j-1} + GU_{n-j+2}.$$

Thus, from (2.30) we obtain

$$(2.31) \quad 0 \leq S_1 \leq (j-1)(GU_{j-1} + GU_{n-j+2}).$$

By (2.19) we can assume that  $\Gamma(\theta) = \Gamma(\theta_n)$  and  $\Gamma(1 - \theta) = \Gamma(1 - \theta_n)$  are uniformly bounded above and below for all values of  $n$ . To estimate  $S_1$ , we use (2.31), (2.23), (2.24). We apply Stirling's formula [4, (8.327)]

$$(2.32) \quad \Gamma(t) = \sqrt{(2\pi)t^{t-(1/2)}} e^{-t} (1 + O(1/t)), \quad t \rightarrow \infty,$$

and (2.27)–(2.29) to obtain

$$(2.33) \quad S_1 = o(1) \quad \text{as } n \rightarrow \infty.$$

We now assume  $S_2$ . From (2.26), (2.2.9) we know that if  $n$  is sufficiently large, then  $j < n/2 < n - j$  and hence, writing  $m = [n/2]$ ,

$$(2.34) \quad -S_2 > -GU_m > 0.$$

Using (2.23), (2.24), (2.32), and Lemma 1, we have, as  $n \rightarrow \infty$ ,

$$(2.35) \quad \begin{aligned} & -GU_m \Gamma(\theta)^2 \Gamma(1 - \theta)^2 \\ &= \frac{h \Gamma(j - 1 + \theta)^2 \Gamma(n - j + 2 - \theta)^2}{(m - j + 1 - \theta) \Gamma(m)^2 \Gamma(n - m + 1)^2} \\ &\geq \frac{h \Gamma(j - 1 + \theta)^2 \Gamma(n - j + 2 - \theta)^2}{((n/2) - j + 1 - \theta) \Gamma(n/2)^2 \Gamma((n + 3)/2)^2} \\ &= cK_n (1 + o(1)), \quad \text{where } K_n \text{ is defined by (2.5)} \\ &\rightarrow \infty. \end{aligned}$$

From (3.34), (3.35) and (2.19) we have

$$(2.36) \quad -S_2 \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

The conclusion of Theorem 1 now follows from (2.30), (2.33) and (2.36).

Before stating the next result, we need a definition.

**DEFINITION.** A sequence  $\{t_n : n = 1, 2, 3, \dots\}$  of real numbers is *uniformly distributed modulo 1* if, for every pair  $a, b$  of real numbers with  $0 \leq a < b \leq 1$ , we have  $\lim_{N \rightarrow \infty} A([a, b]; N)/N = b - a$  where  $A([a, b]; N)$  is the cardinality of the set  $\{t_n : n = 1, 2, \dots, N\} \cap [a, b]$  and  $\{t_n\}$  is the fractional part of  $t_n$ .

The work by Kuipers and Niederreiter [5] is the standard reference on uniformly distributed sequences. We will write “u.d. mod 1” for the phrase “uniformly distributed modulo 1”.

**COROLLARY 1.** Assume (2.10)–(2.13). Let  $x$  be a positive number such that

$$(2.14a) \quad \text{the sequence } \{x/h_n : n = 1, 2, 3, \dots\} \text{ is u.d. mod 1}$$

and let  $g(t) \equiv t$ .

Then the sequence  $\{H_{2n-1}(g, x): n = 1, 2, 3, \dots\}$  is unbounded.

PROOF. Since (2.14a) implies (2.14), the proof is immediate.

Theorem 1 prompts the question about what happens when  $x = 0$ . If we consider  $E_n(0)$  in (2.17) and use (2.15), (2.16), (2.20) with  $x = 0$ , we find that

$$-E_n(0) = \sum_{k=1}^n x_k l_k(0)^2 = h \sum_{k=1}^n V_k$$

where  $V_k = \Gamma(n + 1)^2 / (k\Gamma(k)^2\Gamma(n - k + 1)^2)$  ( $k = 1, 2, \dots, n$ ). So, by (2.13) we find that

$$-E_n(0) > hV_n = nh \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Thus, if conditions (2.10), (2.11), (2.13) hold, then the sequence  $\{H_{2n-1}(g, 0): n = 1, 2, 3, \dots\}$  is unbounded.

### 3. Another divergence theorem

The second major result of this paper is the following.

THEOREM 2. Suppose that, for  $n = 1, 2, 3, \dots$

$$x_k = x_{k,n} = kh_n \quad (k = 1, 2, 3, \dots, n)$$

where  $h_n > 0$  ( $n = 1, 2, 3, \dots$ ),  $\lim_{n \rightarrow \infty} h_n = 0$ , and  $\lim_{n \rightarrow \infty} nh_n = \infty$ . Let  $g(t) \equiv t$ .

Then the sequence  $\{H_{2n-1}(g, x): n = 1, 2, 3, \dots\}$  diverges almost everywhere in  $[0, \infty)$ .

PROOF. We begin by outlining the method of the proof. For  $n = 1, 2, 3, \dots$ , we let

$$r_n = \frac{1}{(2n^3)}, \quad B_n = \bigcup_{k=1}^n (x_{k,n} - r_n, x_{k,n} + r_n)$$

and

$$(3.1) \quad B = \{x: x \in B_n \text{ for infinitely many values of } n\}.$$

Denoting the Lebesgue measure of a set  $X$  by  $m(X)$ , we note that

$$\sum_{n=1}^{\infty} m(B_n) < \infty$$

and hence, by the Borel-Cantelli lemma,  $m(B) = 0$ .



We will show that

$$(3.2) \quad x \notin B \Rightarrow \limsup_{n \rightarrow \infty} |H_{2n-1}(g, x) - g(x)| = \infty$$

from which the conclusion of the theorem will follow immediately.

Assume that

$$(3.3) \quad x \in [0, \infty) \setminus B.$$

Then, by (3.1) there is an integer  $n(x)$  such that if  $n \geq n(x)$  then  $x \notin B_n$ . By restricting  $n$  to such values, we may assume henceforth that

$$(3.4) \quad |x - x_{k,n}| \geq r_n \quad (k = 1, 2, \dots, n)$$

and that, at any stage,  $n$  is sufficiently large for our purposes.

We now follow the argument employed to prove Theorem 1, step by step, with some minor modifications. The proof is unchanged up to and including (2.18). To replace (2.19), observe that (3.4) implies that

$$1/(2n^3h) \leq \theta \leq (2n^3 - 1)/(2n^3h).$$

We can use the identity

$$\Gamma(\theta)\Gamma(1 - \theta) = \pi/(\sin \pi\theta) \quad (0 < \theta < 1)$$

[4, formula 8.334(3)] and replace (2.19) by

$$(3.5) \quad \pi \leq \Gamma(\theta)\Gamma(1 - \theta) \leq n^3h.$$

The argument from (2.20)–(2.31) can be used without change. To estimate  $S_1$ , use the left-hand inequality of (3.5) and obtain (2.33). To estimate  $S_2$ , use (2.34) and the right-hand inequality in (3.5). We then obtain, instead of (2.35),

$$-GU_m > K_n(1 + o(1))/(n^3h), \quad \text{as } n \rightarrow \infty.$$

We now use the proof of Lemma 1 without any change until the last line to show that, as  $n \rightarrow \infty$ ,

$$K_n/(n^3h) \rightarrow \infty$$

and this gives us (2.36). Thus, assuming (3.3) we have

$$\limsup_{n \rightarrow \infty} |H_{2n-1}(g, x) - g(x)| = \infty$$

and so Theorem 2 is established.

#### 4. Examples

It is interesting to compare Theorem 1 with Theorem 2. Theorem 1 provides us with a way of determining if the sequence

$$S(x) = \{H_{2n-1}(g, x) : n = 1, 2, 3, \dots\}$$

diverges for a given value of  $x$  but it does not give us any of how many such values there may be. Theorem 2 tells us that  $S(x)$  diverges almost everywhere in  $[0, \infty)$  but it does not tell us for which values  $x$  the sequence diverges. Between the two results, we can often manage to sort out many particular situations. We now present some examples to show how the theorems may be used.

**EXAMPLE 1.** Consider the case when  $H_n = 1/(\sqrt{an + b})$  where  $a > 0$ ,  $b$  are constants. It is clear that the only difficulty in applying Theorem 1 is checking (2.14). Often, checking (2.14a) in Corollary 1 is easier: this can be done by referring to results in Kuipers and Niederreiter [5]. In this example, we find from [5, Corollary 2.1] that, for all  $x > 0$ , (2.14a) is satisfied and hence  $S(x)$  diverges for all  $x > 0$ . Thus, by the remarks at the end of Section 2,  $S(x)$  diverges for all  $x \geq 0$ . In passing, we note that in Berman's result, the divergence did not occur at the point  $x = 0$  in the interval  $[-1, 1]$ .

**EXAMPLE 2.** Consider the case when  $h_n = 1/(\log n)$ ,  $n > 1$ . From [5, Theorem 2.6] we find that, for all  $x > 0$ , condition (2.14a) is not satisfied. In this case, while Theorem 2 assures us that  $S(x)$  diverges for almost all  $x > 0$ , Corollary 1 does not help us to identify any such value of  $x$ . However, by using another argument from [5, Example 2.5] we can show that for  $x > 0$  the sequence  $\{x \log n\}: n = 2, 3, \dots\}$  is dense in  $[0, 1]$  and hence has a limit point in  $(0, 1)$ . Thus,  $S(x)$  diverges for all  $x > 0$  (and hence all  $x \geq 0$ ).

**EXAMPLE 3.** Consider the case when  $h_n = 1/[\sqrt{n}]$  where  $[t]$  is the integer part of  $t$ . If  $m$  is a natural number, then, for  $n$  sufficiently large, we can find  $k \in \{1, 2, \dots, n\}$  such that  $m = kh_n$  and hence  $H_{2n-1}(g, m) = g(m)$ . Thus  $S(x)$  converges whenever  $x$  is a natural number. From this example we see that in general, the "a.e. divergence" in Theorem 2 cannot be replaced by "everywhere divergence".

**EXAMPLE 4.** Consider the case when  $h_n = 1$  ( $n = 1, 2, 3, \dots$ ). This is not covered by any results in this paper, but it is worth mentioning that, in this case  $S(x)$  diverges for all values of  $x \geq 0$  unless  $x$  is a natural number and hence, ultimately a node of interpolation. Modifying the proof of Theorem 1 will show this.

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### References

- [1] D. L. Berman, 'Divergence of the Hermite-Fejér interpolation process' (Russian), *Uspehi Mat. Nauk* **13** (80) (1958), no. 3, 143–148.
- [2] L. Fejér, 'Über Interpolation', *Göttinger Nachrichten* **1** (1916), 66–91.
- [3] H. H. Gonska and H. B. Knoop, 'On Hermite-Fejér interpolation: a bibliography (1914–1987)', *Studia Sci. Math. Hungar.*, to appear.
- [4] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series, and products*, (Corrected and enlarged edition, Academic Press New York, 1980).
- [5] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences* (Wiley, New York, 1974).
- [6] T. M. Mills, 'Some techniques in approximation theory', *Math. Sci.* **5** (1980), 105–120.
- [7] Paul Otto Runck, 'Sur la convergence des polynomes d'interpolation de Lagrange et d'Hermite aux noeuds equidistants' *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Unvers. Sci. Terre* **245** (1957), 1211–1213.
- [8] G. Szegő, 'Über gewisse Interpolationspolynome, die zu den Jacobischen und Laguerreschen Abszissen gehören', *Math. Z.* **35** (1932), 579–602.
- [9] Gábor Szegő, *Orthogonal polynomials* (4th ed., Colloq. Publ. 23, Amer. Math. Soc., Providence, R. I., 1975).

Towers, Perrin, Forster and Crosby  
356 Collins Street  
Melbourne  
Victoria 3000  
Australia

Bendigo College of Advanced  
Education  
P. O. Box 199, Bendigo  
Victoria 3550  
Australia

Mathematical Institute  
Hungarian Academy of Sciences  
P. O. Box 127  
Budapest 1364  
Hungary