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EXISTENCE, STABILITY AND OSCILLATION PROPERTIES OF SLOW-DECAY POSITIVE SOLUTIONS OF SUPERCRITICAL ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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Dedicated to Petr Zabreiko on the occasion of his seventy-fifth birthday with admiration and gratitude

Abstract We prove the existence of a family of slow-decay positive solutions of a supercritical elliptic equation with Hardy potential

$$-\Delta u + \frac{c}{|x|^2}u = u^p \quad \text{in } \mathbb{R}^N,$$

and study the stability and oscillation properties of these solutions. We also show that if the equation on $\mathbb{R}^N$ has a stable slow-decay positive solution, then for any smooth compact $K \subset \mathbb{R}^N$ a family of the exterior Dirichlet problems in $\mathbb{R}^N \setminus K$ admits a continuum of stable slow-decay infinite-energy solutions.

Keywords: supercritical elliptic equations; Hardy potential; slow-decay solutions; stable solutions; Joseph–Lundgren critical exponent

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1. Introduction

Our starting point is a superlinear elliptic problem in the entire space:

$$-\Delta u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 1$ and $N \geqslant 3$. In what follows we denote the *critical Sobolev exponent* by $p_S := (N+2)/(N-2)$. It is well known that for $p < p_S$ (1.1) has no positive solutions. For finite-energy solutions this is an easy consequence of Pohozaev's identity. For positive solutions without decay assumptions at ∞ this is a deep result of Gidas and Spruck [**9**].

For $p = p_S$ all positive solutions of (1.1) are given up to translations by a one-parameter family

$$W_\lambda(|x|) := \lambda^{(N-2)/2} W_1(\lambda|x|) \quad (\lambda > 0),$$

where $W_1(x) := (1 + (N(N-2))^{-1}|x|^2)^{-(N-2)/2}$ is a rescaled minimizer of the Sobolev inequality.

For $p > p_S$ the structure of the solution set of (1.1) is more complex. First we note that, for all $p > N/(N-2)$, (1.1) possesses an explicit *singular* radial positive solution

$$U_\infty(x) := C_p|x|^{-2/(p-1)}, \quad C_p := \left(\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right)^{1/(p-1)}.$$

Observe that if $p > p_S$, then $U_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and, hence, U_∞ is a weak solution of (1.1) in the entire space $\mathbb{R}^N$, despite the singularity at the origin. However, U_∞ is an infinite-energy solution because of its slow decay at infinity for $p > p_S$.

The set of all radially symmetric solutions of (1.1) can be analysed through phase-plane analysis after applying Fowler's transformation (see [**17**, pp. 50–55]). In particular, if $p > p_S$, then (1.1) admits a radial positive solution $U_1(|x|)$ such that $U_1(0) = 1$. It is known that $U_1(|x|)$ is monotone decreasing and that

$$\lim_{|x|\to\infty} \frac{U_1(|x|)}{|x|^{-2/(p-1)}} = C_p;$$

however, U_1 has no explicit representation in terms of elementary functions. Taking into account the scaling invariance, one concludes that rescalings of U_1 are also solutions of (1.1), so (1.1) possesses a one-parameter continuum of radial positive solutions

$$U_\lambda(|x|) = \lambda^{2/(p-1)} U_1(\lambda|x|) \quad (\lambda > 0). \tag{1.2}$$

One can show that the singular solution U_∞ is the limit of the family (U_λ), in the sense that, for any $x \neq 0$,

$$\lim_{\lambda\to\infty} U_\lambda(|x|) = U_\infty(|x|)$$

holds. In addition, it is known that, given $0 < \lambda_1 < \lambda_2 \leqslant \infty$, the solutions $U_{\lambda_1}(r)$ and $U_{\lambda_2}(r)$ in the range $p_S < p < p_{\mathrm{JL}}$ intersect each other infinitely many times as $r \to \infty$, while for $p \geqslant p_{\mathrm{JL}}$ the solutions are strictly ordered, that is, $U_{\lambda_1}(r) < U_{\lambda_2}(r)$ for all $r \geqslant 0$. Here,

$$p_{\mathrm{JL}} := \begin{cases} \dfrac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}} & \text{if } N > 10, \\ \infty & \text{if } N \leqslant 10, \end{cases}$$

is the *Joseph–Lundgren stability exponent*, introduced in [**13**]. The exponent p_{JL} controls various oscillation and stability properties of the solutions U_λ, which are particularly important in the study of the time-dependent parabolic version of (1.1); see [**11, 20**] or [**17**, pp. 50–55] for a discussion.

We are interested in a perturbation of (1.1) by the Hardy inverse square potential, that is, the equation

$$-\Delta u + \frac{\mu}{|x|^2} u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.3}$$

where $\mu > -C_{\mathrm{H}}$ and $C_{\mathrm{H}} := (N-2)^2/4$ is the *Hardy critical constant*, i.e. the optimal constant in the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x \geqslant C_{\mathrm{H}} \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^2} \,\mathrm{d}x \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.4}$$

The Hardy potential provides an important example of a long-range potential, that is, a potential that modifies the asymptotic decay rate of solutions at ∞ and their behaviour at the origin (see, for example, [**1**, **10**]).

For $p \neq p_S$ a Pohozaev-type identity shows that, similarly to (1.1), (1.3) has no finite-energy solutions [**19**]. For $p = p_S$, (1.3) admits an explicit one-parameter family of finite-energy radial solutions (see [**1**, **19**]). However, the structure of positive solutions of (1.3) in the *critical regime* $p = p_S$ is not fully understood. It is known that for large values of $\mu > 0$ (1.3) admits non-radial solutions that are distinct modulo rescalings from the radial solutions (see [**1**, **19**]). See [**12**] for recent results and a discussion of open questions in this direction.

In the present work we consider (1.3) in the *supercritical regime* $p > p_S$. In § 2 we outline the problem and discuss the basic properties of the explicit singular solution similar to U_∞. In § 3, for optimal ranges of p and μ we establish (Theorem 3.1) the existence of a one-parameter family $(U_\lambda)_{\lambda>0}$ of infinite-energy solutions of (1.3), which coincides with (1.2) when $\mu = 0$. We also discuss the stability properties of these solutions. The presence of the Hardy potential produces a range of new critical exponents related to stability, which do not have immediate analogues in the unperturbed case of (1.1). Finally, in § 4, we discuss (1.3) in exterior domains. In Theorems 4.1 and 4.3 we prove the non-existence of certain classes of slow-decay positive solutions, which justify the optimality of critical exponents introduced in § 3. In Theorem 4.5 we show that if the singular solution U_∞ of the problem on $\mathbb{R}^N$ is stable, then for any smooth compact set $K \subset \mathbb{R}^N$ a family of exterior Dirichlet problems for (1.3) in $\mathbb{R}^N \setminus K$ admits a continuum of stable slow-decay infinite-energy solutions. In particular, our results in § 4 show that the stability of slow-decay positive solutions of (1.3) is a robust property, which in some sense remains stable under perturbations of domain in the equation.

2. Equations with Hardy potential

We study the equation

$$-\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{2.1}$$

where $K = \{0\}$, or $\{0\} \in K$ and K is a connected compact set with the smooth boundary ∂K, $p > 1$, $N \geqslant 3$, $\nu > 0$ and $\nu_* := (N-2)/2$, so ν_*^2 is the Hardy critical constant in (1.4).

By a solution of (2.1) we understand a classical solution $u \in C^2(\mathbb{R}^N \setminus K)$, with no *a priori* assumption on the decay of $u(x)$ at ∞. We say that u is a weak solution of (2.1) in $\mathbb{R}^N$ if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\int \nabla u \cdot \nabla \varphi \, \mathrm{d}x + (\nu^2 - \nu_*^2) \int \frac{u\varphi}{|x|^2} \, \mathrm{d}x = \int u^p \varphi \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

Note that, for $\nu < \nu_*$, solutions of (2.1) must have a singularity at the origin (see Lemma 2.5); however, this singularity might be compatible with the concept of a weak solution in $\mathbb{R}^N$.

We say that a solution u of (2.1) in $\mathbb{R}^N \setminus K$ has *finite energy* if $u \in D_0^1(\mathbb{R}^N \setminus K)$, the completion of $C_c^\infty(\mathbb{R}^N \setminus K)$ with respect to the norm $\|\nabla \varphi\|_{L^2}$. We say that a solution u of (2.1) is *stable* in $\mathbb{R}^N \setminus K$ if the formal second variation at u of the energy that corresponds to (2.1) is non-negative definite, that is,

$$\int |\nabla \varphi|^2 \, \mathrm{d}x + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} \, \mathrm{d}x - p \int u^{p-1} \varphi^2 \, \mathrm{d}x \geqslant 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus K). \tag{2.2}$$

A solution $u > 0$ of (2.1) is called *semi-stable* in $\mathbb{R}^N \setminus K$ if it is stable in $\mathbb{R}^N \setminus B_R$ for some $R > 0$. A solution $u > 0$ of (2.1) is called *unstable* if it is not semistable. Note that these definitions do not require u to be a finite-energy solution. See [**8**] for the discussion of various notions of stability of infinite-energy solutions of elliptic equations of type (1.1).

2.1. Explicit radial solution

For $\nu > 0$, introduce the critical exponents

$$p_* := 1 + \frac{2}{\nu_* + \nu} \quad \text{and} \quad p^* := \begin{cases} 1 + \dfrac{2}{\nu_* - \nu} & \text{if } \nu < \nu_*, \\ \infty & \text{if } \nu \geqslant \nu_*. \end{cases}$$

Clearly, $p_* < p_S < p^*$. For $p > p_*$, set

$$U_\infty(x) := C_{p,\nu} |x|^{-2/(p-1)}, \quad \text{where } C_{p,\nu}^{p-1} := \nu^2 - \left(\nu_* - \frac{2}{p-1}\right)^2.$$

A direct computation shows that U_∞ is a positive solution of (2.1) for all $p_* < p < p^*$, while for $p \notin [p_*, p^*]$ the coefficient $C_{p,\nu}$ becomes negative. Note that $U_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ for $p > p_S$, that is, U_∞ is a weak solution of (2.1) in $\mathbb{R}^N$. However U_∞ is an infinite-energy solution because of its slow decay at ∞.

The importance of the solution U_∞ is due to the fact that it will be used as an elementary building block for constructing further solutions of (2.1). In order to do this, it is essential to understand the stability properties of U_∞.

Lemma 2.1. *Let $p \in (p_*, p^*)$ and $\nu > 0$. The solution U_∞ is stable if and only if*

$$p C_{p,\nu}^{p-1} \leqslant \nu^2, \tag{2.3}$$

while if (2.3) fails, then U_∞ is unstable.

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Proof. The formal second variation of the energy that corresponds to (2.1) at U_∞ is given by

$$\int |\nabla\varphi|^2\,\mathrm{d}x + (\nu^2 - \nu_*^2 - pC_{p,\nu}^{p-1})\int \frac{|\varphi|^2}{|x|^2}\,\mathrm{d}x.$$

Thus, the assertion follows directly from the fact that ν_*^2 is the optimal constant in the Hardy inequality (1.4).

Taking into account the scaling invariance of Hardy's inequality we also conclude that if (2.3) fails, then U_∞ must be unstable. □

The inequality (2.3) amounts to a third-degree algebraic expression, for which closed-form solutions could be obtained using Cardano's formulae; however, the explicit expressions for solutions are tedious. Below we present a qualitative analysis of (2.3). Set $s := -2/(p-1)$, so (2.3) transforms into

$$\frac{(s+\nu_*)^2(s-2)+2\nu^2}{|s|} \leqslant 0 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}).$$

Define

$$\theta(s) := (s+\nu_*)^2(s-2).$$

Solving (2.3) for $p_* < p < p^*$ is then equivalent to classifying the roots of the equation

$$\theta(s) = -2\nu^2 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}), \tag{2.4}$$

and solving the inequality

$$\theta(s) \leqslant -2\nu^2 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}). \tag{2.5}$$

Note that $\theta(0) = -2\nu_*^2$ and that θ has two critical points: a local maximum at $s_{\max} := -\nu_*$ with $\theta(s_{\max}) = 0$ and a local minimum at $s_{\min} := -(\nu_* - 4)/3$ with $\theta(s_{\min}) = -\frac{4}{27}(2+\nu_*)^3$. Define

$$\bar\nu := \sqrt{\frac{2}{27}(2+\nu_*)^3} = \sqrt{2\left(\frac{N+2}{6}\right)^3}.$$

Clearly, for every $\nu > 0$, (2.4) has exactly one root $\sigma_\#$ in the interval $(-\nu_* - \nu, -\nu_*)$. To analyse the roots of (2.5) in the interval $(-\nu_*, \min\{-\nu_* + \nu, 0\})$, we distinguish between the cases $s_{\min} < 0$ and $s_{\min} \geqslant 0$.

In the case $s_{\min} \geqslant 0$ (that is, $3 \leqslant N \leqslant 10$) we have the following.

(i) If $\nu \geqslant \nu_*$, then (2.4) has no roots in $(-\nu_*, 0)$ and (2.5) holds for all $s \in (-\nu_* - \nu, \sigma_\#]$.

(ii) If $0 < \nu < \nu_*$, then (2.4) has exactly one root $\sigma_- \in (-\nu_*, -\nu_* + \nu)$ and (2.5) holds for all $s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, -\nu_* + \nu)$.

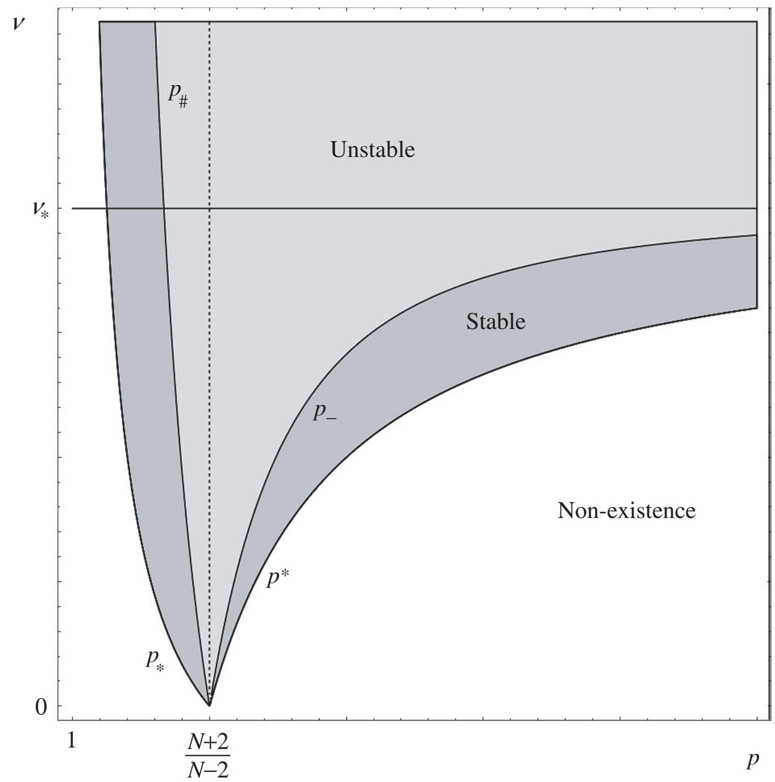

Figure 1. Stability properties of U_∞ in the case $3 \leqslant N \leqslant 10$.

In the case $s_{\min} < 0$ (that is, $N \geqslant 11$) we have the following.

(i) If $\nu > \bar{\nu}$, then (2.4) has no roots in $(-\nu_*, 0)$, so (2.5) holds for all $s \in (-\nu_* - \nu, \sigma_\#]$.

(ii) If $\nu_* < \nu \leqslant \bar{\nu}$, then (2.4) has exactly two roots σ_- and σ_+ in $(-\nu_*, 0)$ and $-\nu_* < \sigma_- \leqslant s_{\min} \leqslant \sigma_+ < 0$, so (2.5) holds for all $s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, \sigma_+]$.

(iii) If $0 < \nu \leqslant \nu_*$, then (2.4) has exactly one root σ_- in $(-\nu_*, 0)$ and $\sigma_- \in (-\nu_*, s_{\min})$, so (2.5) holds for all $s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, 0)$.

In what follows we define

$$p_\# = 1 - \frac{2}{\sigma_\#}, \qquad p_- := 1 - \frac{2}{\sigma_-}, \qquad p_+ := 1 - \frac{2}{\sigma_+},$$

and note that

$$1 < p_* < p_\# < p_S < p_- \leqslant p_+ < p^*$$

for all values of $N \geqslant 3$ and $\nu > 0$, when all the exponents are well defined. The above analysis then leads to the following characterization, equivalent to (2.3), of the stability properties of the solution U_∞ in terms of the original parameters p and ν.

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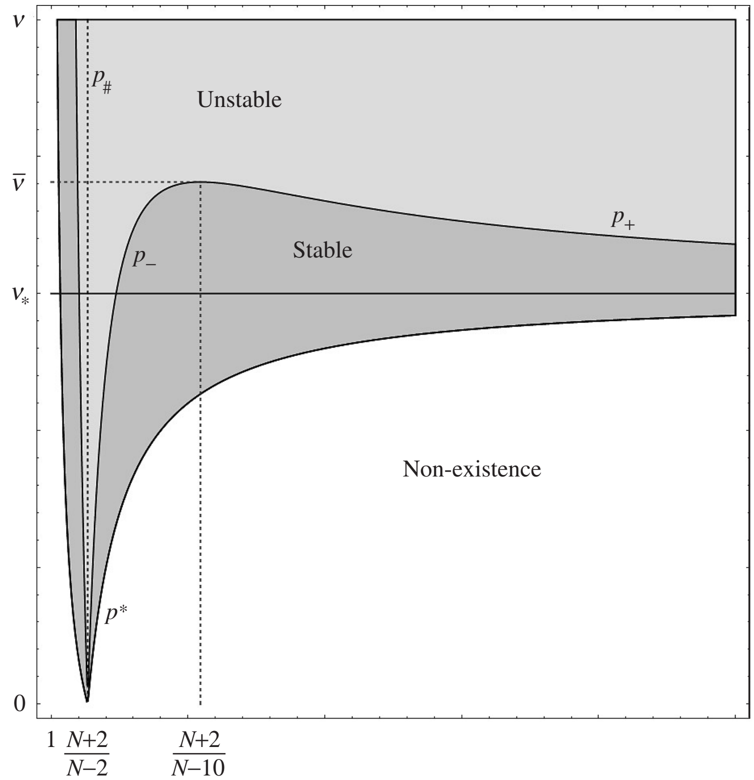

Figure 2. Stability properties of U_∞ in the case $N \geqslant 11$.

Lemma 2.2. *Let $p \in (p_*, p^*)$ and $\nu > 0$.*

(a) *If $\nu_* < \nu \leqslant \bar{\nu}$ and $N \geqslant 11$, then the solution U_∞ is stable for $p \in (p_*, p_\#] \cup [p_-, p_+]$ and unstable for $p \in (p_\#, p_-) \cup (p_+, p^*)$.*

(b) *If $0 < \nu < \nu_*$ and $N \geqslant 3$ or $\nu = \nu_*$ and $N \geqslant 11$, then the solution U_∞ is stable for $p \in (p_*, p_\#] \cup [p_-, p^*)$ and unstable for $p \in (p_\#, p_-)$.*

(c) *If $\nu \geqslant \nu_*$ and $3 \leqslant N \leqslant 10$ or $\nu \geqslant \bar{\nu}$ and $N \geqslant 11$, then the solution U_∞ is stable for $p \in (p_*, p_\#]$ and unstable for $p \in (p_\#, \infty)$.*

Remark 2.3. In the pure Laplacian case $\nu = \nu_*$ one calculates the explicit values

$$p_\# = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}, \qquad p_- = \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}};$$

here p_- is defined only for $N \geqslant 11$. Thus, for the Laplacian the exponent p_- coincides with the Joseph–Lundgren stability exponent p_{JL} (see [**13**] or [**17**, p. 50]), while the exponent $p_\#$ is known to appear in the context of local singularities of solutions of (1.1) (see [**16**, Lemma 5]).

Remark 2.4. If $N \geqslant 11$ and $\nu = \bar{\nu}$, then $p_- = p_+ = (N+2)/(N-10)$ is the only supercritical value of p where U_∞ is stable. Note, in addition, that $\lim_{\nu\to\infty} p_\#(\nu) = 1$, and also that if $3 \leqslant N \leqslant 10$, then $\lim_{\nu\uparrow\nu_*} p_-(\nu) = \infty$, while for $N \geqslant 11$, $\lim_{\nu\downarrow\nu_*} p_+(\nu) = \infty$.

2.2. Slow and fast decay solutions

Clearly, a solution u of (2.1) is a positive superharmonic of the linear Hardy operator, that is, u satisfies the linear inequation

$$-\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus K. \tag{2.6}$$

As a consequence, solutions of (2.1) with $\nu^2 < \nu_*^2$ are always singular at the origin, while for $\nu^2 > \nu_*^2$ solutions might vanish at the origin. More precisely, the following local decay properties for positive superharmonics of Hardy's operator hold (see [**15**]).

Lemma 2.5. *If $u > 0$ satisfies (2.6) in a neighbourhood of the origin, then*

$$\liminf_{|x|\to 0} \frac{u(x)}{|x|^{-\nu_*+\nu}} > 0, \qquad \liminf_{|x|\to 0} \frac{u(x)}{|x|^{-\nu_*-\nu}} < \infty. \tag{2.7}$$

If $u > 0$ satisfies (2.6) in an exterior domain, then

$$\liminf_{|x|\to \infty} \frac{u(x)}{|x|^{-\nu_*-\nu}} > 0, \qquad \liminf_{|x|\to 0} \frac{u(x)}{|x|^{-\nu_*+\nu}} < \infty. \tag{2.8}$$

Bidaut-Véron and Véron [**1**, Theorem 3.3] proved that the structure of the solution set of (2.1) in exterior domains that decay at ∞ no slower than U_∞ is essentially determined by the solutions of the equation

$$-\Delta_{S^{N-1}}\omega + C_{p,\nu}^{p-1}\omega = \omega^p, \quad \omega > 0 \quad \text{in } S^{N-1} \tag{2.9}$$

on the sphere S^{N-1}.

Lemma 2.6 (Bidaut-Véron and Véron [1, Theorem 3.3]). *Let $p \neq p_S$. If $u > 0$ satisfies (2.1) in $\mathbb{R}^N \setminus K$ and*

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{-2/(p-1)}} < \infty, \tag{2.10}$$

then either

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{-\nu_*-\nu}} = c \quad \textit{(fast decay)} \tag{2.11}$$

or there exists a positive solution $\omega(\cdot)$ of (2.9) such that

$$\lim_{|x|\to\infty} \frac{u(|x|,\cdot)}{|x|^{-2/(p-1)}} = \omega(\cdot) \quad \textit{(slow decay)} \tag{2.12}$$

in the $C^k(S^{N-1})$ topology for any $k \in \mathbb{N}$.

Remark 2.7. Clearly, $C_{p,\nu}$ is a constant solution of (2.9). For $1 < p < (N+1)/(N-3)$ it is known (see [**9**] or [**1**, Corollary 6.1]) that $C_{p,\nu}$ is the only solution of (2.9), provided that

$$(p-1)C_{p,\nu}^{p-1} \leqslant N-1, \tag{2.13}$$

while if (2.13) fails, then (2.9) admits non-constant solutions (see [**1**, Corollary 6.1], [**19**, Theorem 0.5] and [**12**, Theorem 1.3]). A similar result holds for some values $p > (N+1)/(N-3)$ (see [**3**]). The complete structure of the solution set of (2.9) is not yet fully understood (see [**2**, **12**] for some recent results in this direction).

Remark 2.8. If $\nu > 0$ and $p < p_S$, then (2.10) always holds (see [**1**, Remark 3.2]).

We classify the positive solutions of (2.1) into *fast-* and *slow-decay solutions* according to the alternatives (2.11) and (2.12). Note that for $p > p_S$ slow-decay solutions are always infinite-energy solutions, because of the slow decay rate (2.12) at ∞.

3. Radial slow-decay solutions in $\mathbb{R}^N$

The radial positive solutions $u(|x|) > 0$ of (2.1) in $\mathbb{R}^N \setminus \{0\}$ correspond to the positive solutions $U(r) = u(r)$ of the initial-value problem

$$-U'' - \frac{N-1}{r}U' + \frac{\nu - \nu_*}{r^2}U = U^p \quad (r > 0), \tag{3.1}$$

which can be studied through the phase-plane analysis.

The existence of a family of regular at the origin slow-decay solutions of (3.1) in the Laplacian case $\nu = \nu_*$ is well known and goes back at least to [**13**].

Theorem 3.1. *Let $p_S < p < p^*$. For any $\lambda > 0$ (3.1) then admits a unique positive solution $U_\lambda \in C^2(0,\infty)$ such that*

$$\lim_{r\to 0} \frac{U_\lambda(r)}{r^{-\nu_*+\nu}} = \lambda, \qquad \lim_{r\to\infty} \frac{U_\lambda(r)}{r^{-2/(p-1)}} = C_{p,\nu}. \tag{3.2}$$

Moreover,

$$U_\lambda(r) = \lambda^{2/(p-1)} U_1(\lambda r) \quad \forall \lambda > 0. \tag{3.3}$$

Furthermore, for $\lambda \in (0,\infty]$ the following properties hold.

(i) *If $pC_{p,\nu}^{p-1} \leqslant \nu^2$, then the solutions U_λ are stable and ordered in the sense that $0 < \lambda_1 < \lambda_2 \leqslant \infty$ implies that $U_{\lambda_1}(r) < U_{\lambda_2}(r)$ for every $r \geqslant 0$ and, in addition,*

$$\lim_{r\to\infty} \frac{U_{\lambda_2}(r) - U_{\lambda_1}(r)}{r^{-\nu_*}} > 0. \tag{3.4}$$

(ii) *If $pC_{p,\nu}^{p-1} > \nu^2$, then the solutions U_λ are unstable and oscillate in the sense that $0 < \lambda_1 < \lambda_2 \leqslant \infty$ implies that $U_{\lambda_2}(r) - U_{\lambda_1}(r)$ changes sign in $(R, +\infty)$ for arbitrary $R > 0$.*

The proof of the theorem follows the exposition in [**17**, pp. 50–53], with minor adjustments needed to accommodate $\nu \neq \nu_*$. We present the sketch of the arguments for the convenience of readers.

Proof of Theorem 3.1. Using the transformation

$$w(t) = r^{2/(p-1)} U(r), \quad t = \log(r), \tag{3.5}$$

problem (3.1) becomes an autonomous second-order differential equation

$$w'' + 2\beta w' + w^p - \gamma w = 0, \quad t \in \mathbb{R}, \tag{3.6}$$

where, since $p_S < p < p^*$,

$$\beta := \nu_* - \frac{2}{p-1} > 0 \quad \text{and} \quad \gamma = C_{p,\nu}^{p-1} = \nu^2 - \left(\nu_* - \frac{2}{p-1}\right)^2 > 0.$$

Set

$$\mathcal{E}(w) = \mathcal{E}(w, w') := \tfrac{1}{2}|w'|^2 - \tfrac{1}{2}\gamma w^2 + \frac{1}{p+1} w^{p+1}.$$

Then, $\mathcal{E}$ is a Lyapunov function for (3.6) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(w(t)) = -2\beta(w'(t))^2 \leqslant 0.$$

Set $x := w$ and $y := w'$. Problem (3.6) can then be written as an autonomous first-order system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -2\beta y + \gamma x - x^p \end{pmatrix} =: \Phi(x, y),$$

which possesses two equilibria

$$(0,0) \quad \text{and} \quad (\gamma^{1/(p-1)}, 0)$$

in the half-space $\{(x, y) \colon x \geqslant 0\}$. Define

$$A_0 := \nabla\Phi(0,0) = \begin{pmatrix} 0 & 1 \\ \gamma & -2\beta \end{pmatrix}, \qquad A_* := \nabla\Phi(\gamma, 0) = \begin{pmatrix} 0 & 1 \\ -(p-1)\gamma & -2\beta \end{pmatrix}.$$

The matrix A_0 has two real eigenvalues

$$\alpha_\pm := -\beta \pm \sqrt{\beta^2 + \gamma} = \frac{2}{p-1} - \nu_* \pm \nu,$$

such that $\alpha_- < 0 < \alpha_+$. The corresponding eigenvectors are $(1, \alpha_+)$ and $(1, \alpha_-)$, that is, $(0,0)$ is a saddle point of the vector field Φ. The matrix A_* has two eigenvalues

$$\alpha_\pm^* := -\beta \pm \sqrt{\beta^2 - (p-1)\gamma};$$

the corresponding eigenvectors are $(1, \alpha_+^*)$ and $(1, \alpha_-^*)$. Clearly, $\mathrm{Re}(\alpha_\pm^*) < 0$, so $(\gamma^{1/(p-1)}, 0)$ is always an attractor. Note also that $\alpha_\pm^*$ is real if and only if $pC_{p,\nu}^{p-1} \leqslant \nu^2$.

Using the Lyapunov function $\mathcal{E}$ one can show that the trajectory tangent at the origin to the eigenvector $(1, \alpha_+)$ is a heteroclinic orbit that connects the equilibria $(0,0)$ and $(\gamma^{1/(p-1)}, 0)$ (see [**17**, p. 52]). Moreover, since $(0,0)$ is a hyperbolic saddle-point, the uniqueness of such a heteroclinic orbit follows by standard arguments. The corresponding solution $w(t)$ exists for all $t \in \mathbb{R}$ and satisfies that

$$\lim_{t\to-\infty} w(t) = 0, \qquad \lim_{t\to+\infty} w(t) = \gamma^{1/(p-1)}. \tag{3.7}$$

Moreover, we can assume that $w(t)$ satisfies the normalization condition

$$\lim_{t\to-\infty} \frac{w(t)}{\mathrm{e}^{\alpha_+ t}} = 1. \tag{3.8}$$

Since (3.6) is autonomous, $w(t+\theta)$ is also a solution of (3.6) that corresponds to the same heteroclinic orbit, for any $\theta \in \mathbb{R}$. Given $\theta \in \mathbb{R}$, set $\lambda := \mathrm{e}^{\theta}$. Then,

$$U_\lambda(r) := r^{-2/(p-1)} w(\log(\lambda r)) = \lambda^{2/(p-1)} U(\lambda r),$$

and U_λ satisfies (3.2) in view of (3.8) and (3.7), that is, U_λ is the required solution of (3.1). The uniqueness of U_λ follows from the uniqueness of $w(t)$ since (3.5) defines a one-to-one correspondence between solutions of (3.1) and (3.6).

To understand the oscillation and stability properties of U_λ note that the eigenvalues $\alpha^*_\pm$ are real if and only if

$$\beta^2 \geqslant (p-1)\gamma,$$

which is equivalent to the stability condition (2.3). Note that then

$$\alpha_- < \alpha^*_- \leqslant -\nu_* \leqslant \alpha^*_+ < \alpha_+.$$

If the roots $\alpha^*_\pm$ are real, then arguments similar to those in [**17**, p. 53] show that the trajectory $w(t)$ is monotone increasing in t for all $t \in \mathbb{R}$. Hence, the solutions $U_\lambda(r)$ are monotone increasing in λ. In particular, $U_\lambda(r) < U_\infty(r)$ for any $\lambda > 0$ and the solutions U_λ are ordered. Furthermore, in view of (2.3) the solution U_∞ is stable. Since $U_\lambda(r) < U_\infty(r)$, we obtain that

$$pU_\lambda^{p-1}(|x|) \leqslant pU_\infty^{p-1}(|x|) = p\gamma|x|^2 \leqslant \nu^2|x|^2.$$

By Hardy's inequality we conclude that

$$\int |\nabla\varphi|^2 \, \mathrm{d}x + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} \, \mathrm{d}x - p \int U_\lambda^{p-1}(|x|)\varphi^2 \geqslant 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, that is, U_λ is a stable solution of (2.1). In addition, similarly to [**17**, Remark 9.4], we conclude that

$$\lim_{t\to\infty} \frac{w'(t)}{w(t) - \gamma^{1/(p-1)}} = \alpha^*_+ \geqslant -\beta,$$

which, after returning to the original variables and combined with (3.3), implies (3.4).

If $\alpha_\pm^*$ are complex, then, similarly to in [**17**, p. 52], one can see that the trajectory $(x(t), y(t))$ spirals infinitely many times around the attractor $(\gamma, 0)$, which suggests that the solutions U_λ oscillate in the sense of Theorem 3.1 (ii). The detailed proof of the oscillation and instability of U_λ when $\alpha_\pm^*$ are complex is a particular case of a more general Theorem 4.3, which will be proved in the next section. □

Remark 3.2. In the subcritical case $p_* < p \leqslant p_S$, (3.1) has no positive slow-decay solution that satisfies (3.2). Indeed, if $p = p_S$, then $\beta = 0$, $\mathrm{Re}(\alpha_\pm^*) = 0$ and the stationary point $(\gamma^{1/(p-1)}, 0)$ is a centre. One can show that the trajectory tangent at the origin to the eigenvector $(1, \alpha_+)$ is a homoclinic orbit. This homoclinic orbit corresponds to an explicit one-parameter family of finite-energy solutions of (3.1) (see [**19**, pp. 253–254]). If $p_* < p < p_S$, then $\beta > 0$, $\mathrm{Re}(\alpha_\pm^*) > 0$ and the stationary point $(\gamma^{1/(p-1)}, 0)$ is repelling. Hence, a heteroclinic between $(\gamma^{1/(p-1)}, 0)$ and $(0, 0)$ originates at $(\gamma^{1/(p-1)}, 0)$ and converges to $(0, 0)$ tangentially to the eigenvector $(1, \alpha_-)$. This heteroclinic orbit corresponds to a positive solution of (3.1) that decays at infinity as $O(|x|^{-\nu_* - \nu})$ and has a singularity at the origin of order $O(|x|^{-2/(p-1)})$.

4. Slow decay solutions in exterior domains

We first justify that the value of the non-existence exponent p^* is sharp. The result, which first appeared in [**1**, Remark 3.2], is an immediate consequence of Lemma 2.6.

Theorem 4.1. *Let $p \geqslant p^*$. Equation (2.1) then has no slow-decay solutions in $\mathbb{R}^N \setminus \bar{B}_R$ for arbitrary $R > 0$.*

Proof. Simply note that for $p > p^*$ one has that $C_{p,\nu}^{p-1} \leqslant 0$ and, hence, the equation (2.9) on the sphere does not have any positive solution. The conclusion then follows from Lemma 2.6. □

Remark 4.2. If $p > p^*$ then the slow decay rate is incompatible with the upper bound (2.8) of Lemma 2.5. This argument, however, does not apply when $p = p_*$.

Next we justify the sharpness of the stability and non-oscillation condition (2.3). The result below extends the oscillation statement of Theorem 3.1 beyond the radial setting. See also [**20**, Proposition 3.5] for related results in the pure Laplacian case $\nu = \nu_*$.

Theorem 4.3. *Let $p > p_S$, let $\nu > 0$ and let $pC_{p,\nu}^{p-1} > \nu^2$. Let $U_* > 0$ be a subsolution of (2.1) such that*

$$\liminf_{|x|\to\infty} \frac{U_*(x)}{|x|^{-2/(p-1)}} \geqslant C_{p,\nu}. \tag{4.1}$$

Then U_* is unstable. Furthermore, if $u > 0$ is a supersolution of (1.1), then either $u = U_*$ or $(u - U_*)_- \neq 0$ in $\mathbb{R}^N \setminus \bar{B}_R$, for arbitrary large $R > 0$.

Proof. From (4.1) we obtain that

$$pU_*^{p-1}(x) \geqslant (\nu^2 + \varepsilon)|x|^{-2} \quad (|x| > R_\varepsilon)$$

for some $\varepsilon > 0$ and $R_\varepsilon \geqslant R$. Assume that U_* is semistable, that is, there exists $R > 0$ such that (2.2) holds in $\mathbb{R}^N \setminus \bar{B}_R$. But then we arrive at

$$\int |\nabla\varphi|^2 \, \mathrm{d}x + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} \, \mathrm{d}x \geqslant (\nu^2 + \varepsilon) \int \frac{\varphi^2}{|x|^2} \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \bar{B}_{R_\varepsilon}),$$

a contradiction to Hardy's inequality. We conclude that U_* is unstable.

Furthermore, set $h = u - U_*$ and assume that $h \geqslant 0$ in $\mathbb{R}^N \setminus \bar{B}_R$ for some $R > 1$. By convexity and (4.1) we then obtain that

$$\begin{aligned} -\Delta h + \frac{\nu^2 - \nu_*^2}{|x|^2} h &\geqslant u^p - U_*^p = (U_* + h)^p - U_*^p \\ &\geqslant p U_*^{p-1} h \geqslant \frac{p C_{p,\nu}^{p-1}}{|x|^2} h \geqslant \frac{\nu^2 + \varepsilon^2}{|x|^2} h \quad (|x| > R_\varepsilon). \end{aligned} \tag{4.2}$$

It is well known that such an inequation has no positive solutions (see [**15**, Corollary 3.2]). We conclude that either $h = 0$ or h changes sign in $\mathbb{R}^N \setminus \bar{B}_{R_\varepsilon}$. □

Remark 4.4. The above result does not exclude the possibility that $u < U_*$ in an exterior domain. The latter is, however, not possible in the case when both U_* and u are slow-decay solutions. In particular, since all the solutions U_λ satisfy (4.1), the above result includes the oscillation statement (ii) of Theorem 3.1.

We next show that if the stability assumption (2.3) holds, then the slow-decay solutions of (2.1) in exterior domains are well ordered in a certain sense. We consider the exterior boundary-value problem for (2.1)

$$\left. \begin{aligned} -\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p, \quad u > 0 \quad &\text{in } \mathbb{R}^N \setminus K, \\ u = \psi \qquad\qquad &\text{on } \partial K, \end{aligned} \right\} \tag{4.3}$$

where $K \ni \{0\}$ is a connected compact set with the smooth boundary ∂K, and $\psi \in C(\partial K)$ is a non-negative continuous function.

Theorem 4.5. *Let $p > p_S$, let $\nu > 0$ and let $p C_{p,\nu}^{p-1} \leqslant \nu^2$. Let $U_* > 0$ be a slow-decay solution of (2.1) in $\mathbb{R}^N \setminus K$ such that, for some $R > 0$,*

$$U_*(x) \leqslant U_\infty(x) \quad (|x| > R)$$

holds. Given $\psi \in C(\partial K)$ such that

$$0 \leqslant \psi(x) \leqslant U_*(x) \quad \text{on } \partial K,$$

problem (4.3) admits a slow-decay solution U_*^ψ such that

$$0 < U_*^\psi \leqslant U_* \quad \text{in } \mathbb{R}^N \setminus K.$$

Moreover,

$$\lim_{|x| \to \infty} \frac{U_*(x) - U_*^\psi(x)}{|x|^{-\nu_*}} = 0. \tag{4.4}$$

Proof. We construct a subsolution $\underline{U}$ and a supersolution $\bar{U}$ such that

$$0 \leqslant \underline{U} \leqslant \bar{U} \leqslant U_* \quad \text{and} \quad \underline{U} = \bar{U} = \psi \quad \text{on } \partial K.$$

The existence of a solution U_*^ψ between $\underline{U}$ and $\bar{U}$ then follows via the classical sub- and supersolution argument (see [**14**, Theorem 38.1] or [**4**]).

Subsolution $\underline{U}$. Let $h_\psi > 0$ be the minimal positive solution to the problem

$$-\Delta h + \frac{\nu^2 - \nu_*^2}{|x|^2} h = pU_*^{p-1} h \quad \text{in } \mathbb{R}^N \setminus K, \qquad h = U_* - \psi \quad \text{on } \partial K. \tag{4.5}$$

The existence of such a solution is ensured by the Lax–Milgram theorem: indeed, by the assumptions

$$pU_*^{p-1}(x) \leqslant pU_\infty^{p-1}(x) \leqslant pC_{p,\nu}^{p-1} |x|^{-2} \leqslant \nu^2 |x|^{-2}. \tag{4.6}$$

Hence, the corresponding quadratic form to (4.5) is coercive on the Sobolev space $D_0^1(\mathbb{R}^N \setminus K)$. Moreover, from Lemma 2.6 we conclude that given a large $R > 0$ there exists $m \in (0, \nu^2]$ such that

$$pU_*^{p-1}(x) \geqslant m|x|^{-2} \quad (|x| > R).$$

A standard application of the comparison principle for Hardy operators (see Lemma 2.5 and [**15**, Lemma A.8]) then implies the two-sided bound

$$c|x|^{\alpha'_-} \leqslant h_\psi \leqslant C|x|^{\alpha^*_-} \quad (|x| > R),$$

where α^*_- is the smallest root of

$$-(\alpha + \nu_* - \nu)(\alpha + \nu_* + \nu) = pC_{p,\nu}^{p-1}$$

and α'_- is the smallest root of

$$-(\alpha + \nu_* - \nu)(\alpha + \nu_* + \nu) = m.$$

Note that $0 < m \leqslant pC_{p,\nu}^{p-1} \leqslant \nu^2$, so both equations have real roots and

$$-\nu_* - \nu < \alpha'_- \leqslant \alpha^*_- < -\nu_* < -\frac{2}{p-1}.$$

Set

$$\underline{U} := U_* - h_\psi.$$

Then,

$$\lim_{|x| \to \infty} \frac{\underline{U}(x)}{U_*(x)} = 1,$$

and by convexity a direct computation shows that

$$-\Delta \underline{U} = U_*^p - pU_*^{p-1} h_\psi \leqslant (U_* - h_\psi)^p = \underline{U}^p \quad \text{in } \mathbb{R}^N \setminus K,$$

that is, $\underline{U}$ is the required subsolution. In addition,

$$\lim_{|x| \to \infty} \frac{U_*(x) - \underline{U}(x)}{|x|^{-\nu_*}} = \lim_{|x| \to \infty} \frac{h_\psi(x)}{|x|^{-\nu_*}} = 0,$$

which implies (4.4).

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Supersolution $\bar{U}$. Let $\eta_\psi > 0$ be the minimal solution to the problem

$$-\Delta\eta + \frac{\nu^2 - \nu_*^2}{|x|^2} h = U_*^{p-1}\eta \quad \text{in } \mathbb{R}^N \setminus K, \qquad \eta = U_* - \psi \quad \text{on } \partial K.$$

Note that $U_*^{p-1} \leqslant pU_*^{p-1}$. Hence, solution η_ψ exists simply because (4.6) applies. Moreover, a comparison argument similar to the ones above shows that

$$0 < \eta_\psi < h_\psi \quad \text{in } \mathbb{R}^N \setminus K.$$

Define

$$\bar{U} := U_* - \eta_\psi.$$

Then,

$$\lim_{|x|\to\infty} \frac{\bar{U}(x)}{U_*(x)} = 1$$

and

$$-\Delta\bar{U} = U_*^{p-1}(U_* - \eta_\psi) \geqslant (U_* - \eta_\psi)^{p-1}(U_* - \eta_\psi) = \bar{U}^p \quad \text{in } \mathbb{R}^N \setminus K,$$

that is, $\bar{U}$ is the required supersolution. □

The next result shows that under suitable assumptions on the boundary data the exterior problem (4.3) admits a continuum of distinct slow-decay positive solutions, which in a certain sense could be interpreted as a perturbation of the family of slow-decay solutions $(U_\lambda)_{\lambda>0}$ constructed in Theorem 3.1.

Corollary 4.6. *Let $p > p_S$, let $\nu > 0$ and let $pC_{p,\nu}^{p-1} \leqslant \nu^2$. Then, for every $\psi \in C(\partial K)$ such that*

$$0 \leqslant \psi(x) < U_\infty(x) \quad \text{on } \partial K, \tag{4.7}$$

problem (4.3) admits a continuum of distinct positive slow-decay solutions.

Proof. Consider the family of slow-decay solutions $(U_\lambda)_{\lambda>0}$, constructed in Theorem 3.1. In view of (4.7) there exists $\lambda_\psi > 0$ such that, for all $\lambda > \lambda_\psi$,

$$0 \leqslant \psi(x) < U_\lambda(x) < U_\infty(x) \quad \text{on } \partial K.$$

Let $\lambda_1 \in (\lambda_\psi, \infty]$. In Theorem 4.5, choose $U_* := U_{\lambda_1}$ and note that in view of (4.4) and (3.4) the solution $U_{\lambda_1}^\psi$ given by Theorem 4.5 is distinct, with U_λ for any $\lambda > \lambda_\psi$, or with $U_{\lambda_2}^\psi$ for any $\lambda_2 > \lambda_\psi$, $\lambda_2 \neq \lambda_1$. In such a way, we have obtained a family of distinct slow-decay solutions $(U_\lambda^\psi)_{\lambda\in(\lambda_\psi,\infty]}$. □

Remark 4.7. In particular, if $p > p_S$ and $pC_{p,\nu}^{p-1} \leqslant \nu^2$, then the problem

$$-\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p \quad \text{in } \mathbb{R}^N \setminus K, \qquad u = 0 \quad \text{on } \partial K \tag{4.8}$$

admits a continuum of distinct positive slow-decay solutions $(U_\lambda^0)_{\lambda\in(0,\infty]}$. This partially extends the result in [**6**, Theorem 1], obtained in the pure Laplacian case $\nu = \nu_*$. Note, however, that in [**6**] the existence of a continuum of slow-decay solutions was proved for the whole range of supercritical exponents $p > p_S$, including the most challenging unstable regime $pC_{p,\nu_*}^{p-1} > \nu_*^2$. The techniques in [**6**] (see also [**5**] and a survey [**7**]) are based on linearization and perturbation arguments combined with a sophisticated machinery of harmonic expansions. Such considerations go beyond the scope of the present work.

Remark 4.8. In the pure Laplacian case $\nu = \nu_*$ it is known that if K is star shaped with respect to ∞, then (4.8) has no positive solutions in the subcritical range $1 < p \leqslant p_S$ (see [**18**, Theorem 2]). This suggests that the non-uniqueness statement of Corollary 4.6 cannot be extended beyond the supercritical range of exponents.

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